Path ideals of graphs

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Simplicial complexes

A simplicial complex on the vertex set V is a collection Δ of subsets of V such that if $F \in \Delta$ and $G \subset F$, then $G \in \Delta$.

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A vertex cover of Δ is a subset A of V, with the property that for every facet F_i there is a vertex $x_j \in A$ such that $x_j \in F_i$.
A minimal vertex cover of Δ is a subset A of V such that A is a vertex cover and no proper subset of A is a vertex cover of Δ.

The smallest cardinality of a minimal vertex cover of Δ is called the vertex covering number of Δ.

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Let $R = k[x_1, \dots, x_n]$, where k is a field.

For a simplicial complex Δ with vertex set $\{x_1, \dots, x_n\}$, the Stanley-Reisner ideal of Δ is defined as:

$$I_\Delta = (\prod_{x \in F} x : F \notin \Delta).$$

Example

Let $\Delta = \langle \{x_1, x_2, x_4\}, \{x_2, x_3\}, \{x_3, x_4\} \rangle$, *then*

$$I_\Delta = (x_2x_3x_4, x_1x_3) \subseteq k[x_1, \dots, x_4]$$

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Facet ideals

The facet ideal of Δ is

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Facet ideals

For a squarefree monomial ideal $I = (M_1, ..., M_q)$ we define $\delta_{\mathcal{F}}(I)$ to be the simplicial complex over a set of vertices $\{v_1, ..., v_n\}$ with facets $F_1, ..., F_q$, where for each i, $F_i = \{v_j \ : \ x_j | M_i, 1 \leq j \leq n\}$. We call $\delta_{\mathcal{F}}(I)$ the facet complex of I.

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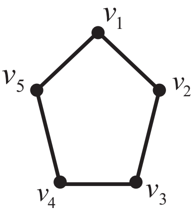
Edge ideals

Let $G = (V, E)$ be a finite simple graph with vertex set $V = \{x_1, \ldots, x_n\}$ and edge set E.
Associated to G is a monomial ideal

$$I(G) = (x_i x_j : \{x_i, x_j\} \in E),$$

called the edge ideal of G.

Edge ideals



$$I(G) = (x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_1).$$

Path ideals

Let $G = (V, E)$ be a directed graph with vertex set $V = \{x_1, \ldots, x_n\}$ and edge set E. Fix an integer $2 \leq t \leq n$. Associated to G is a monomial ideal

$$I_t(G) = (x_{i_1} \cdots x_{i_t} : x_{i_1}, \ldots, x_{i_t} \text{ is a path of length t in } G),$$

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We have $I_2(G) = I(G)$. So $I_t(G)$ is sometimes called the generalized edge ideal of G.

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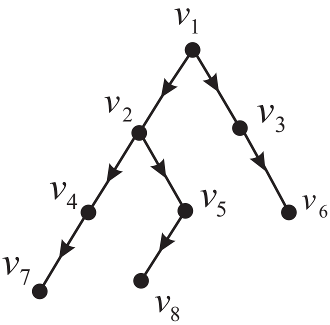

$$I_3(G) = (x_1x_3x_6, x_1x_2x_4, x_2x_4x_7, x_1x_2x_5, x_2x_5x_8).$$

Path ideals

For a directed graph G, the simplicial complex $\Delta_t(G)$ is defined to be

$$\Delta_t(G) = \langle \{v_{i_1}, \ldots, v_{i_t}\} : v_{i_1}, \ldots, v_{i_t} \text{ is a path of length t in } G \rangle.$$

So, we have

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A finitely generated graded module M over R is said to satisfy the Serre's condition S_r if depth $M_P \geq \min(r, \dim M_P)$, for all $P \in \mathrm{Spec}(R)$.

