What pullback constructions can do for you

Parviz Sahandi,
University of Tabriz

IPM 9-10 Azar 1390

1. Definition and some properties of pullbacks
   - How can we construct an example
   - Find an example

2. An example of a *w*-Jaffard domain that is not a Jaffard domain
   - Definition of Jaffard domain
   - Definition of *w*-Jaffard domain
   - The desired example

Pullback diagram

In this talk, we shall discuss pullback diagrams of the following type:

$$\begin{array}{ccc} R & \rightarrow & D \\ \downarrow & & \downarrow \\ T & \xrightarrow{\varphi} & k. \end{array}$$

Where T is a domain, φ is a homomorphism from T onto a field k with $\ker(\varphi) = M$, D is a proper subring of k, and $R = \varphi^{-1}(D)$. We shall refer to this as a diagram of type $\square$.

Some properties of pullback constructions

Gilmer(1968), Fontana(1980)

In a diagram of type $\square$ we have:

1. $R/M \cong D$ and $M = (R : T)$, (It follows that $qf(R) = qf(T)$, and that each fractional ideal of T is a fractional ideal of R).
2. If T is quasilocal, then M is a divided prime ideal of R, and so each prime ideal of R is comparable with M. If in addition $k = qf(D)$, then $R_M = T$.
3. If T is quasilocal, then $\dim(R) = \dim(D) + \dim(T)$.
4. If T and D are quasilocal, then R is quasilocal.

Some properties of pullback constructions

Gilmer(1968), Fontana(1980)

In a diagram of type □ we have:

1. $R/M \cong D$ and $M = (R : T)$, (It follows that $qf(R) = qf(T)$, and that each fractional ideal of T is a fractional ideal of R).
2. If T is quasilocal, then M is a divided prime ideal of R, and so each prime ideal of R is comparable with M. If in addition $k = qf(D)$, then $R_M = T$.
3. If T is quasilocal, then $\dim(R) = \dim(D) + \dim(T)$.
4. If T and D are quasilocal, then R is quasilocal.

Some properties of pullback constructions

Gilmer(1968), Fontana(1980)

In a diagram of type $\square$ we have:

1. $R/M \cong D$ and $M = (R : T)$, (It follows that $qf(R) = qf(T)$, and that each fractional ideal of T is a fractional ideal of R).
2. If T is quasilocal, then M is a divided prime ideal of R, and so each prime ideal of R is comparable with M. If in addition $k = qf(D)$, then $R_M = T$.
3. If T is quasilocal, then $\dim(R) = \dim(D) + \dim(T)$.
4. If T and D are quasilocal, then R is quasilocal.

Some properties of pullback constructions

Gilmer(1968), Fontana(1980)

In a diagram of type $\square$ we have:

1. $R/M \cong D$ and $M = (R : T)$, (It follows that $qf(R) = qf(T)$, and that each fractional ideal of T is a fractional ideal of R).
2. If T is quasilocal, then M is a divided prime ideal of R, and so each prime ideal of R is comparable with M. If in addition $k = qf(D)$, then $R_M = T$.
3. If T is quasilocal, then $\dim(R) = \dim(D) + \dim(T)$.
4. If T and D are quasilocal, then R is quasilocal.

Some properties of pullback constructions

Gilmer(1968), Fontana(1980)

In a diagram of type $\square$ we have:

1. $R/M \cong D$ and $M = (R : T)$, (It follows that $qf(R) = qf(T)$, and that each fractional ideal of T is a fractional ideal of R).
2. If T is quasilocal, then M is a divided prime ideal of R, and so each prime ideal of R is comparable with M. If in addition $k = qf(D)$, then $R_M = T$.
3. If T is quasilocal, then $\dim(R) = \dim(D) + \dim(T)$.
4. If T and D are quasilocal, then R is quasilocal.

