The 2nd Seminar on
Combinatorial Commutative Algebra, February 8 and 9, 2012, IPM, Tehran, Iran

Squarefree Vertex Cover Algebras

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In this paper we introduce squarefree vertex cover algebras. We study the question when these algebras coincide with the ordinary vertex cover algebras and when these algebras are standard graded. In this context we exhibit a duality theorem for squarefree vertex cover algebras.
This talk is based on a joint work with Farhad Rahmati.

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Extensions of Stanley-Reisner Theory: Cell Complexes and More

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Stanley-Reisner rings and resolutions are usually associated to simplicial complexes. We show how one can do this as well for polyhedral complexes.

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A Generalization of k-Cohen-Macaulay Simplicial Complexes

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Combinatorial commutative algebra is a branch of commutative algebra in which ideas from combinatorics, commutative algebra, homological algebra and topology meet together to develop a beautiful theory. Among active topics of this branch, the theory of Cohen-Macaulay and Buchsbaum simplicial complexes, have received significant attention. In fact, there are still many unknown questions in this topic which need further investigation.
In this talk we introduce a new class of simplicial complexes, called $k - \mathrm{CM}_t$ simplicial complexes. This class is not only a generalization of the notions of Cohen-Macaulay and Buchsbaum simplicial complexes, but it is also a generalization of k-Cohen-Macaulay connectivity introduced in [1].
As one of the main results, we show that for a $(d-1)$-dimensional simplicial complex, the following conditions are equivalent ($\mathrm{CM}_t := 1 - \mathrm{CM}_t$):

(1) Δ is CM_t over a filed K;

(2) Δ is pure and $\widetilde{H}_i(\mathrm{link}_\Delta(\sigma)\ ; K) = 0$ for any $\sigma \in \Delta$ with $\sharp\sigma \geq t$ and $i < d - \sharp\sigma - 1$;

(3) $H_i(|\Delta|, |\Delta| \setminus p\ ; K) = 0$ for all $p \in |\Delta| \setminus |\Delta_{t-2}|$ and all $i < d-1$, where Δ_{t-2} is the $(t-2)$-skeleton of Δ.

We also show that if Δ and Δ' are two simplicial complexes of dimensions $d-1$ and $d'-1$, respectively, then the join of these two simplicial complexes is CM_t if and only if Δ is $\mathrm{CM}_{t-d'}$ and Δ' is CM_{t-d}.
This is a joint work with S. Yassemi and R. Zaare Nahandi.

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Algebraic Properties of Product of Graphs

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Let G and H be two simple graphs and let $G * H$ denotes the graph theoretical product of G by H. In this paper we provide graded Betti numbers, Castelnuovo-Mumford regularity, projective dimension, h-vector and Hilbert series of $G * H$ in terms of that information of G and H. More generally, we present explicit formulae to compute graded Betti numbers, h-vector, and Hilbert series of disjoint union of complexes. We will also prove that the family of graphs for which the regularity of each edge ring equals the maximum number of pairwise 3-disjoint edges, is closed under product of graphs.

Aluffi Torsion-free Graphs

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Let R be a ring and $J \subset I$ ideals of R. The symmetric algebra $\mathcal{S}_R(I)$ maps surjectively both to the Rees algebra $\mathcal{R}_R(I)$ and (by functoriality of the symmetric algebra) to $\mathcal{S}_{R/J}(I/J)$. The Aluffi algebra of I/J is

$$\mathcal{A}_{R/J}(I/J) :\ = \mathcal{S}_{R/J}(I/J) \otimes_{\mathcal{S}_R(I)} \mathcal{R}_R(I).$$

The Aluffi algebra is squeezed as $\mathcal{S}_{R/J}(I/J) \twoheadrightarrow \mathcal{A}_{R/J}(I/J) \twoheadrightarrow \mathcal{R}_{R/J}(I/J)$ and is moreover a residue ring of the ambient Rees algebra $\mathcal{R}_R(I)$. The Kernel of the right hand surjection is the so called module of Valabrega-Vall which is

$$\mathcal{W}(J \subset I) = \bigoplus_{t \geq 0} \frac{J \cap I^t}{JI^{t-1}}.$$

A. Nasrollah Nejad and A. Simis proved that the Valabrega-Valla module is torsion of the Aluffi algebra. A pair of ideals $J \subset I$ of a Notherian ring R is said to be *Aluffi torsion-free* if the map $\mathcal{A}_{R/J}(I/J) \twoheadrightarrow \mathcal{R}_{R/J}(I/J)$ is injective or equivalently $J \cap I^n = JI^{n-1}$ for any positive integer n. In this talk, we consider $J := I(G)$ the edge ideal of the graph G and I stand for the Jacobian ideal of J. We give some necessary and sufficient condition for graphs equivalent to Aluffi torsion-free property. Finally, we present several examples of graphs which are Aluffi torsion-free or not. This work based on joint work with Rashid Zaare Nahandi.

