Boij-Söderberg theory

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Conjectures

Conjecture 1

All extremal rays of pure type do exist: For any sequence of integers $d_0 < d_1 < \cdots < d_c$ there exists a CM-module of codimension c with resolution

$$S(-d_0)^{\beta_0} \leftarrow S(-d_1)^{\beta_1} \leftarrow \cdots \leftarrow S(-d_c)^{\beta_c}.$$

Conjecture 2

Pure diagrams account for *all* the extremal rays: There are no more extremal rays in the cone of Betti diagrams than those coming from pure diagrams.

Conjectures

Conjecture 3

The algorithm always works in order to write a Betti diagram β as a positive linear combination of pure diagrams: It gives a chain of degree sequences $\mathbf{d}^1 < \mathbf{d}^2 < \cdots < \mathbf{d}^r$ such that

$$\beta = c_1\pi(\mathbf{d}^1) + c_2\pi(\mathbf{d}^2) + \cdots + c_r\pi(\mathbf{d}^r).$$

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Note. Conjecture 3 implies conjecture 2.

Koszul complexes

Example

Pure resolution of type $(0, 1, 2, 3)$.

$$S \leftarrow S(-1)^3 \leftarrow S(-2)^3 \leftarrow S(-3).$$

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Let $S = Symm(V)$ where V is a k-vector space of dimension n. Consider $\wedge^p V$ to have degree p. Pure resolution of type $(0, 1, 2, \ldots, n)$:

$$S \leftarrow S \otimes_k V \leftarrow S \otimes_k \wedge^2 V \leftarrow \cdots \leftarrow S \otimes_k \wedge^n V.$$

Powers of maximal ideals

Example

Let $m \subseteq k[x_1, x_2, x_3]$ be the maximal ideal. Resolution of m^2 has type $(0, 2, 3, 4)$:

$$S \leftarrow S(-2)^6 \leftarrow S(-3)^8 \leftarrow S(-4)^3.$$

Let $S_r(V)$ be the r'th graded piece of $S = Symm(V)$. Let m the maximal ideal in S. The resolution of m^r has type $(0, r, r+1, r+2, \ldots, n+r-1)$:

$$S \leftarrow S \otimes_k S_r(V) \leftarrow S \otimes_k S_{r,1}(V) \leftarrow \cdots \leftarrow S \otimes_k S_{r,1,\ldots,1}(V) \leftarrow \cdots .$$

Representations of $GL(V)$

The general linear group $GL(V)$ consists of all automorphisms of the vector space V. Let $n = \dim_k V$. For every partition into n parts

$$\lambda : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n,$$

there is an irreducible $GL(V)$-representation, denoted $S_\lambda(V)$.

Example

$$\underbrace{S_{1,1,\ldots,1}}_{r \text{ copies of } 1}(V) = \wedge^r V, \quad S_{r,0,\ldots,0}(V) = S_r(V).$$

Pure resolutions of type length three

char.$k = 0$ (Eisenbud, F. , Weyman)

Let $S = k[x_1, x_2, x_3]$. Want pure resolution:

$$S^{\beta_0} \leftarrow S(-e_1)^{\beta_1} \leftarrow S(-e_1 - e_2)^{\beta_2} \leftarrow S(-e_1 - e_2 - e_3)^{\beta_3}.$$

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Try:

$$\begin{aligned} S \otimes_k S_{\lambda_1,\lambda_2,\lambda_3} \leftarrow S \otimes_k S_{\lambda_1+e_1,\lambda_2,\lambda_3} &\leftarrow S \otimes_k S_{\lambda_1+e_1,\lambda_2+e_2,\lambda_3} \\ &\leftarrow S \otimes_k S_{\lambda_1+e_1,\lambda_2+e_2,\lambda_3+e_3}. \end{aligned}$$

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This works! Let:

$$\lambda_3 = 0, \quad \lambda_2 = e_3 - 1, \quad \lambda_1 = (e_2 - 1) + (e_3 - 1).$$

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The construction generalizes to resolutions of any length n.