Some properties of pullback constructions

Gilmer(1968), Fontana(1980)

In a diagram of type □ we have:

1. $R/M \cong D$ and $M = (R : T)$, (It follows that $qf(R) = qf(T)$, and that each fractional ideal of T is a fractional ideal of R).
2. If T is quasilocal, then M is a divided prime ideal of R, and so each prime ideal of R is comparable with M. If in addition $k = qf(D)$, then $R_M = T$.
3. If T is quasilocal, then $\dim(R) = \dim(D) + \dim(T)$.
4. If T and D are quasilocal, then R is quasilocal.

Some properties of pullback constructions

Gilmer(1968), Fontana(1980)

In a diagram of type $\square$ we have:

1. $R/M \cong D$ and $M = (R : T)$, (It follows that $qf(R) = qf(T)$, and that each fractional ideal of T is a fractional ideal of R).
2. If T is quasilocal, then M is a divided prime ideal of R, and so each prime ideal of R is comparable with M. If in addition $k = qf(D)$, then $R_M = T$.
3. If T is quasilocal, then $\dim(R) = \dim(D) + \dim(T)$.
4. If T and D are quasilocal, then R is quasilocal.

Some properties of pullback constructions

Gilmer(1968), Fontana(1980)

In a diagram of type □ we have:

1. For each $P \in \mathrm{Spec}(R)$ with $M \not\subseteq P$, there is a unique $Q \in \mathrm{Spec}(T)$ such that $Q \cap R = P$, and this Q satisfies $T_Q = R_P$.
2. If $P \in \mathrm{Spec}(R)$ and $P \supseteq M$, then there is a unique $Q \in \mathrm{Spec}(D)$ such that $P = \varphi^{-1}(Q)$.Moreover, the following of canonical homomorphisms

$$\begin{array}{ccc} R_P & \rightarrow & D_Q \\ \downarrow & & \downarrow \\ T_M & \xrightarrow{\varphi} & k. \end{array}$$

is a pullback diagram.

Some properties of pullback constructions

Gilmer(1968), Fontana(1980)

In a diagram of type $\square$ we have:

1. For each $P \in \mathrm{Spec}(R)$ with $M \not\subseteq P$, there is a unique $Q \in \mathrm{Spec}(T)$ such that $Q \cap R = P$, and this Q satisfies $T_Q = R_P$.
2. If $P \in \mathrm{Spec}(R)$ and $P \supseteq M$, then there is a unique $Q \in \mathrm{Spec}(D)$ such that $P = \varphi^{-1}(Q)$.Moreover, the following of canonical homomorphisms

$$\begin{array}{ccc} R_P & \rightarrow & D_Q \\ \downarrow & & \downarrow \\ T_M & \xrightarrow{\varphi} & k. \end{array}$$

is a pullback diagram.

Some properties of pullback constructions

Gilmer(1968), Fontana(1980)

In a diagram of type $\square$ we have:

1. For each $P \in \text{Spec}(R)$ with $M \not\subseteq P$, there is a unique $Q \in \text{Spec}(T)$ such that $Q \cap R = P$, and this Q satisfies $T_Q = R_P$.
2. If $P \in \text{Spec}(R)$ and $P \supseteq M$, then there is a unique $Q \in \text{Spec}(D)$ such that $P = \varphi^{-1}(Q)$. Moreover, the following of canonical homomorphisms

$$\begin{array}{ccc} R_P & \rightarrow & D_Q \\ \downarrow & & \downarrow \\ T_M & \xrightarrow{\varphi} & k. \end{array}$$

is a pullback diagram.

Some properties of pullback constructions

Gilmer(1968), Fontana(1980)

In a diagram of type $\square$ we have:

1. For each $P \in \mathrm{Spec}(R)$ with $M \not\subseteq P$, there is a unique $Q \in \mathrm{Spec}(T)$ such that $Q \cap R = P$, and this Q satisfies $T_Q = R_P$.
2. If $P \in \mathrm{Spec}(R)$ and $P \supseteq M$, then there is a unique $Q \in \mathrm{Spec}(D)$ such that $P = \varphi^{-1}(Q)$.Moreover, the following of canonical homomorphisms

$$\begin{array}{ccc} R_P & \rightarrow & D_Q \\ \downarrow & & \downarrow \\ T_M & \xrightarrow{\varphi} & k. \end{array}$$

is a pullback diagram.

