Squarefree vertex cover algebras

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Vertex cover algebras

- Let Δ be a simplicial complex on $[n]$, and k a nonnegative integer. A k-cover of Δ is a nonzero vector $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{N}^n$ with $\sum_{i \in F} c_i \geq k$ for all $F \in \mathcal{F}(\Delta)$. The k-cover $\mathbf{c}$ is called squarefree if $c_i \in \{0, 1\}$ for all i.

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- A k-cover $\mathbf{c}$ is decomposable if there exist an i-cover $\mathbf{a}$ and a j-cover $\mathbf{b}$ as follows:
  1. $\mathbf{c} = \mathbf{a} + \mathbf{b}$;
  2. $k = i + j$;

- Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K.
- We denote by $J_k(\Delta)$ the K-vector space spanned by the monomials $x^{\mathbf{c}}$ where $\mathbf{c}$ is a k-cover.

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Vertex cover algebra

$$A(\Delta) = \bigoplus_{k \geq 0} J_k(\Delta)t^k \subseteq S[t]$$

- For any subset $F \subseteq [n]$, let P_F denotes the prime ideal of S generated by the variables x_i with $i \in F$. Then

$$I(\Delta)^\vee = \bigcap_{F \in \mathcal{F}(\Delta)} P_F.$$

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  where $I(\Delta)$ is the facet ideal of Δ.
- $A(\Delta)$ is the symbolic Rees algebra of $I(\Delta)^{\vee}$.

Squarefree vertex cover algebra

- Let $B(\Delta)$ be the S-subalgebra of $S[t]$ generated by the elements $x^{\mathbf{c}}t^k$ where $\mathbf{c}$ is a squarefree k-cover. The algebra $B(\Delta)$ is called the squarefree vertex cover algebra of Δ.

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- Observe that $B(\Delta)$ is a graded S-algebra,

$$B(\Delta) = \bigoplus_{k \geq 0} L_k(\Delta) t^k.$$

Each $L_k(\Delta)$ is a monomial ideal in S and $L_k(\Delta) \subseteq J_k(\Delta)$.

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- **Proposition.**
  (i) $L_k(\Delta)^{sq} = \bigcap_{F \in \mathcal{F}(\Delta)} P_F{}^{\langle k \rangle}$.

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- The k-th squarefree power of a monomial ideal I, denoted by $I^{\langle k \rangle}$, is defined to be $(I^k)^{sq}$.
- **Proposition.**
  - **(i)** $L_k(\Delta)^{sq} = \bigcap_{F \in \mathcal{F}(\Delta)} {P_F}^{\langle k \rangle}$.
  - **(ii)** The algebra $B(\Delta)$ is standard graded if and only if

$$\bigcap_{F \in \mathcal{F}(\Delta)} {P_F}^{\langle k \rangle} = \Big(\bigcap_{F \in \mathcal{F}(\Delta)} P_F \Big)^{\langle k \rangle}.$$

- **Theorem.** Let Δ be of dimension $d-1$, and $k \in \{1, \ldots, d\}$. Then

$$L_k(\Delta)^{sq} \subseteq I(\Delta^{(d-k)})^{\vee} \quad \text{for all} \quad k.$$

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Furthermore, the following conditions are equivalent:

(i) Δ is a pure simplicial complex;
(ii) $L_k(\Delta)^{sq} = I(\Delta^{(d-k)})^{\vee}$ for some $k \neq 1$;
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- **Corollary.** Let Δ be a pure simplicial complex of dimension $d-1$. Then

$$L_j(\Delta^{(d-i)})^{sq} = L_i(\Delta^{(d-j)})^{sq}$$

- **Corollary.** Let Δ be a pure simplicial complex of dimension $d - 1$. Then

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- Let $P = \{p_1, \ldots, p_m\}$ be a finite poset and $d \geq 1$ an integer. Let $\Delta_d(P)$ be the simplicial complex for which $I(\Delta_d(P))^{\vee} = H_d(P)$, the generalized Hibi ideal. Then

$$I(\Delta_d(P)^{(k)})^{\vee} = (I(\Delta_d(P))^{\vee})^{\langle d-k \rangle} \quad \text{for all} \quad k$$

  Therefore, $B(\Delta_d(P))$ is standard graded.

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- **Corollary.** Let Δ be a pure simplicial complex of dimension $d-1$. Then

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- Combinatorially, that means every k-cover of Δ can be written as a sum of k 1-covers.
- Ideal-theoretically, that means the symbolic powers of $I(\Delta)^\vee$ coincide with the ordinary powers.
- By a result of Herzog, Hibi and Trung, the Rees algebra of $I(\Delta)^\vee$ is a Cohen-Macaulay normal domain. Therefore, the associated graded ring of $I(\Delta)^\vee$ is also Cohen-Macaulay.