Eagon-Northcott complex

Divided powers $D_r(V) = S_r(V^*)^*$. Let

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Let A and B be vector spaces with $b = \dim_k B \leq \dim_k A$.

$$S \leftarrow \wedge^b A \otimes \tilde{D}_0(B^*) \otimes S(-b) \leftarrow \wedge^{b+1} A \otimes \tilde{D}_1(B^*) \otimes S(-b-1)$$
$$\leftarrow \wedge^{b+2} A \otimes \tilde{D}_2(B^*) \otimes S(-b-2) \leftarrow \cdots$$

Extends to family of complexes

Pure resolutions with two linear parts

Buchsbaum-Rim complex:

$$S_1(B) \otimes S \leftarrow A \otimes S_0(B) \otimes S(-1)$$
$$\leftarrow \quad \wedge^{b+1} A \otimes \tilde{D}_0(B^*) \otimes S(-b-1) \leftarrow \wedge^{b+2} A \otimes \tilde{D}_1(B^*) \otimes S(-b-2)$$

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Further complexes:

$$S_2(B) \otimes S \leftarrow A \otimes S_1(B) \otimes S(-1) \leftarrow \wedge^2 A \otimes S_0(B) \otimes S(-2)$$
$$\leftarrow \quad \wedge^{b+2} A \otimes \tilde{D}_0(B^*) \otimes S(-b-2) \leftarrow \wedge^{b+3} A \otimes \tilde{D}_1(B^*) \otimes S(-b-3)$$

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$$S_p(B) \otimes S \leftarrow \cdots \leftarrow \wedge^p A \otimes S_0(B) \otimes S(-p)$$
$$\leftarrow \quad \wedge^{p+b} A \otimes \tilde{D}_0(B^*) \otimes S(-p-b) \leftarrow \wedge^{p+b+1} A \otimes \tilde{D}_1(B^*) \otimes S(-p-b$$

Pure resolutions with three linear parts
(Eisenbud, Schreyer), (Berkesch, Erman, Kummini, Sam)

Two vector spaces B_1 and B_2 of dimensions b_1 and b_2.

$$S_p(B_1) \otimes S_q(B_2) \otimes S \leftarrow A \otimes S_{p-1}(B_1) \otimes S_{q-1}(B_2) \otimes S(-1) \leftarrow \cdots$$
$$\leftarrow \wedge^p A \otimes S_0(B_1) \otimes S_{q-p}(B_2) \otimes S(-p)$$

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twist jump

$$\leftarrow \wedge^{p+b_1} A \otimes \tilde{D}_0(B_1^*) \otimes S_{q-p-b_1}(B_2) \otimes S(-p-b_1)$$

$$\leftarrow \wedge^{p+b_1+1} A \otimes \tilde{D}_1(B_1^*) \otimes S_{q-p-b_1-1}(B_2) \otimes S(-p-b_1-1) \leftarrow \cdots$$

$$\leftarrow \wedge^q A \otimes \tilde{D}_{q-p-b_1}(B_1^*) \otimes S_0(B_2) \otimes S(-q)$$

twist jump

$$\leftarrow \wedge^{q+b_2} \otimes \tilde{D}_{q-p-b_1-b_2}(B_1^*) \otimes \tilde{D}_0(B_2^*) \otimes S(-q-b_2) \cdots$$

$$\leftarrow \wedge^{q+b_2+1} \otimes \tilde{D}_{q-p-b_1+b_2+1}(B_1^*) \otimes \tilde{D}_1(B_2^*) \otimes S(-q-b_2-1) \cdots$$

The simplicial fan

Fix two degree sequences

$$\mathbf{a} = (a_0 < a_1 < \ldots < a_n), \quad \mathbf{b} = (b_0 < b_1 < \ldots < b_n).$$

Consider chain of degree sequences

$$\mathbf{a} \leq \mathbf{d}^1 < \mathbf{d}^2 < \cdots < \mathbf{d}^r \leq \mathbf{b}.$$