CPI extension

Boisen and Sheldon (1977)

The notion of CPI (complete pre-image) extension of a domain R with respect to a prime ideal P of R; this is denoted $R(P)$ and is defined by the following pullback diagram of type $\square$: Here φ is the canonical homomorphism.

$$\begin{array}{ccc} R(P) & \rightarrow & R/P \\ \downarrow & & \downarrow \\ R_P & \xrightarrow{\varphi} & R_P/PR_P. \end{array}$$

So that $R(P)/PR_P \cong R/P$ and $R(P)_{PR_P} = R_P$.

Classical D+M constructions

In a pullback diagram of type $\square$, the case where $T = V$ is a valuation domain of the form $K + M$, where K is a field and M is the maximal ideal of V is of crucial interest, known as classical "$D + M$" construction.

$$\begin{array}{ccc} D+M & \rightarrow & D \\ \downarrow & & \downarrow \\ K+M & \xrightarrow{\varphi} & K. \end{array}$$

The earliest use of the D+M construction in the literature seem due to Krull at 1936. Using the technique of D+M construction Krull gives an example of a one-dimensional quasilocal integrally closed domain that is not a valuation ring.

Classical D+M constructions

In a pullback diagram of type $\square$, the case where $T = V$ is a valuation domain of the form $K + M$, where K is a field and M is the maximal ideal of V is of crucial interest, known as classical "$D + M$" construction.

$$\begin{array}{ccc} D+M & \rightarrow & D \\ \downarrow & & \downarrow \\ K+M & \xrightarrow{\varphi} & K. \end{array}$$

The earliest use of the D+M construction in the literature seem due to Krull at 1936. Using the technique of D+M construction Krull gives an example of a one-dimensional quasilocal integrally closed domain that is not a valuation ring.

How can we use to produce desired examples

Suppose that we need to have a 2-dimensional valuation domain.

So that we need to know the behavior of pullback constructions with valuation domains.

Gilmer (1968)

Consider a diagram of type $\square$. Then R is a valuation domain if and only if D and T are valuation domains and $k = qf(D)$.

How can we use to produce desired examples

Suppose that we need to have a 2-dimensional valuation domain.

So that we need to know the behavior of pullback constructions with valuation domains.

Gilmer (1968)

Consider a diagram of type □. Then R is a valuation domain if and only if D and T are valuation domains and $k = qf(D)$.

How can we use to produce desired examples

Suppose that we need to have a 2-dimensional valuation domain.

So that we need to know the behavior of pullback constructions with valuation domains.

Gilmer (1968)

Consider a diagram of type $\square$. Then R is a valuation domain if and only if D and T are valuation domains and $k = qf(D)$.

How can we use to produce desired examples

Suppose that we need to have a 2-dimensional valuation domain.

So that we need to know the behavior of pullback constructions with valuation domains.

Gilmer (1968)

Consider a diagram of type $\square$. Then R is a valuation domain if and only if D and T are valuation domains and $k = qf(D)$.