- Furthermore, by a result of Huneke, Ulrich and Vasconcelos, the associated graded ring of $I(\Delta)^\vee$ is reduced, and $I(\Delta)^\vee$ is normally torsion free.

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- (Herzog, Hibi, Trung and Zheng) In addition, assume that $I(\Delta)^\vee$ is generated in one degree. Let $k[I(\Delta)^\vee]$ denote the toric ring generated by the monomial generators of $I(\Delta)^\vee$. Then $k[I(\Delta)^\vee]$ is a normal Cohen-Macaulay domain.

Questions

(1) Is $A(\Delta)$ standard graded if and only if $B(\Delta)$ is standard graded?

(2) When do we have $A(\Delta) = B(\Delta)$?

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  Let Δ be the simplicial complex with the following facets:

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  However $B(\Delta)$ is standard graded.

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- **Theorem.** Let G be a graph, and suppose Δ is the simplicial complex with $I(\Delta) = I(G)^\vee$. Then $B(\Delta)$ is standard graded if and only if $A(\Delta)$ is standard graded.

- A subcomplex Γ of Δ, denoted by $\Gamma \subseteq \Delta$, is a simplicial complex such that $\mathcal{F}(\Gamma) \subseteq \mathcal{F}(\Delta)$.

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- A cycle of length r of Δ is a sequence $i_1, F_1, i_2, \ldots, F_r, i_{r+1} = i_1$ where $F_j \in \mathcal{F}(\Delta)$, $i_j \in [n]$ and $v_j, v_{j+1} \in F_j$ for $j = 1, \ldots, r$.

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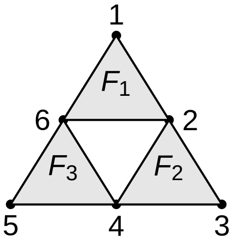
- A cycle is called special if each facet of the cycle contains exactly two vertices of the cycle.

- **Theorem.** Let Δ be a simplicial complex. Then the following conditions are equivalent:
  - **(i)** The algebra $B(\Gamma)$ is standard graded for all $\Gamma \subseteq \Delta$;
  - **(ii)** The algebra $A(\Gamma)$ is standard graded for all $\Gamma \subseteq \Delta$;
  - **(iii)** Δ has no special odd cycles.

Equality of the algebras

- **Theorem.** Let G be a finite graph on $[n]$. Then the following conditions are equivalent:
  - **(i)** $A(G) = B(G)$;
  - **(ii)** For every cycle C of G of odd length and for every vertex i of G there exist a vertex j of the cycle C such that $\{i, j\}$ is an edge of G.

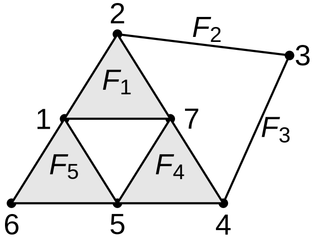
The facets of Δ form the special odd cycle $6, F_1, 2, F_2, 4, F_3, 6$ of length 3 and the equality $B(\Delta) = A(\Delta)$ holds.



This figure shows a simplicial complex Δ of dimension 2 such that

$$1, F_1, 2, F_2, 3, F_3, 4, F_4, 5, F_5, 1$$

is a special odd cycle of length 5. In this case $B(\Delta) \neq A(\Delta)$.



Equality of the algebras

- Let Δ be a simplicial complex with $\mathcal{F}(\Delta) = \{F_1, \ldots, F_m\}$. We say that Δ has the strict intersection property if
(i) $|F_i \cap F_j| \leq 1$ for all $i \neq j$;
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- We define the intersection graph G_Δ of such a simplicial complex Δ as follows:
  $V(G_\Delta) = \{v_1, \ldots, v_m\}$
  $E(G_\Delta) = \{\{v_i, v_j\} : \ i \neq j \quad \text{and} \quad F_i \cap F_j \neq \emptyset\}$

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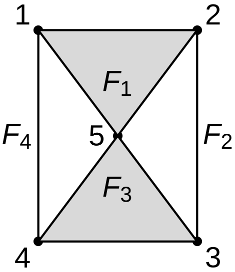
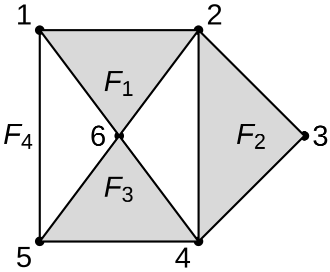
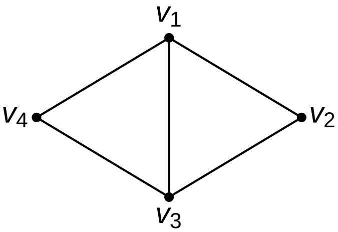
- Let Δ be a simplicial complex satisfying the strict intersection property and suppose that no two cycles of G_Δ have precisely two edges in common. Then $B(\Delta) = A(\Delta)$ if and only if each connected component of G_Δ is a bipartite graph or an odd cycle.







