Extensions of Stanley-Reisner theory: Cell complexes and beyond

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Polyhedral cell complexes

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k a field.
Form the k-vector space with basis the cells of dimension p,

$$\mathcal{C}_p(\Gamma) = \bigoplus_{F \text{ cell of dimension } p} k \cdot F.$$

Homology and cohomology

1. Augmented chain complex:

$$\mathcal{C}.(\Gamma) \; : \; \cdots \to \mathcal{C}_i(\Gamma) \xrightarrow{d} \mathcal{C}_{i-1}(\Gamma) \to \cdots \to \mathcal{C}_0(\Gamma) \to k,$$

with differential

$$d(F) = \bigoplus_{G \text{ is } p-1-\text{face of } F} sgn(G, F) \cdot G.$$

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2. Augmented cochain complex:

$$\mathcal{C}^{\cdot}(\Gamma) = \mathrm{Hom}(\mathcal{C}.(\Gamma), k).$$

Reduced cohomology $\tilde{H}^i(\Gamma) = H^i(\mathcal{C}.(\Gamma))$.

Squarefree modules

Let ϵ by i'th unit vector in $\mathbb{N}^n$. An $\mathbb{N}^n$-graded S-module M is *squarefree* if for $\mathbf{b} \in \mathbb{N}^n$ the multiplication

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Identify subset, f.ex. $R = \{1, 2, 5\} \subseteq [6]$, with its characteristic vector $\mathbf{r} = (1, 1, 0, 0, 1, 0)$ in $\mathbb{N}^6$. Let $M_R = M_{\mathbf{r}}$.

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The module M is determined by M_R where $R \subseteq [n]$, and by the multiplications between them.

Alexander dual

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- The multiplication

$$(M^*)_R \xrightarrow{\cdot x_i} (M^*)_{R \cup \{i\}}$$

is the dual map of the multiplication

$$M_{R^c \setminus \{i\}} \xrightarrow{\cdot x_i} M_{R^c}.$$

Free squarefree modules

Free squarefree module: $\oplus_{R \subseteq [n]} S^{\beta_R} \cdot R$.

Example

$S.\{1,2,4\} = S.(1,1,0,1,0)$ is a free squarefree module.
$S.(1,2,0,1,0)$ is a free module but *not* squarefree.

Standard duality

$\mathbf{F}_{sq}$ is the category of *complexes* of free squarefree modules.

Example

The enriched chain and cochain complexes $\mathcal{E}(\Gamma)$ and $\mathcal{E}^\vee(\Gamma)$ are in $\mathbf{F}_{sq}$.

Standard duality $\mathbb{D} : \mathbf{F}_{sq} \to \mathbf{F}_{sq}$ defined by

$$\mathbb{D}(\mathcal{P}^\cdot) = \mathrm{Hom}_S(\mathcal{P}^\cdot, \omega_S).$$

Alexander duality

Alexander duality $\mathbb{A} : \mathbf{F}_{sq} \to \mathbf{F}_{sq}$.
First form Alexander dual $(\mathcal{P}^{\cdot})^*$. Then take a minimal squarefree resolution

$$\mathcal{Q}^{\cdot} \to (\mathcal{P}^{\cdot})^*.$$

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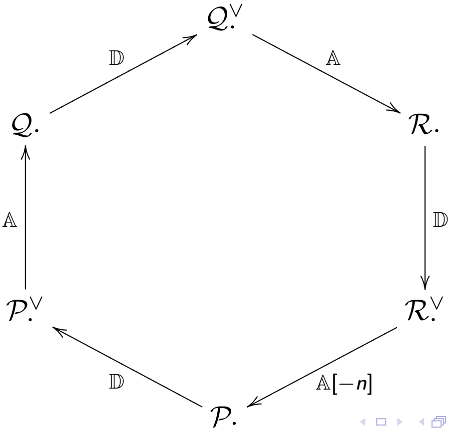
Example