The example

Let k be a field and X and Y be indeterminates over k. Let $T := k(X)[Y]_{(Y)}$ which is a 1-dimensional valuation domain. Note that $T = k(X) + M$, where $M := Yk(X)[Y]_{(Y)}$ is the maximal ideal of T. Let $D := k[X]_{(X)}$ and consider

$$\begin{array}{ccc} R := k + M' & \rightarrow & k[X]_{(X)} \\ \downarrow & & \downarrow \\ T = k(X) + M & \xrightarrow{\varphi} & k(X). \end{array}$$

where $M' = Xk[X]_{(X)} + Yk(X)[Y]_{(Y)}$. Since D and T are valuation domains and $k(X) = qf(D)$, then R is a valuation domain. Now the dimension of R:

$$\dim(R) = \dim(D) + \dim(T) = 1 + 1 = 2.$$

The example

Let k be a field and X and Y be indeterminates over k. Let $T := k(X)[Y]_{(Y)}$ which is a 1-dimensional valuation domain. Note that $T = k(X) + M$, where $M := Yk(X)[Y]_{(Y)}$ is the maximal ideal of T. Let $D := k[X]_{(X)}$ and consider

$$\begin{array}{ccc} R := k + M' & \rightarrow & k[X]_{(X)} \\ \downarrow & & \downarrow \\ T = k(X) + M & \xrightarrow{\varphi} & k(X). \end{array}$$

where $M' = Xk[X]_{(X)} + Yk(X)[Y]_{(Y)}$. Since D and T are valuation domains and $k(X) = qf(D)$, then R is a valuation domain. Now the dimension of R:

$$\dim(R) = \dim(D) + \dim(T) = 1 + 1 = 2.$$

The example

Let k be a field and X and Y be indeterminates over k. Let $T := k(X)[Y]_{(Y)}$ which is a 1-dimensional valuation domain. Note that $T = k(X) + M$, where $M := Yk(X)[Y]_{(Y)}$ is the maximal ideal of T. Let $D := k[X]_{(X)}$ and consider

$$\begin{array}{ccc} R := k + M' & \rightarrow & k[X]_{(X)} \\ \downarrow & & \downarrow \\ T = k(X) + M & \xrightarrow{\varphi} & k(X). \end{array}$$

where $M' = Xk[X]_{(X)} + Yk(X)[Y]_{(Y)}$. Since D and T are valuation domains and $k(X) = qf(D)$, then R is a valuation domain. Now the dimension of R:

$$\dim(R) = \dim(D) + \dim(T) = 1 + 1 = 2.$$

The example

Let k be a field and X and Y be indeterminates over k. Let $T := k(X)[Y]_{(Y)}$ which is a 1-dimensional valuation domain. Note that $T = k(X) + M$, where $M := Yk(X)[Y]_{(Y)}$ is the maximal ideal of T. Let $D := k[X]_{(X)}$ and consider

$$\begin{array}{ccc} R := k + M' & \rightarrow & k[X]_{(X)} \\ \downarrow & & \downarrow \\ T = k(X) + M & \xrightarrow{\varphi} & k(X). \end{array}$$

where $M' = Xk[X]_{(X)} + Yk(X)[Y]_{(Y)}$. Since D and T are valuation domains and $k(X) = qf(D)$, then R is a valuation domain. Now the dimension of R:

$$\dim(R) = \dim(D) + \dim(T) = 1 + 1 = 2.$$

The example

Let k be a field and X and Y be indeterminates over k. Let $T := k(X)[Y]_{(Y)}$ which is a 1-dimensional valuation domain. Note that $T = k(X) + M$, where $M := Yk(X)[Y]_{(Y)}$ is the maximal ideal of T. Let $D := k[X]_{(X)}$ and consider

$$\begin{array}{ccc} R := k + M' & \rightarrow & k[X]_{(X)} \\ \downarrow & & \downarrow \\ T = k(X) + M & \xrightarrow{\varphi} & k(X). \end{array}$$

where $M' = Xk[X]_{(X)} + Yk(X)[Y]_{(Y)}$. Since D and T are valuation domains and $k(X) = qf(D)$, then R is a valuation domain. Now the dimension of R:

$$\dim(R) = \dim(D) + \dim(T) = 1 + 1 = 2.$$

The example

Let k be a field and X and Y be indeterminates over k. Let $T := k(X)[Y]_{(Y)}$ which is a 1-dimensional valuation domain. Note that $T = k(X) + M$, where $M := Yk(X)[Y]_{(Y)}$ is the maximal ideal of T. Let $D := k[X]_{(X)}$ and consider

