CM$_t$ SIMPLICIAL COMPLEXES

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Abstract. Cohen-Macaulay-ness is a distinguished property in many areas of mathematics including algebraic geometry, commutative algebra and algebraic combinatorics. In algebraic combinatorics, Cohen-Macaulay and Buchsbaum simplicial complexes have been extensively studied. The CM$_t$ property unifies and naturally generalizes both Cohen-Macaulay and Buchsbaum properties.

In this talk, after recalling necessary preliminaries, some extensions of basic properties of Cohen-Macaulay and Buchsbaum simplicial complexes on CM$_t$ simplicial complexes will be discussed. In particular, a characterization of bipartite CM$_t$ graphs will be outlined as an extension of a result of Herzog and Hibi in the Cohen-Macaulay case. This will cover some joint work with H. Haghighi and S. Yassemi.

1. Preliminaries and notation

Let Δ be a simplicial complex on a finite set $V = \{x_1, \cdots, x_n$ of n elements. Hence, Δ is a collection of subsets of V such that (a) if $F \in \Delta$ and $G \subset F$, then $G \in \Delta$, and (b) $\{x\} \in \Delta$ for all $x \in V$. Elements of Δ are called the *faces* of Δ. Any maximal face (with respect to inclusion) is called a *facet* of Δ. For a face F, *dimension* of F is

$$\dim F = \#F - 1$$

and *dimension* of Δ is

$$\dim\Delta = \max_{F\in\Delta} \dim F.$$

A simplicial complex is called *pure* if all its facets have the same dimension. For an integer $r \geq 0$, the $r-$*skeleton* of Δ is the simplicial complex

$$\{F \in \Delta : \dim F \leq r\}.$$

For any face $F \in \Delta$, the *link* of F in Δ is defined as:

$$\text{link}_{\Delta}(F) = \{G \in \Delta | G \cup F \in \Delta, G \cap F = \varnothing\}.$$

Let e_i be the ith unit coordinate vector in$\mathbb{R}^n$. For a face $F \in \Delta$, let

$$|F| = \text{convex hull of } \{e_i : x_i \in F\}$$

and define the *geometric realization* of Δ to be

$$|\Delta| = \cup_{F\in\Delta} |F|.$$

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Let R be a commutative ring with unit. For Δ the *augmented chain complex* over R

$$C_\bullet : 0 \longrightarrow C_n(\Delta) \xrightarrow{\partial_n} C_{n-1}(\Delta) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1(\Delta) \xrightarrow{\partial_1} C_0(\Delta) \xrightarrow{\varepsilon} C_{-1}(\Delta) = R \longrightarrow 0$$

is associated, where $C_q(\Delta)$ is the free R-module of rank equal to the number of faces of Δ of dimension q and where ∂_q and ε are defined as

$$\partial_q(\{v_0, \cdots, v_q\}) = \sum_{i=0}^{n} (-1)^i \{v_0, \cdots, \widehat{v_i}, \cdots, v_q\}.$$

Here $\widehat{v_i}$ means to 'delete' v_i. The qth *reduced simplicial homology group* of Δ with coefficients in A, $\widetilde{H}_q(\Delta)$, is the qth homology group of this complex. If $R = k$ is a field, the vector space $\widetilde{H}_q(\Delta; k) = H_q(C_\bullet \otimes k)$ is called the qth *reduced simplicial homology* of Δ with coefficients in k.

Given a simplicial complex Δ on the vertex set $\{x_1, \cdots, x_n\}$ and a field k, the *StanleyReisner* ideal or the *face* ideal of Δ is the monomial ideal generated by the square-free monomials corresponding to the non-faces of Δ:

$$I_\Delta = (x_{i_1} \cdots x_{i_r} : \{i_1, \cdots, i_r\} \nsubseteq \Delta).$$

Then, the *StanleyReisner* ring, or, *face* ring of Δ denoted by $k[\Delta]$, is

$$k[\Delta] = k[x_1, \cdots, x_n]/I_\Delta.$$

A simplicial complex Δ is called Cohen-Macaulay (Buchsbaum) over k if its face ring is a CohenMacaulay (Buchsbaum, respectively) ring .