- $\mathcal{P}^{\cdot} = S.(1,1,0,1,0)$.
- Alexander dual $(\mathcal{P}^{\cdot})^* = S/(x_1, x_2, x_4)$.
- Minimal squarefree resolution of $S/(x_1, x_2, x_4)$ is:

$$S \to S^3 \to S^3 \to S.$$

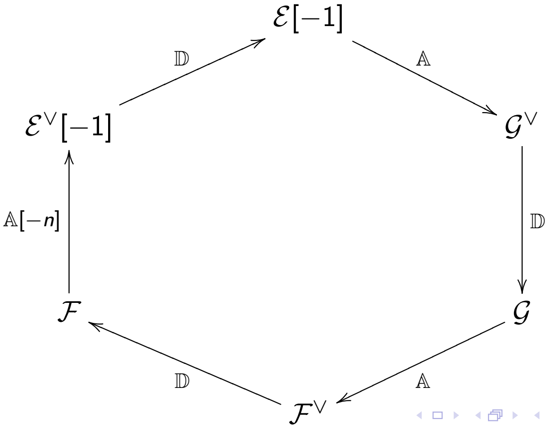
- Then Alexander dual $\mathcal{Q}^{\cdot} = \mathbb{A}(\mathcal{P}^{\cdot})$ is this complex.

Hexagon of complexes
Yanagawa



For simplicial complexes

$\mathcal{E}$ is the enriched chain complex, $\mathcal{F}^\vee$ the resolution of the Stanley-Reisner ring.



Stanley-Reisner complex instead of Stanley-Reisner resolution

For cell complexes Γ one has enriched chain complex $\mathcal{E}(\Gamma)$. May turn the wheel back and get a complex

$$\mathcal{F}^{\vee} : \cdots \rightarrow F_p \rightarrow \cdots \rightarrow F_1 \rightarrow F_0.$$

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Point:

Stanley-Reisner theory can be done equally well for polyhedral complexes as for simplicial complexes!

Numerical invariants

Symmetry breakdown.

A complex $\mathcal{P}^{\cdot}$ of free graded S-modules

$$\mathcal{P}^{\cdot} : \cdots \to \oplus S(-j)^{b^i_j} \to \cdots$$

comes with three sets of numerical invariants:

- The graded Betti numbers b^i_j.

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- The cohomology modules and their Hilbert functions

$$c^i_j = \dim_k H^i(\mathrm{Hom}(\mathcal{P}^\cdot, \omega_S))_j.$$

Squarefree invariants

A complex $\mathcal{P}^\cdot$ of free squarefree S-modules

$$\mathcal{P}^\cdot \; : \; \cdots \to \oplus_R (S \otimes_k B_R^i) \to \cdots .$$

Three sets of invariants for subsets $R \subseteq [n]$.

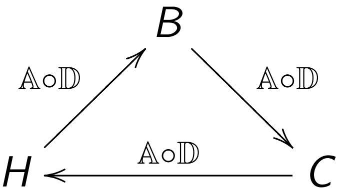
- Betti spaces B_R^i.
- Homology spaces $H_R^i = H^i(\mathcal{P}^\cdot)_R$.
- Cohomology spaces $C_R^i = H^i \mathrm{Hom}(\mathcal{P}^\cdot, \omega_S)_R$.

Rotations of invariants

Perfect symmetry!

The functor $\mathbb{A} \circ \mathbb{D}$ cyclically rotates the homological invariants.

- $B^i_R(\mathbb{A} \circ \mathbb{D}(\mathcal{P}^\cdot)) = H^{i+r}_{R^c}(\mathcal{P}^\cdot)$.
- $H^i_R(\mathbb{A} \circ \mathbb{D}(\mathcal{P}^\cdot)) = C^i_R(\mathcal{P}^\cdot)$.
- $C^i_R(\mathbb{A} \circ \mathbb{D}(\mathcal{P}^\cdot)) = B^{i+r}_{R^c}(\mathcal{P}^\cdot)$.