$$\begin{array}{ccc} R := k + M' & \rightarrow & k[X]_{(X)} \\ \downarrow & & \downarrow \\ T = k(X) + M & \xrightarrow{\varphi} & k(X). \end{array}$$

where $M' = Xk[X]_{(X)} + Yk(X)[Y]_{(Y)}$. Since D and T are valuation domains and $k(X) = qf(D)$, then R is a valuation domain. Now the dimension of R:

$$\dim(R) = \dim(D) + \dim(T) = 1 + 1 = 2.$$

Some other ideal theoretic properties

Consider a diagram of type $\square$.

Prüfer and pullback

Then R is a Prüfer domain if and only if D and T are Prüfer domains and $k = qf(D)$.

Noetherian and pullback

Then R is a Noetherian domain if and only if T is Noetherian, $D = F$ is a field and $[k : F] < \infty$.

A question

Sahandi (2009)

Is there an example of a *w*-Jaffard domain that is not a Jaffard domain?

Jaffard domain

The valuative dimension of a domain D was defined by Jaffard as: $\dim_v(D) := \sup\{\dim(V) | V \text{ is a valuation overring of } D\}$.

Jaffard (1959)

- $\dim(D) \leq \dim_v(D)$;
- $\dim(D) = \dim_v(D) \Leftrightarrow \dim(D[X_1, \ldots, X_k]) = k + \dim(D)$ for all $k \in \mathbb{N}$.

Anderson, Bouvier, Dobbs, Fontana and Kabbaj (1988)

Definition: An integral domain D is called a Jaffard domain if $\dim(D) < \infty$ and $\dim(D) = \dim_v(D)$.

Jaffard domain

The valuative dimension of a domain D was defined by Jaffard as: $\dim_v(D) := \sup\{\dim(V) | V \text{ is a valuation overring of } D\}$.

Jaffard (1959)

- $\dim(D) \leq \dim_v(D)$;
- $\dim(D) = \dim_v(D) \Leftrightarrow \dim(D[X_1, \ldots, X_k]) = k + \dim(D)$ for all $k \in \mathbb{N}$.

Anderson, Bouvier, Dobbs, Fontana and Kabbaj (1988)

Definition: An integral domain D is called a Jaffard domain if $\dim(D) < \infty$ and $\dim(D) = \dim_v(D)$.

Jaffard domain

The valuative dimension of a domain D was defined by Jaffard as: $\dim_v(D) := \sup\{\dim(V) | V \text{ is a valuation overring of } D\}$.

Jaffard (1959)

- $\dim(D) \leq \dim_v(D)$;
- $\dim(D) = \dim_v(D) \Leftrightarrow \dim(D[X_1, \ldots, X_k]) = k + \dim(D)$ for all $k \in \mathbb{N}$.

Anderson, Bouvier, Dobbs, Fontana and Kabbaj (1988)

Definition: An integral domain D is called a Jaffard domain if $\dim(D) < \infty$ and $\dim(D) = \dim_v(D)$.

Anderson, Bouvier, Dobbs, Fontana and Kabbaj (1988)

Theorem: For a diagram of type $(\square)$, let $F = qf(D)$ and $d := \text{tr.deg.}(k/F)$. Then:

- $\dim(R) = \max\{\dim(T), \dim(D) + \dim(T_M)\}$.
- $\dim_v(R) = \max\{\dim_v(T), \dim_v(D) + \dim_v(T_M) + d\}$.

w-operation

Let D be an integral domain with quotient field K.

The *v*-operation on D is defined as $E^v := (E^{-1})^{-1}$, with $E^{-1} := (D : E) := \{x \in K | xE \subseteq D\}$ for each fractional ideal E of D.

The ring of fractions $\mathrm{Na}(D, v) := D[X]_{N_v}$, is called the *v*-Nagata ring of D, where $N_v := \{f \in D[X] \mid f \neq 0 \text{ and } c_D(f)^v = D\}$.

