Algebraic Properties of product of graphs

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$\mathbb{K}[G] = R/I(G)$: *edge ring* of G.

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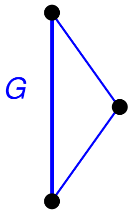

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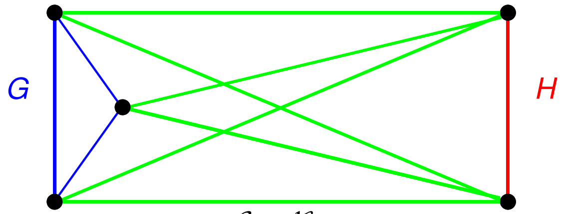

$\mathcal{C}_3 * \mathcal{K}_2$

Notations of special graphs

$\overline{G}$: complement of a graph G,

$\mathcal{K}_n$: complete graph on n vertices,

$\mathcal{C}_n$: cycle graph on n vertices,

$\mathcal{W}_n$: wheel graph on $n+1$ vertices,

$\mathcal{S}_n$: star graph on $n+1$ vertices,

$\mathcal{K}_{m,n}$: complete bipartite graph whose partitions have m and n vertices.

$\mathcal{K}_{n_1,\ldots,n_r}$: complete multipartite graph whose partitions have $n_1,\ldots,n_r$ vertices.

Stanley-Reisner ideal

The *Stanley-Reisner ideal* (or the *non-face ideal*) of Δ, denoted by I_Δ, is

$$I_\Delta = (x_{i_1} \cdots x_{i_j} \mid i_1 < i_2 < \cdots < i_j,\ \{x_{i_1}, \ldots, x_{i_j}\} \notin \Delta).$$

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$\mathbb{K}[\Delta] = R/I_\Delta$: *Stanley-Reisner ring* of Δ.

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$\mathcal{F}(\Delta \cup \Delta') = \mathcal{F}(\Delta) \cup \mathcal{F}(\Delta')$, i.e,

F is a face of $\Delta \cup \Delta'$ if and only if F is a face of Δ or Δ'.

Alexander dual and cover ideal

$I = (x_{11}x_{12}\cdots x_{1t_1}, x_{21}x_{22}\cdots x_{2t_2}, \ldots, x_{n1}x_{n2}\cdots x_{nt_n})$: square-free monomial ideal.

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The *cover ideal* of a graph G is defined to be the Alexander dual of $I(G)$, i.e,

$$I(G)^\vee = (x_F \mid F \text{ is a (minimal) vertex cover of } G),$$

where $x_F = \prod_{x_i \in F} x_i$.

Betti numbers and regularity

Let M be an arbitrary graded R-module, and let

$$0 \to \bigoplus_j R(-j)^{\beta_{t,j}(M)} \to \bigoplus_j R(-j)^{\beta_{t-1,j}(M)} \to \cdots \to \bigoplus_j R(-j)^{\beta_{0,j}(M)} \to 0$$

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Hochster's formula. For $i > 0$, the $\mathbb{N}$-graded Betti numbers $\beta_{i,j}^{\mathbb{K}}$ of a simplicial complex Δ (over the field $\mathbb{K}$) are given by

$$\beta_{i,j}^{\mathbb{K}}(\Delta) = \sum_{\substack{W \subseteq V(\Delta) \\ |W|=j}} \dim_{\mathbb{K}} \widetilde{H}_{j-i-1}(\Delta|_W; \mathbb{K}),$$

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Lemma. Let Δ_1 and Δ_2 be two simplicial complexes with disjoint vertex sets having m and n vertices, respectively. Then the $\mathbb{N}$-graded Betti numbers $\beta_{i,d}(\Delta_1 \cup \Delta_2)$ can be expressed as

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$$\begin{cases} \sum_{j=0}^{d-2}\{\binom{n}{j}\beta_{i-j,d-j}(\Delta_1) + \binom{m}{j}\beta_{i-j,d-j}(\Delta_2)\} & d\neq i+1 \\ \sum_{j=0}^{d-2}\{\binom{n}{j}\beta_{i-j,d-j}(\Delta_1) + \binom{m}{j}\beta_{i-j,d-j}(\Delta_2)\}+\sum_{j=1}^{d-1}\binom{m}{j}\binom{n}{d-j} & d=i+1\,. \end{cases}$$

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Lemma. Let G and H be two simple graphs whose vertex sets are disjoint. Then $\Delta_{G*H} = \Delta_G \cup \Delta_H$ is the disjoint union of two simplicial complexes.