In his 1974 thesis, Gerald Reisner gave a complete characterization of Cohen-Macaulay simplicial complexes. This was soon followed up by more precise homological results about face rings due to Melvin Hochster. Then Richard Stanley found a way to prove the Upper Bound Conjecture for simplicial spheres, which was open at the time, using the face ring construction and Reisner's criterion of Cohen-Macaulay-ness. Stanley's idea of translating difficult conjectures in algebraic combinatorics into statements from commutative algebra and proving them by means of homological techniques was the origin of the rapidly developing field of combinatorial commutative algebra.

2. CM$_t$ simplicial complexes

In the hierarchy of families of simplicial complexes with respect to Cohen-Macaulay property, Buchsbaum complexes appear right after Cohen-Macaulay ones. Natural families of simplicial complexes in this hierarchy, as we will see, are CM$_t$ simplicial complexes, namely, simplicial complexes which are pure and Cohen-Macaulay in codimension t. The concept of CM$_t$ simplicial complexes was introduced in [27] which is the pure version of simplicial complexes Cohen-Macaulay in codimension t studied by Miller, Novik and Swartz [30].

Definition 2.1. *Let k be a field, Δ a simplicial complex of dimension $(d-1)$ over k. Let t be an integer $0 \le t \le d-1$. Then Δ is called* CM$_t$ *over k if Δ is pure and $\mathrm{link}_\Delta(F)$ is Cohen-Macaulay over k for any $F \in \Delta$ with $\#F \ge t$.*

We will adopt the convention that for $t \leq 0$, CM_t means CM_0. Note that from the results by Reisner [17] and Schenzel [19] it follows that CM_0 is the same as Cohen-Macaulay-ness and CM_1 is identical with the Buchsbaum property. It is also clear that for any $i \leq j$, CM_i implies CM_j.

Example 2.2. *Let Δ be the union of two $(d-1)$-simplices that intersect in a $(t-2)$-dimensional face where $1 \leq t \leq d-1$. Then Δ is a CM_t complex which is not a CM_{t-1} complex. In fact, if Γ is a finite union of $(d-1)$-simplices where any two of them intersect in a face of dimension at most $t-2$, then Γ is a CM_t complex, and if at least two of the simplices have a $(t-2)$-dimensional face in common, then Γ is not CM_{t-1}. These include simplicial complexes corresponding to the transversal monomial ideals which happen to have linear resolutions* [24].

Note that the condition $t < d-1$ is necessary because the union of two $(d-1)$-simplices which intersect in a $(d-2)$-dimensional face, is Cohen-Macaulay.

It is known that the links of a Cohen-Macaulay simplicial complex are also Cohen-Macaulay, see [11]. A similar property holds for CM_t complexes.

Lemma 2.3. *(*[27, Lemma 2.3]*) Let $t \geq 1$ and let Δ be a nonempty complex. Then the following are equivalent:*

(i) *Δ is a CM_t complex.*
(ii) *Δ is pure and $link_\Delta(v)$ is CM_{t-1} for every vertex $v \in \Delta$.*

We recall Reisner's characterization of Cohen-Macaulay simplicial complexes [17, Theorem 1].

Theorem 2.4. *Let Δ be a simplicial complex of dimension $(d-1)$. Then the following are equivalent:*

(i) *Δ is Cohen-Macaulay over k.*
(ii) *$\tilde{H}_i(link_\Delta(F); k) = 0$ for any $F \in \Delta$ and all $i < dim(link_\Delta(F))$.*

In analogy with the above result, the following theorem provides equivalent conditions for CM_t complexes.

Theorem 2.5. *Let Δ be a simplicial complex of dimension $(d-1)$. Then the following are equivalent:*

(i) *Δ is CM_t over k;*
(ii) *Δ is pure and $\tilde{H}_i(link_\Delta(F); k) = 0$ for all $F \in \Delta$ with $\#F \geq t$ and $i < d - \#F - 1$.*

We state a result due to Munkres [14, Corollary 3.4] which shows that Cohen-Macaulayness is a topological property.