A graded R-module M is called sequentially Cohen-Macaulay (resp. S_r) (over k) if there exists a finite filtration of graded R-modules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

such that each M_i/M_{i-1} is Cohen-Macaulay (resp. S_r), and the Krull dimensions of the quotients are increasing:

$$\dim(M_1/M_0) < \dim(M_2/M_1) < \cdots < \dim(M_r/M_{r-1})$$

Suppose that I is a homogeneous ideal of R whose generators all have degree d. Then I has a linear resolution if for all $i \geq 0$, $\beta_{i,j}(I) = 0$ for all $j \neq i + d$.

Some algebraic preliminaries

Let I be a squarefree monomial ideal. The Alexander dual of $I = (x_{1,1} \cdots x_{1,s_1}, \ldots, x_{t,1} \cdots x_{t,s_t})$ is the ideal

$$I^{\vee} = (x_{1,1}, \ldots, x_{1,s_1}) \cap \cdots \cap (x_{t,1}, \ldots, x_{t,s_t}).$$

Yanagawa-Terai

Let $r \geq 2$. A simplicial complex Δ is S_r if and only if the minimal free resolution of $I_\Delta^\vee$ is linear in the first r steps.

Path ideals of cycles

It is known that

$$C_n \text{ is unmixed} \iff n = 3,\ 4,\ 5 \text{ or } 7.$$

Theorem

Let $t \geq 3$. Then $I_t(C_n)$ is unmixed if and only if $n = 2t + 1$ or $t \leq n \leq \lfloor 3t/2 \rfloor + 1$.

Path ideals of cycles

Francisco -Van Tuyl

Let $n \geq 3$. Then C_n is (sequentially) Cohen-Macaulay if and only if $n = 3$ or 5.

Path ideals of cycles

Theorem

Let $t \geq 2$. Then $R/I_t(C_n)$ is Cohen-Macaulay if and only if $n = t$ or $t + 1$ or $2t + 1$.

Haghighi et al.

One has C_n is S_2 if and only if $n = 3, 5$ or 7.

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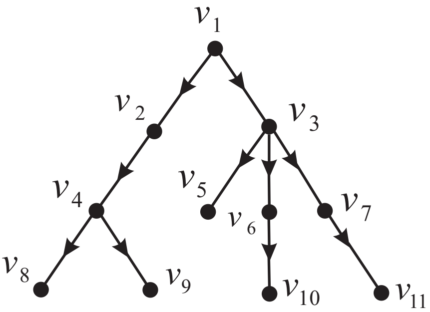
Theorem

Let $3 \leq t \leq n$ and $r \geq 2$. Then $R/I_t(C_n)$ is S_r if and only if it is Cohen-Macaulay or $\lceil \frac{n}{n-t-1} \rceil \geq r+3$.

Path ideals of trees



Francisco -Van tuyl

If G is a tree, then G is sequentially Cohen-Macaulay.

He -Van tuyl

Let G be a directed tree and $t \geq 2$. Then $R/I_t(G)$ is sequentially Cohen-Macaulay.

Theorem

Let I be a squarefree monomial ideal in R. Then R/I is Cohen-Macaulay if and only if R/I is sequentially Cohen-Macaulay and I is unmixed.

Haghighi et al.

Let I be a squarefree monomial ideal in R. Then R/I is S_r if and only if R/I is sequentially S_r and I is unmixed.

The clique complex of a finite graph G is the simplicial complex $\Delta(G)$ whose faces are the cliques of G.

Herzog et al.

Let G be a chordal graph on the vertex set V. Let $F_1, \ldots, F_m$ be the facets of $\Delta(G)$ which admit a free vertex. Then the following conditions are equivalent:
(i) G is Cohen-Macaulay.
(ii) G is unmixed.
(iii) V is the disjoint union of $F_1, \ldots, F_m$.

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level(v) :=the length of the unique path starting at the root and ending at v minus one.

Note that by removing leaves at level strictly less than $(t-1)$ from a tree Γ and repeating this process, one obtains a tree denoted by $C(\Gamma)$. This process is called cleaning process and the tree $C(\Gamma)$ is called the clean form of Γ.