The *w* operation on D is defined as follows:
$E^w := E\mathrm{Na}(D, v) \cap K$, for all fractional ideal E of D.

w-operation

Let D be an integral domain with quotient field K.

The *v*-operation on D is defined as $E^v := (E^{-1})^{-1}$, with $E^{-1} := (D : E) := \{x \in K | xE \subseteq D\}$ for each fractional ideal E of D.

The ring of fractions $\text{Na}(D, v) := D[X]_{N_v}$, is called the *v*-Nagata ring of D, where $N_v := \{f \in D[X] \mid f \neq 0 \text{ and } c_D(f)^v = D\}$.

The *w* operation on D is defined as follows:
$E^w := E\text{Na}(D, v) \cap K$, for all fractional ideal E of D.

w-operation

Let D be an integral domain with quotient field K.

The v-operation on D is defined as $E^v := (E^{-1})^{-1}$, with $E^{-1} := (D : E) := \{x \in K | xE \subseteq D\}$ for each fractional ideal E of D.

The ring of fractions $\text{Na}(D, v) := D[X]_{N_v}$, is called the v-Nagata ring of D, where $N_v := \{f \in D[X] \mid f \neq 0 \text{ and } c_D(f)^v = D\}$.

The w operation on D is defined as follows:
$E^w := E\text{Na}(D, v) \cap K$, for all fractional ideal E of D.

w-operation

Let D be an integral domain with quotient field K.

The v-operation on D is defined as $E^v := (E^{-1})^{-1}$, with $E^{-1} := (D : E) := \{x \in K | xE \subseteq D\}$ for each fractional ideal E of D.

The ring of fractions $\text{Na}(D, v) := D[X]_{N_v}$, is called the v-Nagata ring of D, where $N_v := \{f \in D[X] \mid f \neq 0 \text{ and } c_D(f)^v = D\}$.

The w operation on D is defined as follows:
$E^w := E\text{Na}(D, v) \cap K$, for all fractional ideal E of D.

w-dim(D) and w-dim$_v(D)$

Definition: The w-Krull dimension of D is defined as

$$w\text{-}\dim(D) := \sup\left\{ n \,\middle|\, \begin{array}{c} (0) = P_0 \subset P_1 \subset \cdots \subset P_n \text{ where } P_i \\ \text{is a prime ideal of } D \text{ s.t. } P_i^w = P_i \end{array} \right\}.$$

Jaffard (1960)

Definition: We say that a valuation overring V of D is a w-valuation overring of D provided $F^w \subseteq FV$, for each fractional ideal F of D.

S. (2009)

Definition: The w-valuative dimension of D, is defined as:

$$w\text{-}\dim_v(D) := \sup\{\dim(V) | V \text{ is a } w\text{-valuation overring of } D\}.$$

w-dim(D) and w-dim$_v(D)$

Definition: The w-Krull dimension of D is defined as

$$w\text{-}\dim(D) := \sup\left\{ n \,\middle|\, \begin{array}{l} (0) = P_0 \subset P_1 \subset \cdots \subset P_n \text{ where } P_i \\ \text{is a prime ideal of } D \text{ s.t. } P_i^w = P_i \end{array} \right\}.$$

Jaffard (1960)

Definition: We say that a valuation overring V of D is a w-valuation overring of D provided $F^w \subseteq FV$, for each fractional ideal F of D.

S. (2009)

Definition: The w-valuative dimension of D, is defined as:

$$w\text{-}\dim_v(D) := \sup\{\dim(V) \mid V \text{ is a } w\text{-valuation overring of } D\}.$$

w-dim(D) and w-dim$_v(D)$

Definition: The w-Krull dimension of D is defined as

$$w\text{-}\dim(D) := \sup\left\{ n \,\middle|\, \begin{array}{l} (0) = P_0 \subset P_1 \subset \cdots \subset P_n \text{ where } P_i \\ \text{is a prime ideal of } D \text{ s.t. } P_i^w = P_i \end{array} \right\}.$$

Jaffard (1960)

Definition: We say that a valuation overring V of D is a w-valuation overring of D provided $F^w \subseteq FV$, for each fractional ideal F of D.