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On Binomial Edge Ideals

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Let G be a finite simple graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and let $S = k[x_1, \ldots, x_n, y_1, \ldots, y_n]$ be the polynomial ring over a field k. Then the **binomial edge ideal** of G in S, denoted by J_G, is generated by binomials of the form $f_{ij} = x_i y_j - x_j y_i$, where $i < j$ and $\{v_i, v_j\}$ is an edge of G. We study some algebraic properties of this ideal. We characterize all graphs whose binomial edge ideals have linear resolutions.
This talk is based on a joint work with D. Kiani.

Skeletons of $\mathbb{Z}^n$-graded Modules

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Let Δ be a simplicial complex of dimension $d-1$ on the vertex set $[n] = \{1,\dots,n\}$, K a field and $K[\Delta]$ the Stanley–Reisner ring of Δ. The depth of $K[\Delta]$ can be expressed in terms of the skeletons of Δ, as has been shown by D. Smith [4, Theorem 3.7] for pure simplicial complexes, and by Hibi [1, Corollary 2.6] in general. The jth skeleton of Δ is the simplicial subcomplex $\Delta^{(j)} = \{F \in \Delta\colon\ |F| \leq j\}$ of Δ. The result is that $\operatorname{depth} K[\Delta] = \max\{j\colon\ \Delta^{(j)} \text{ is Cohen–Macaulay}\}$. This result was generalized by Herzog et. al [2] for monomial ideals in general.

First note that one has the following chain of Stanley–Reisner ideals $I_\Delta = I_{\Delta^d} \subset I_{\Delta^{d-1}} \subset \cdots \subset I_{\Delta^0} \subset S$ with $\dim S/I_{\Delta^{(j)}} = j$ for all j. For an arbitrary monomial ideal $I \subset S$ we defined in a natural way a similar chain of monomial ideals $I = I_d \subset I_{d-1} \subset \cdots \subset I_0 \subset S$ with $\dim S/I_j = j$ for all j, and of course this chain should satisfy the condition that $\operatorname{depth} S/I = \max\{j\colon\ S/I_j \text{ is Cohen–Macaulay}\}$. Such a natural chain of monomial ideals with these properties indeed exists. The ideal I_j is called the jth *skeleton ideal* of I.

For the construction of the skeleton ideals of I Herzog et. al consider the so-called *characteristic poset* $P^g_{S/I}$ introduced in [3]. Here $g \in \mathbb{N}^n$ is an integer vector such that $g \geq a$ for all a for which x^a belongs to the minimal set of monomial generators of I, and $P^g_{S/I}$ is the (finite) poset of all $b \in \mathbb{N}^n$ such that $b \leq g$ and $x^b \notin I$. The partial order on $\mathbb{N}^n$ is defined as follows: $a \leq b$ if and only if $a(i) \leq b(i)$ for $i = 1,\dots,n$. In case of a Stanley–Reisner ideal I_Δ and $g = (1,1,\dots,1)$ this poset is just the face poset of Δ. For each $b \in \mathbb{N}^n$, let $\rho(b) = |\{j\colon\ b(j) = g(j)\}|$. It has been shown in [3, Corollary 2.6] that $\dim S/I = \max\{\rho(b)\colon\ b \in P^g_{S/I}\}$. The integer function ρ was used to define the skeleton ideals of I.

In this talk we generalize this result and define the skeleton submodules of M for any finitely generated $\mathbb{Z}^n$-graded S-module M with the property that $\dim M_a \leq 1$ for all $a \in \mathbb{Z}^n$.

