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Aluffi Torsion-free Graphs

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Definition

Let R be a Notherian ring and $J \subset I$ ideals of R. The ***Aluffi algebra*** *of I/J is*

$$\mathcal{A}_{R/J}(I/J) := \mathcal{S}_{R/J}(I/J) \otimes_{\mathcal{S}_R(I)} \mathcal{R}_R(I).$$

Lemma

Let $J \subset I$ be ideal of the ring R. There are natural R/J-algebra isomorphisms

$$\mathcal{A}_{R/J}(I/J) \simeq \frac{\mathcal{R}_R(I)}{(J, \tilde{J})\,\mathcal{R}_R(I)}$$

where J is in degree 0 and $\tilde{J}$ is in degree 1.

Let $J \subset I \subset R$ be ideals. One has

$$\mathcal{A}_{R/J}(I/J) \simeq \mathcal{R}_R(I)/(J, \tilde{J})\mathcal{R}_R(I) = \bigoplus_{t \geq 0} I^t/JI^{t-1}.$$

This follows immediately that the Aluffi algebra surjects onto the Rees algebra

$$\mathcal{R}_{R/J}(I/J) = \bigoplus_{t \geq 0} (I^t, J)/J \simeq \bigoplus_{t \geq 0} I^t/J \cap I^t$$

The kernel of this surjection is the so-called *module of Valabrega–Valla*.

$$\mathcal{W}_{J \subset I} = \bigoplus_{t \geq 2} \frac{J \cap I^t}{JI^{t-1}}.$$

Theorem

Let $J \subset I \subsetneq R$ be ideals of the Noetherian ring R. If I/J has a regular element then the R/J-torsion of the Aluffi algebra of I/J is the module of Valabrega-Valla. If J is besides a prime ideal then the R/J-torsion of $\mathcal{A}_{R/J}(I/J)$ is a minimal prime ideal.

Definition

A pair of ideals $J \subset I$ of a ring R is said to be Aluffi torsion-free if the map $\mathcal{A}_{R/J}(I/J) \twoheadrightarrow \mathcal{R}_{R/J}(I/J)$ is injective. Equivalently a pair of ideals $J \subset I$ is Aluffi Torsion-free if

$$J \cap I^n = JI^{n-1}$$

for all positive integers n.

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Nasrollah Nejad and Simis give necessary and sufficient conditions for Aluffi torsion-freeness of ideals $J \subset I$ in terms of I-standard basis of J , relation type number of I/J over R/J and the Artin-Rees number of J relative to I.

Geometric Setting

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Let $X \overset{i}{\hookrightarrow} Y \overset{j}{\hookrightarrow} Z$ be closed embeddings of schemes with $J \subset I \subset R$ the ideal sheaves of Y and X in Z, respectively. Let $\widetilde{Z} = \mathrm{Proj}(\mathcal{R}_R(I)) \overset{\pi}{\to} Z$ be the blowup of Z along X and $\widetilde{Y} = \mathrm{Proj}(\mathcal{R}_{R/J}(I/J))$ be the blowup of Y along X. Note that $\widetilde{Y}$ embeds in $\widetilde{Z}$ as the strict transform of Y under $\widetilde{Z} \overset{\pi}{\to} Z$. Let $E = \pi^{-1}(X)$ be the exceptional divisor of the blowup. Then, E is a subscheme of $\pi^{-1}(Y)$. Let $\mathfrak{R} = \mathfrak{R}(E, \pi^{-1}(Y))$ be the residual scheme of E in $\pi^{-1}(Y)$. Here "residual" is taken in the sense of W. Fulton. In terms of the ideal sheaves, $\mathfrak{R}$ is characterized by the equation $\mathcal{I}_{\mathfrak{R}}.\mathcal{I}_E = \mathcal{I}_{\pi^{-1}(Y)}$, where $\mathcal{I}_E, \mathcal{I}_{\pi^{-1}(Y)}$ are respectively the ideals of E and $\pi^{-1}(Y)$ in $\tilde{Z}$. Aluffi proved that $\mathrm{Proj}(\mathcal{A}_{R/J}(I/J)) = \mathfrak{R}(E, \pi^{-1}(Y))$.

W.Fulton shows that if i and j are regular embeddings, then $\mathfrak{R} = \widetilde{Y}$ which is equivalent to say that $J \cap I^n = JI^{n-1}$ for all sufficiently large n.

S. Keel shows that this result holds as long as $X \hookrightarrow Y$ is a linear embedding and $Y \hookrightarrow Z$ is a regular embedding.