S. (2009)

Definition: The w-valuative dimension of D, is defined as:

$$w\text{-}\dim_v(D) := \sup\{\dim(V) \mid V \text{ is a } w\text{-valuation overring of } D\}.$$

w-Jaffard domain

S. (2009)

Theorem: $w\text{-dim}(D) \leq w\text{-dim}_v(D)$.

S. (2009)

Definition: The domain D is said to be a w-Jaffard domain, if $w\text{-dim}(D) < \infty$ and $w\text{-dim}(D) = w\text{-dim}_v(D)$.

S. (2009)

Theorem: If D is a Krull domain then it is a w-Jaffard.

w-Jaffard domain

S. (2009)

Theorem: $w\text{-dim}(D) \leq w\text{-dim}_v(D)$.

S. (2009)

Definition: The domain D is said to be a w-Jaffard domain, if $w\text{-dim}(D) < \infty$ and $w\text{-dim}(D) = w\text{-dim}_v(D)$.

S. (2009)

Theorem: If D is a Krull domain then it is a w-Jaffard.

w-Jaffard domain

S. (2009)

Theorem: $w\text{-}\dim(D) \leq w\text{-}\dim_v(D)$.

S. (2009)

Definition: The domain D is said to be a w-Jaffard domain, if $w\text{-}\dim(D) < \infty$ and $w\text{-}\dim(D) = w\text{-}\dim_v(D)$.

S. (2009)

Theorem: If D is a Krull domain then it is a w-Jaffard.

S. (2011)

Theorem: For a diagram of type ($\square$), let $F = qf(D)$ and $d := \text{tr.deg.}(k/F)$. Then:

- $w\text{-}\dim(R) = \max\{w\text{-}\dim(T), w\text{-}\dim(D) + \dim(T_M)\}$.
- $w\text{-}\dim_v(R) = \max\{w\text{-}\dim_v(T), w\text{-}\dim_v(D) + \dim_v(T_M) + d\}$.

The example

Let K be a field. Let (V_1, M_1) and (V_2, M_2) be incomparable valuation domains of $K(W, X, Y, Z)$, such that $\dim(V_1) = 1$ and $\dim(V_2) = 3$ such that $V_1/M_1 \cong K(W, X, Z)$. Then $T := V_1 \cap V_2$ is a 3 dimensional Prüfer domain with $\mathfrak{m}_1 := M_1 \cap T$ and $\mathfrak{m}_2 := M_2 \cap T$ as maximal ideals such that $T_{\mathfrak{m}_1} = V_1$ and $T_{\mathfrak{m}_2} = V_2$. Note that $T/\mathfrak{m}_1 \cong K(W, X, Z)$

$$\begin{array}{ccc} R := \varphi^{-1}(K[W, X]) & \rightarrow & D := K[W, X] \\ \downarrow & & \downarrow \\ T & \xrightarrow{\varphi} & T/\mathfrak{m}_1. \end{array}$$

Notice that $d := tr.deg.(K(W, X, Z)/K(W, X)) = 1$.

The example

Let K be a field. Let (V_1, M_1) and (V_2, M_2) be incomparable valuation domains of $K(W, X, Y, Z)$, such that $\dim(V_1) = 1$ and $\dim(V_2) = 3$ such that $V_1/M_1 \cong K(W, X, Z)$. Then $T := V_1 \cap V_2$ is a 3 dimensional Prüfer domain with $\mathfrak{m}_1 := M_1 \cap T$ and $\mathfrak{m}_2 := M_2 \cap T$ as maximal ideals such that $T_{\mathfrak{m}_1} = V_1$ and $T_{\mathfrak{m}_2} = V_2$. Note that $T/\mathfrak{m}_1 \cong K(W, X, Z)$

$$\begin{array}{ccc} R := \varphi^{-1}(K[W,X]) & \rightarrow & D := K[W,X] \\ \downarrow & & \downarrow \\ T & \xrightarrow{\varphi} & T/\mathfrak{m}_1 . \end{array}$$

Notice that $d := tr.deg.(K(W,X,Z)/K(W,X)) = 1$.