The intersection graph of Δ_1 and Δ_2

Vertex cover algebra of Borel sets

- A subset $\mathcal{B} \in 2^{[n]}$ is called Borel, if whenever $F \in B$ and $i < j$ for some $i \in [n] \setminus F$ and $j \in F$, then $(F \setminus \{j\}) \cup \{i\} \in \mathcal{B}$.

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- Elements $F_1, \ldots, F_m \in \mathcal{B}$ are called Borel generators of $\mathcal{B}$, denoted by $\mathcal{B} = B(F_1, \ldots, F_m)$, if $\mathcal{B}$ is the smallest Borel subset of $2^{[n]}$ such that $F_1, \ldots, F_m \in \mathcal{B}$.

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- A Borel set B is called principal, if there exists $F \in \mathcal{B}$ such that $\mathcal{B} = B(F)$.

Vertex cover algebra of Borel sets

- A squarefree monomial ideal $I \subseteq S$ is called a (principal) squarefree Borel ideal, if there exists a (principal) Borel set $\mathcal{B} \subseteq 2^{[n]}$ such that

$$I = (\{x_F : F \in \mathcal{B}\}).$$

If $\mathcal{B} = B(F_1, \ldots, F_m)$, then the monomials $x_{F_1}, \ldots, x_{F_m}$ are called the Borel generators of I.

Vertex cover algebra of Borel sets

- **Theorem.** Let $\mathcal{B} = B(F)$ be a principal Borel set with Borel generator $F = \{i_1 < i_2 < \cdots < i_d\}$, and let Δ be the simplicial complex with $\mathcal{F}(\Delta) = \mathcal{B}$. Then the S-algebra $B(\Delta)$ is generated by the elements $x_H t^k$, for k=1,. . . ,d, where

$$H \in B(\{q, q+1, \ldots i_{k+q-1}\}: \; q = 1, \ldots, d-k+1).$$

Vertex cover algebra of Borel sets

- In order to compute the generators of $B(\Delta)$ in more general case, one can use the fact $(I+J)^{\vee} = I^{\vee} \cap J^{\vee}$ for all monomial ideals I and J.

Vertex cover algebra of Borel sets

- In order to compute the generators of $B(\Delta)$ in more general case, one can use the fact $(I+J)^\vee = I^\vee \cap J^\vee$ for all monomial ideals I and J.
- **Proposition.** Let $\mathcal{B} = B(F_1, \ldots, F_m)$ be a Borel set such that $|F_i| = |F_j|$ for all i, j, and suppose Δ is a simplicial complex with $\mathcal{F}(\Delta) = \mathcal{B}$. Then $L_k(\Delta)^{sq}$ is a squarefree Borel ideal for all k.

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- **Theorem.** Let $\mathcal{B} = B(F)$ be a principal Borel set with Borel generator $F = \{i_1 < i_2 < \cdots < i_d\}$, and let Δ be the simplicial complex with $\mathcal{F}(\Delta) = \mathcal{B}$. Then $B(\Delta^{(j)}) = A(\Delta^{(j)})$ for every $j = 0, \ldots, d-1$.

Vertex cover algebra of Borel sets

- **Corollary.** Let Σ_n denote the simplex of all subsets of $[n]$. Then the S-algebra $A(\Sigma_n^{(d-1)})$ is minimally generated by the monomials $x_{j_1}x_{j_2}\cdots x_{j_{n-d+k}}t^k$, where $k = 1,\ldots,d$ and $1 \leq j_1 < j_2 < \cdots < j_{n-d+k} \leq n$.

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- **Proposition.** Let $\mathcal{B} = B(F)$ be a principal Borel set with Borel generator $F = \{i_1 < i_2 < \cdots < i_d\}$, and let Δ be the simplicial complex with $\mathcal{F}(\Delta) = \mathcal{B}$. Then $x_1x_2\cdots x_{i_d}t^d$ belongs to the minimal set of monomial generators of $A(\Delta)$ if and only if $i_1 \neq 1$.