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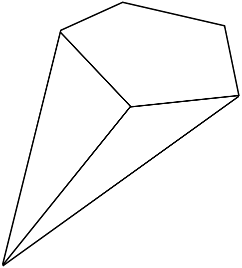
Get pure diagrams

$$\pi(\mathbf{d}^1), \pi(\mathbf{d}^2), \ldots, \pi(\mathbf{d}^r).$$

These are linearly independent so they generate a *simplicial cone*.
Varying over *all* chains, we get a *simplicial fan F*

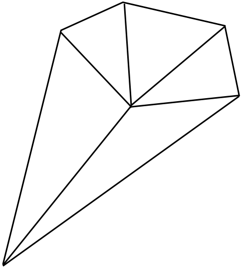
Geometric version of Boij-Söderberg conjectures



Cone B of Betti diagrams

BS conjecture 1 says:
$|F| \subseteq B$.

BS conjecture 3 says:
$B \subseteq |F|$.



Simplicial fan F generated by pure diagrams

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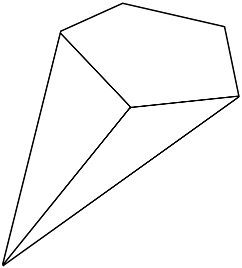

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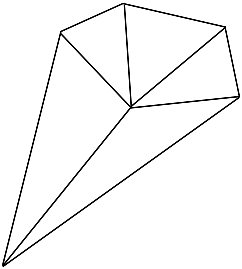
Conclusion: $B = |F|$.

Note that BS conjecture 3 implies BS conjecture 2.



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(Eisenbud, Schreyer)

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- For each exterior facet equation h and Betti diagram β of a Cohen-Macaulay module, show that $h(\beta) \geq 0$.

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- Find equations h of supporting hyperplanes for the exterior facets of F.
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- This shows that $B \subseteq |F|$.

Facet equations

Exterior facets have equations:

$$\sum_{\substack{i=0,\ldots,c \\ a_i \leq j \leq b_i}} c_{ij}\beta_{ij} = 0,$$

which we represent by an array:

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There are three types of exterior facets. Types 1 and 2 have the simple equations $\beta_{ij} = 0$ for suitable i and j.

Facet equations of type 3

Example

There is a maximal chain

$$\mathbf{a} = (0,1,3) < (0,2,3) < (0,2,4) < (0,3,4) = \mathbf{b}.$$

Get four-dimensional simplicial cone generated by

$$\pi(0,1,3) = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \pi(0,2,3), \pi(0,2,4), \pi(0,3,4).$$

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The facet generated by

$$\pi(0,1,3), \pi(0,2,3), \pi(0,3,4)$$

is an exterior facet of type 3.

Facet equations of type 3

Exterior facets of type 3 occurs when the maximal chain is

$$\cdots < (\ldots, r{-}1, r, \ldots) < (\ldots, r{-}1, r{+}1, \ldots) < (\ldots, r, r{+}1, \ldots) < \cdots,$$

and we form the simplicial cone generated by the pure diagrams of these elements, except $f = (\ldots, r-1, r+1, \ldots)$.

Equations of exterior facets of type 3

Example

$$D : \cdots < (-1,0,1,3) < \overset{f=}{(-1,0,2,3)} < (-1,1,2,3) < \cdots$$

Equation of hyperplane $h_{D,f}$ is the following diagram rotated 90° degrees counterclockwise.

	6	5	4	3	2	1	0	-1	-2	-3	
$\cdots$	5	0	-3	-4	-3	0	5	12	21	32	$\cdots$
$\cdots$	-12	-5	0	3	4	3	0	-5	-12	-21	$\cdots$
$\cdots$	21	12	5	0	-3	-4	-3	0	5	12	$\cdots$

The numbers in the first row are the values of $(d-1)(d+3)$.

Final stage of proof

When h is the equation of an exterior facet of the fan F. Show that $h(\beta) \geq 0$ for any Betti diagram β of a Cohen-Macaulay module.

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This makes us conclude that $B \subseteq |F|$.