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Let $\beta_{i,j}(G * H)$ denotes the $\mathbb{N}$-graded Betti numbers of Δ_{G*H}. We have the following translation of first lemma.

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Using this formula we get the following known results on Betti numbers of some families of graphs.

Betti numbers of some families of graphs

$$\beta_{i,d}(\mathcal{S}_m) = \begin{cases} 0 & \text{if } \ d \neq i+1 \\ \\ \binom{m}{i} & \text{if } \ d = i+1. \end{cases}$$

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$$\beta_{i,d}(\mathcal{W}_m) = \begin{cases} \beta_{i,d}(\mathcal{C}_m) + \beta_{i-1,d-1}(\mathcal{C}_m) & \text{if } \ d \neq i+1 \\ \beta_{i,d}(\mathcal{C}_m) + \beta_{i-1,d-1}(\mathcal{C}_m) + \binom{m}{i} & \text{if } \ d = i+1. \end{cases}$$

Betti numbers of some families of graphs

Corollary. Let H be a simple graph with $m+1$ vertices and let $x \in V(H)$ be adjacent to all other vertices of H. Then

$$\beta_{i,d}(H) = \begin{cases} \beta_{i,d}(G) + \beta_{i-1,d-1}(G) & \text{if } d \neq i+1 \\ \beta_{i,d}(G) + \beta_{i-1,d-1}(G) + \binom{m}{i} & \text{if } d = i+1, \end{cases}$$

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Corollary. Let G and be a simple graphs with m vertices. Then the $\mathbb{N}$-graded Betti numbers $\beta_{i,d}(G * \mathcal{S}_n)$ may be expressed as

$$\begin{cases} \sum_{j=0}^{d-2} \binom{n+1}{j} \beta_{i-j,d-j}(G) & d \neq i+1 \\ \sum_{j=0}^{d-2} \binom{n+1}{j} \beta_{i-j,d-j}(G) + \binom{m+n+1}{i+1} + \binom{m+n}{i} - \binom{m+1}{i+1} - \binom{n+1}{i+1} & d = i+1. \end{cases}$$

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Remark. Let $s > 0$ be an integer. Previous result gives a procedure to construct a family of graphs for which the regularity of each edge ring equals s. Indeed assume G is a simple graph with $\operatorname{reg}(R/I(G)) = s$ (for example, choose G as the cycle graph over $3s$ vertices). Then for any graph H with $\operatorname{reg}(R/I(H)) \leq s$ one has $\operatorname{reg}(R/I(G*H)) = s$.

Pairwise 3-disjoint edges

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$$\operatorname{reg}(R/I(G)) \geq a(G).$$

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Proof. First note that

$$a(G) = \operatorname{reg}(R/I(G)) \geq \operatorname{reg}(R/I(H)) \geq a(H).$$

Therefore we get

$$\operatorname{reg}(R/I(G * H)) = \operatorname{reg}(R/I(G)) = a(G) = a(G * H).$$

Main result

Let $\mathcal{A}$ be the set of all graphs for which the regularity of each edge ring equals the maximum number of pairwise 3-disjoint edges, i.e.,

$$\mathcal{A} = \{\ G \quad | \quad G \text{ is a simple graph with } \operatorname{reg}(R/I(G)) = a(G)\}.$$

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It is proved that $\mathcal{A}$ contains the following classes of graphs:

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- ✠ very well-covered graphs by Mahmoudi, et al. in(2011).

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$$\operatorname{reg}(R/I(G * H)) = a(G * H).$$

Main result

Remark. Using previous Proposition one can construct graphs that do not belong to the above mentioned families of graphs but the regularity of their edge rings equal the maximum number of pairwise 3-disjoint edges. For example, consider $\overline{\mathcal{C}}_4 * \mathcal{C}_4$. One can easily see that this graph does not belong to the above mentioned families of graphs, but its Castelnuovo-Mumford regularity equals the maximum number of pairwise 3-disjoint edges which is 2.

Translation to cover ideals

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The assertion follows using alternating sum property of Hilbert series.

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Proposition. Let Δ and Δ' be two simplicial complexes with disjoint vertex sets. Also let $\dim(\Delta') \leq \dim(\Delta)$ and assume $n = \dim(\Delta) - \dim(\Delta')$. Then

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Corollary. Let Δ be a simplicial complex and let $\Delta_1, \ldots, \Delta_r$ be connected components of Δ. Then

$$H_{\mathbb{K}[\Delta]}(t) = \sum_{j=1}^{r} H_{\mathbb{K}[\Delta_j]}(t) - (r-1).$$

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In particular,

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Finally

Thanks for your attention