From pure resolutions...

... to pure complexes.

A resolution of a graded S-module of the form

$$S(-d_0)^{\beta_0} \leftarrow S(-d_1)^{\beta_1} \leftarrow \cdots \leftarrow S(-d_r)^{\beta_r}$$

is called a *pure resolution*.

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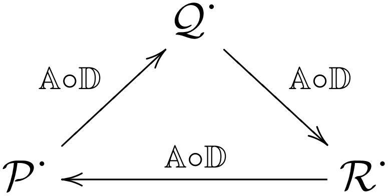
A complex $\mathcal{P}^{\cdot}$ of free squarefree module is *pure* if

$$\mathcal{P}^i = \oplus_{R \subseteq [n]} S \otimes_k B^i_R$$

where all R have the same cardinality d_i.

Resolutions of Cohen-Macaulay modules



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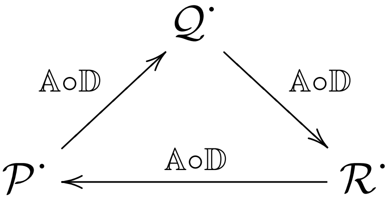

Fact

$\mathcal{P}^\cdot$ is a resolution of a Cohen-Macaulay module if and only if $\mathcal{Q}^\cdot$ and $\mathcal{R}^\cdot$ are linear complexes.

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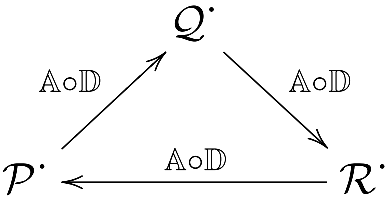

Fact

$\mathcal{P}^{\cdot}$ is a resolution of a Cohen-Macaulay module if and only if $\mathcal{Q}^{\cdot}$ and $\mathcal{R}^{\cdot}$ are linear complexes.

In particular $\mathcal{P}^{\cdot}$ is a pure resolution of a Cohen-Macaulay module if and only if:

$$i.\, \mathcal{P}^{\cdot} \text{ is pure }, \; ii.\, \mathcal{Q}^{\cdot} \text{ is linear }, \; iii.\, \mathcal{R}^{\cdot} \text{ is linear.}$$

Triples of pure complexes

Problem

Construct complexes $\mathcal{P}^{\cdot}$, $\mathcal{Q}^{\cdot}$, and $\mathcal{R}^{\cdot}$ which are all pure.

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Example

$S = k[x_1, x_2, x_3]$.

$$\mathcal{P}^\cdot \quad : \quad S \xleftarrow{[x_1x_2, x_1x_3, x_2x_3]} S(-2)^3$$

$$\mathcal{Q}^\cdot \quad : \quad S^2 \leftarrow S(-2)^3 \xleftarrow{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} S(-3)$$

$$\mathcal{R}^\cdot \quad : \quad S(-1)^3 \leftarrow S(-2)^6 \leftarrow S(-3)^2$$

Resolutions...

... but not necessarily of ideals

Only for few integer sequences $0 = d_0 < d_1 < \cdots < d_r$ can we hope to get a squarefree free resolution

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of a *quotient ring* S/I, i.e. with $\beta_0 = 1$.

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Are there natural classes of squarefree modules which give as interesting resolution theory as what one has for ideals? And which have the potential of giving all (or many) pure resolutions?

Resolutions...
... but not necessarily of ideals

If Δ simplicial complex with SR-ideal $I_\Delta = (x^{F_1}, x^{F_2}, \ldots, x^{F_m})$, then SR-resolution starts with

$$S \leftarrow \oplus_i S(-F_i) \leftarrow \cdots .$$

Choose integers a and b and consider resolutions which start with

$$S^a \xleftarrow{\phi} \oplus_i S(-F_i)^b \leftarrow \cdots$$

where ϕ is a general map.