Vector bundles with supernatural cohomology

There exists a vector bundle $\mathcal{E}$ on $\mathbb{P}^2$ with Hilbert polynomial $\chi\mathcal{E}(d) = (d-1)(d+3)$ whose cohomology table is:

d	$\cdots$	-6	-5	-4	-3	-2	- 1	0	1	2	3	4
$\dim_{\Bbbk} H^2\mathcal{E}(d)$	$\cdots$	21	12	5	0	0	0	0	0	0	0	0
$\dim_{\Bbbk} H^1\mathcal{E}(d)$	$\cdots$	0	0	0	0	3	4	3	0	0	0	0
$\dim_{\Bbbk} H^0\mathcal{E}(d)$	$\cdots$	0	0	0	0	0	0	0	0	5	12	21

Vector bundles with supernatural cohomology

A vector bundle $\mathcal{E}$ on $\mathbb{P}^m$ is said to have *supernatural cohomology* if there are integers $z_1 > z_2 > \cdots > z_m$ such that:

1. The Hilbert polynomial

$$\chi\mathcal{E}(d) = c(d - z_1)\cdots(d - z_m).$$

2. In each column of the cohomology table there is at most one nonzero value.

Theorem

For each sequence $z_1 > \cdots > z_m$ such a bundle exists.

Pairing between betti diagrams and cohomology tables

For a module M let $\beta_{ij} = \beta_{ij}(M)$ be its Betti numbers.

For a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^{n-1}$ let $\gamma_{ij} = H^i(\mathbb{P}^{n-1}, \mathcal{F}(j))$ be its cohomology numbers.

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Let $e \in \mathbb{Z}$ and $0 \leq \tau \leq n-1$, and define $\gamma_{\leq i,d}$ to be $\gamma_{0,d} - \gamma_{1,d} + \cdots + (-1)^i \gamma_{i,d}$. Define the pairing $\langle \beta, \gamma \rangle_{e,\tau}$ as the expression:

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$$
\begin{aligned}
& \sum_{i<\tau, d\in\mathbb{Z}} (-1)^i \beta_{i,d} \gamma_{\leq i,-d} \\
+ & \sum_{d\leq e} (-1)^\tau \beta_{\tau,d} \gamma_{\leq \tau,-d} + \sum_{d>e} (-1)^\tau \beta_{\tau,d} \gamma_{\leq \tau-1,-d} \\
+ & \sum_{d\leq e+1} (-1)^{\tau+1} \beta_{\tau+1,d} \gamma_{\leq \tau,-d} + \sum_{d>e+1} (-1)^{\tau+1} \beta_{\tau+1,d} \gamma_{\leq \tau-1,-d} \\
+ & \sum \quad (-1)^i \beta_{i,d} \gamma_{\leq i-2,-d}
\end{aligned}
$$

Positivity of pairing

Theorem (Eisenbud, Schreyer)

For any module M and any coherent sheaf $\mathcal{F}$ the pairing:

$$\langle \beta(M), \gamma(\mathcal{F}) \rangle_{e,\tau} \geq 0.$$

Conclusion

As we saw earlier in the example:

Theorem

The degree sequence corresponding to an exterior facet of type 3:

$$f = (f_0 < f_1 < \cdots < \underbrace{f_\tau}_{=r-1} < \underbrace{f_{\tau+1}}_{=r+1} < \cdots < f_n).$$

Let $\mathcal{E}$ be the vector bundle on $\mathbb{P}^{n-1}$ with supernatural cohomology and root sequence $-f_0 > -f_1 > \cdots - f_{\tau-1} > -f_{\tau+2} > \cdots > -f_n$.

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Let $\mathcal{E}$ be the vector bundle on $\mathbb{P}^{n-1}$ with supernatural cohomology and root sequence $-f_0 > -f_1 > \cdots - f_{\tau-1} > -f_{\tau+2} > \cdots > -f_n$. Then the hyperplane equation of the exterior facet obtained by omitting the pure diagram $\pi(f)$ is:

$$h_{D,f}(\beta) = \langle \beta, \gamma(\mathcal{E}) \rangle_{e,\tau}$$

where $e = f_\tau$.

Conclusion

We may then finally conlude:

Corollary

$h_{D,f}(\beta) \geq 0$ for all Betti diagrams $\beta = \beta(M)$.

Hence the cone of Betti diagrams B is inside the geometric realization $|F|$ of the fan F.