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Let Γ be a tree over n vertices and $2 \leq t \leq n$. Suppose that $F_1, \ldots, F_m$ are all facets of $\Delta = \Delta_t(C(\Gamma))$ containing a leaf of $C(\Gamma)$ such that each leaf belongs to exactly one of them. If $V(\Delta)$ is the disjoint union of $F_1, \ldots, F_m$, then we say that Γ is t-partitioned (by $F_1, \ldots, F_m$).

Let Γ be a t-partitioned tree (by $F_1, \ldots, F_m$). We define a t-branch of Γ, as a path of length $t+1$, say P, which starts at a vertex of some F_i, like x, and $P \cap F_i = \{x\}$. Then, for each $i = 1, \ldots, m$, we define degree of F_i, as

$\mathrm{Deg}_\Gamma(F_i)$:=the number of vertices of F_i which are the first vertices of a t-branch of Γ.

We define degree of Γ, as

$$\mathrm{Deg}(\Gamma) := \max\{\mathrm{Deg}_\Gamma(F_i) \ : \ 1 \leq i \leq m\}.$$

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Path ideals of trees

We call a t-branch of Γ, initial if it intersects some F_i in the first vertex of F_i. Otherwise, we call it non-initial.

We define level of a t-branch P of Γ, denoted by level(P), as the level of the vertex x, where $P \cap F_i = \{x\}$ for some $i = 1, \ldots, m$.

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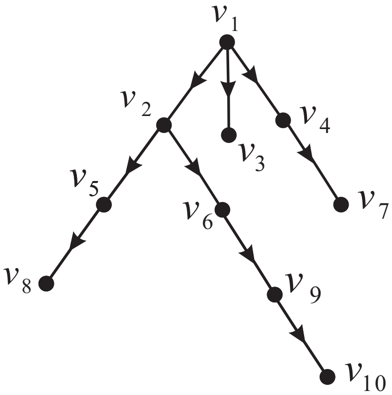
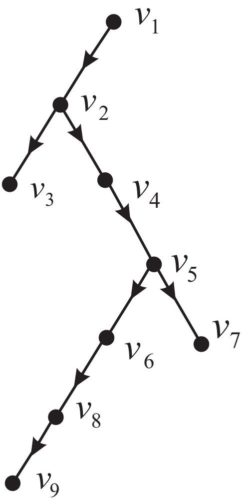
Let Γ be a t-partitioned tree over n vertices and $2 \leq t \leq n$. We say that Γ is fitting t-partitioned, if the following hold:

(1) $\mathrm{Deg}(\Gamma) \leq 1$; and
(2) $\mathrm{level}(P) \leq t-1$, for each non-initial t-branch P of Γ.



Γ_1 is fitting 3-partitioned by $F_1 = \{v_1, v_4, v_7\}$, $F_2 = \{v_2, v_5, v_8\}$ and $F_3 = \{v_6, v_9, v_{10}\}$. We have $\mathrm{Deg}(\Gamma_1) = 1$.

Path ideals of trees



Γ_2 *is* $3-$ *partitioned but not fitting* $3-$ *partitioned.*

Theorem

Let Γ be a tree over n vertices and $2 \leq t \leq n$. Then the following conditions are equivalent:
(i) $I_t(\Gamma)$ is unmixed.
(ii) $R/I_t(\Gamma)$ is Cohen-Macaulay.
(iii) $R/I_t(\Gamma)$ is S_r.
(iv) Γ is fitting t-partitioned.

Herzog et al.

Let G be a chordal graph. Then G is Gorenstein, if and only if G is a disjoint union of edges.

Theorem

Let Γ be a tree over n vertices and $2 \leq t \leq n$. Then $R/I_t(\Gamma)$ is Gorenstein if and only if $C(\Gamma)$ is L_t.

Thanks for your attention.