The example

Let K be a field. Let (V_1, M_1) and (V_2, M_2) be incomparable valuation domains of $K(W, X, Y, Z)$, such that $\dim(V_1) = 1$ and $\dim(V_2) = 3$ such that $V_1/M_1 \cong K(W, X, Z)$. Then $T := V_1 \cap V_2$ is a 3 dimensional Prüfer domain with $\mathfrak{m}_1 := M_1 \cap T$ and $\mathfrak{m}_2 := M_2 \cap T$ as maximal ideals such that $T_{\mathfrak{m}_1} = V_1$ and $T_{\mathfrak{m}_2} = V_2$. Note that $T/\mathfrak{m}_1 \cong K(W, X, Z)$

$$\begin{array}{ccc} R := \varphi^{-1}(K[W,X]) & \rightarrow & D := K[W,X] \\ \downarrow & & \downarrow \\ T & \xrightarrow{\varphi} & T/\mathfrak{m}_1. \end{array}$$

Notice that $d := tr.deg.(K(W, X, Z)/K(W, X)) = 1.$

The example

Let K be a field. Let (V_1, M_1) and (V_2, M_2) be incomparable valuation domains of $K(W, X, Y, Z)$, such that $\dim(V_1) = 1$ and $\dim(V_2) = 3$ such that $V_1/M_1 \cong K(W, X, Z)$. Then $T := V_1 \cap V_2$ is a 3 dimensional Prüfer domain with $\mathfrak{m}_1 := M_1 \cap T$ and $\mathfrak{m}_2 := M_2 \cap T$ as maximal ideals such that $T_{\mathfrak{m}_1} = V_1$ and $T_{\mathfrak{m}_2} = V_2$. Note that $T/\mathfrak{m}_1 \cong K(W, X, Z)$

$$\begin{array}{ccc} R := \varphi^{-1}(K[W, X]) & \rightarrow & D := K[W, X] \\ \downarrow & & \downarrow \\ T & \xrightarrow{\varphi} & T/\mathfrak{m}_1 . \end{array}$$

Notice that $d := tr.deg.(K(W, X, Z)/K(W, X)) = 1$.

R is w-Jaffard

Thus we have:

$$\begin{aligned} w\text{-}\dim(R) &= \max\{w\text{-}\dim(T), w\text{-}\dim(K[W,X]) + \dim(T_{\mathfrak{m}_1})\} \\ &= \max\{3, 1+1\} = 3, \text{ and} \\ w\text{-}\dim_v(R) &= \max\{w\text{-}\dim_v(T), w\text{-}\dim_v(K[W,X]) + \dim_v(T_{\mathfrak{m}_1}) + d\} \\ &= \max\{3, 1+1+1\} = 3. \end{aligned}$$

This means that R is a w-Jaffard domain of w-dimension 3.

R is not Jaffard

But we have

$$\begin{aligned}\dim(R) &= \max\{\dim(T), \dim(K[W,X]) + \dim(T_{\mathfrak{m}_1})\}\\ &= \max\{3, 2+1\} = 3, \text{ and}\\ \dim_v(R) &= \max\{\dim_v(T), \dim_v(K[W,X]) + \dim_v(T_{\mathfrak{m}_1}) + d\}\\ &= \max\{3, 2+1+1\} = 4.\end{aligned}$$

Therefore R is not a Jaffard domain.

THANKS FOR YOUR ATTENTION