Let M be a finitely generated $\mathbb{N}^n$ graded S-module with the property that $\dim_K(M_a) \leq 1$ for each $a \in \mathbb{N}^n$. Suppose $m_1,\dots,m_t$ is a minimal set of generators of M with $\deg m_i = a_i$. We choose $g \in \mathbb{Z}^n$ such that $a_i \leq g$ for all i , and let P^g_M be the set of all $c \in \mathbb{N}^n$ with $c \leq g$ and such that $a_i \leq c$ for some i. The set P^g_M viewed as a subposet of $\mathbb{N}^n$ is a finite poset. This poset is called the *characteristic poset* of M with respect to g. There is a natural choice for g, namely the join of all a_i. Notice that for each $a \in P^g_M$ there is a unique homogenous element $m \in M$ such that $\deg m = a$.

Let $d = \dim M$ the Krull dimension of M. It is easy to see that

$$d = \max\{\rho(b)\colon b \in P^g_M\}. \tag{1}$$

As a consequence of (1) we obtain

Lemma 0.1. *Let $d = \dim M$. Then $\rho(b) \leq d$ for all $b \in \mathbb{N}^n$ for which $m \in M$ and $\deg(m) = b$.*

Let $0 \leq j \leq d$, and $M_{(j)}$ be the submodule of M generated by the components M_b with $\rho(b) > j$. We set $M^{(j)} = M/M_{(j)}$ and called it the jth skeleton of M. The following results is crucial

Theorem 0.2. *For each $0 \leq j \leq d$ if $M^{(j)} \neq M^{(j-1)}$, then $M^{(j)}/M^{(j-1)}$ is Cohen–Macauley of dimension j.*

As a corollary of this result one has that $\operatorname{depth} M = \max\{j : M^{(j)}$ is Cohen–Macauley$\}$.

Let M be a finitely generated $\mathbb{Z}^n$-graded S-module, $m \in M$ be a homogeneous element and $Z \subset X = \{x_1, \ldots, x_n\}$. We denote by $mK[Z]$ the K-subspace of M generated by all homogeneous elements of the form mu, where u is a monomial in $K[Z]$. The K-subspace $mK[Z]$ is called a *Stanley space of dimension* $|Z|$ if $mK[Z]$ is a free $K[Z]$-module.

A decomposition $\mathscr{D}$ of M as a finite direct sum of Stanley spaces is called a *Stanley decomposition* of M. The minimal dimension of a Stanley space in the decomposition $\mathscr{D}$ is called the *Stanley depth* of $\mathscr{D}$, denoted $\operatorname{sdepth} \mathscr{D}$. We set

$$\operatorname{sdepth} M = \max\{\operatorname{sdepth} \mathscr{D} : \mathscr{D} \text{ is a Stanley decomposition of } M\},$$

and call this number the *Stanley depth* of M. A famous conjecture of Stanley [5] asserts that $\operatorname{sdepth} M \geq \operatorname{depth} M$.

As an application we show that the Stanley depth of a finitely generated $\mathbb{N}^n$-graded S-module can be computed in finite number of steps and if the Stanley conjecture holds for all Cohen–Macauley $\mathbb{N}^n$-graded S-module, then it is true for all $\mathbb{N}^n$-graded S-module.

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Cellular Resolutions of Transversal Monomial Ideals

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Let $S = k[x_{11}, ..., x_{1b_1}, ..., x_{n1}, ..., x_{nb_n}], b_i \geq 1$, be the polynomial ring in $m = b_1 + ... + b_n$ indeterminates over a field k. Consider the prime ideals $P_i = (x_{i1}, ..., x_{ib_i})$ for $i = 1, ..., n$. Let

$$I_t = \sum_{1 \leq j1 < ... < jt \leq n} P_{j1 ... P_j t}$$

be the *t-transversal* monomial ideal on $P_1, ..., P_n$. This class includes the class of squarefree Veronese ideals. We construct a polytopal subdivision of the $(n-t)$-simplex that supports a minimal free resolution of the squarefree Veronese ideal. This construction is extended to build up a polytopal cell complex that supports a minimal free resolution of an arbitrary t-transversal monomial ideal.

Regularity of Monomial Ideals Generated in Degree 2 and 3

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In this talk we introduce some operations on the edges of a graph or circuits of a 3-uniform clutter. Then, we prove that linearity and regularity of the resolution of the corresponding ideals are conserved under these operations. We apply the operations to give an alternative prove for Fröberg's Theorem on linearity of resolution of edge ideal of graphs. Also we prove that any clutter corresponding to a triangulation of sphere does not have linear resolution while any proper sub-clutter of it has linear resolution. This is a joint work with Marcel Morales, Abbas Nasrollah Nejad and Ali Akbar Yazdan Pour.