Let $R = k[x_1 \ldots, x_n]$ be the $\mathbb{N}$-graded polynomial ring over a field k, $J \subset R$ be a homogeneous ideal and $I \subset R$ be the Jacobian ideal of J, by which we always mean the ideal $(J, I_r(\Theta))$ where $r = \mathrm{ht}\,(J)$ and Θ stands for the Jacobian matrix of a set of generators of J. More precisely, if $J = (f_1, \ldots, f_s)$, then,

$$\Theta = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_1} & \cdots & \frac{\partial f_s}{\partial x_1} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_1}{\partial x_n} & \frac{\partial f_2}{\partial x_n} & \cdots & \frac{\partial f_s}{\partial x_n} \end{bmatrix}.$$

Lemma

With the above assumptions and notations, if $I_r(\Theta) = \mathfrak{m}^r$, then the pair $J \subseteq I$ is Aluffi torsion-free.

Let I be a monomial ideal in the polynomial ring $k[x_1, \ldots, x_n]$. It is known that the ideal of r-minors of the Jacobian matrix of I is again a monomial ideal (Simis, Kalases). We provide another simple proof for this fact.

Let M be a $m \times n$ matrix and $1 \leq r \leq \min\{m, n\}$ be an integer. A transversal of length r in M or an r-transversal of M is a product of r entries of M with different rows and columns. In other words, an r-transversal of M is product of entries of the main diagonal of an $r \times r$ sub-matrix of M after suitable changes of columns and rows.

Lemma

Let I be an ideal of $k[x_1, \ldots, x_n]$ generated by monomials $m_1, \ldots, m_s$. Let Θ be the Jacobian matrix of I and $1 \leq r \leq \min\{n, s\}$. Then, any r-minor of Θ is a monomial.

Recall that for a finite simple graph G with vertex set $V(G) = \{v_1, \ldots, v_n\}$, an ideal $I(G)$ in the ring $k[x_1, \ldots, x_n]$ is corresponded which is generated by all square-free quadratic monomials $x_i x_j$ provided that $\{v_i, v_j\}$ is an edge in G. This ideal is called the edge ideal of G.

Let v be a vertex in G. Degree of v is number of all vertices adjacent to v. For a subset A of $V(G)$, the set of all vertices adjacent to some vertices in A is called neighborhood of A and denoted by $N(A)$. A subset B of vertices of G is called an independent set if there is no any edge between each two vertices of B.

A matching in G is a subset of edges of G such that there is no any common vertex between any two of them. We identify any edge v_i with the corresponding indeterminate x_i.

Lemma

Let G be a graph with n vertices, $I(G)$ edge ideal of G and Θ the Jacobian matrix of $I(G)$. Let $g \in k[x_1, \ldots, x_n]$ be a monomial and r a positive integer. The following conditions are equivalent.

(i) *g is a r-transversal of Θ.*

(ii) *There are r different edges $e_1 = \{x_{1_1}, x_{1_2}\}, \ldots, e_r = \{x_{r_1}, x_{r_2}\}$ such that vertices $x_{1_1}, \ldots, x_{r_1}$ are different and $g = x_{1_2} \cdots x_{r_2}$.*

Moreover, let the set $\{x_{i_1}, \ldots, x_{i_s}\}$ is independent. Then there is an $r-$transversal of the form $g = x_{i_1}^{\alpha_1} \cdots x_{i_s}^{\alpha_s}$ with $\alpha_j > 0$ for $1 \leq j \leq s$, and $\sum_{j=1}^{s} \alpha_j = r$ if and only if $|N(\{x_{i_1}, \ldots, x_{i_s}\})| \geq r$.

Definition

We say that a graph G is Aluffi torsion-free if the pair $I(G) \subseteq (I(G), I_r(\Theta))$ is Aluffi torsion-free, where r is height of $I(G)$ and Θ is Jacobian matrix of $I(G)$.

Theorem

Let G be a graph and $\mathrm{ht}\,(I(G)) = r > 1$. Then G is not Aluffi torsion-free if and only if there are adjacent vertices x_1, x_2 and other vertices $x_{i_1}, \ldots, x_{i_s}$ for some integer $s \geq 1$, such that

(i) *The sets $\{x_1, x_{i_1}, \ldots, x_{i_s}\}$ and $\{x_2, x_{i_1}, \ldots, x_{i_s}\}$ both are independent, and*

(ii) $|N(\{x_{i_1}, \ldots, x_{i_s}\})| = r - 1$.

Example

(1) *A complete graph K_n for $n > 2$ is Aluffi torsion-free.*

(2) *A complete r-partite graph is Aluffi torsion-free.*

(3) *A complete graph minus edges in a matching is Aluffi torsion-free.*

(4) *The cycles C_3 and C_4 are Aluffi torsion-free. Because C_3 is a complete graph and the C_4 is the complete graph K_4 minus a maximal matching.*

(5) *For each $n \geq 5$, the cycle C_n is not Aluffi torsion-free.*

(6) *Any path P_n is not Aluffi torsion-free.*