Theorem 2.6. *Let Δ be a pure simplicial complex of dimension $(d-1)$. Then the following are equivalent:*

(i) *Δ is Cohen-Macaulay over k.*
(ii) *$\tilde{H}_i(|\Delta|; k) = 0 = H_i(|\Delta|, |\Delta| \setminus p; k)$ for all $p \in |\Delta|$ and all $i < d-1$, where $|\Delta|$ is the geometric realization of Δ. Here, $H_i(|\Delta|, |\Delta| \setminus p; k)$ is the ith reduced relative homology group!*

The following theorem may lead one to believe that the property CM_t is also a topological invariant.

Theorem 2.7. *Let Δ be a pure simplicial complex of dimension $(d-1)$. Then the following are equivalent:*

(i) *Δ is CM_t over k;*
(ii) *$H_i(|\Delta|, |\Delta| \setminus p; k) = 0$ for all $p \in |\Delta| \setminus |\Delta_{t-2}|$ and all $i < d-1$, where Δ_{t-2} is the $(t-2)$-skeleton of Δ and $|\Delta_{t-2}|$ is induced from a fixed geometric realization of Δ.*

Let Δ and Δ' be two simplicial complexes whose vertex sets are disjoint. The simplicial join $\Delta * \Delta'$ is defined to be the simplicial complex whose faces are of the form $F \cup F'$ where $F \in \Delta$ and $F' \in \Delta'$.

The algebraic and combinatorial properties of the simplicial join $\Delta * \Delta'$ through the properties of Δ and Δ' have been studied by a number of authors (see [4], [6], [16], and [1]). For instance, in [6], Fröberg showed that the simplicial join $\Delta * \Delta'$ is Cohen-Macaulay (resp. Gorenstein) if and only if both of them are Cohen-Macaulay (resp. Gorenstein). On the other hand, if the join of two complexes is Buchsbaum, it should indeed be Cohen-Macaulay [31, Theorem 2.6]. Therefore, it is natural to ask about Δ and Δ' when $\Delta * \Delta'$ is CM_t.

If Δ is a CM_r complex of dimension $d-1$ and Δ' is a $\mathrm{CM}_{r'}$ complex of dimension $d'-1$, then their join $\Delta * \Delta'$ is a CM_t complex where $t = \max\{d + r', d' + r\}$ [27, Proposition 2.10]. However, if one of the complexes is Cohen-Macaulay, this result could be strengthened. Below we combine this with relevant known results. The proof is mainly based on the following characterization of CM_t simplicial complexes by Miller, Novik and Swartz which is interestin in its own:

Proposition 2.8. [12, Corollay 7.4] *Let Δ be a simplicial complex of dimension $d-1$ on n vertices and let $R = k[x_1, \cdots, x_n]$. Then Δ is CM_t if and only if Δ is pure and*

$$\dim \mathrm{Ext}^i_R(k[\Delta], R) \leq t$$

for all $i > n - d$, where dim *refers to the Krull dimension.*

Theorem 2.9. *Let Δ and Δ' be two complexes of dimensions $d-1$ and $d'-1$, respectively. Then*

(i) *The join complex $\Delta * \Delta'$ is Cohen-Macaulay if and only if both Δ and Δ' are so.*
(ii) *If Δ is Cohen-Macaulay and Δ' is $CM_{r'}$ for some $r' \geq 1$, then $\Delta * \Delta'$ is $CM_{d+r'}$ (independent of d'). This is sharp, i.e., if Δ' is not $CM_{r'-1}$, then $\Delta * \Delta'$ is not $CM_{d+r'-1}$. In particular, a cone on Δ' is $CM_{r'+1}$.*
(iii) *If Δ is CM_r and Δ' is $CM_{r'}$ for some $r, r' \geq 1$, then $\Delta * \Delta'$ is CM_t where $t = \max\{d + r', d' + r\}$. Conversely, if $\Delta * \Delta'$ is CM_t, then Δ is $CM_{t-d'}$ and Δ' is CM_{t-d}.*

3. CM_t flag complexes and Bipartite CM_t graphs

Let $G = (V, E)$ be a simple graph with vertex set V and edge set E. The *inclusive neighborhood* of $v \in V$ is the set $N[v]$ consisting of v and vertices adjacent to v in G. The *independence complex* of $G = (V, E)$ is the complex $\mathrm{Ind}(G)$ with vertex set V and with faces consisting of independent sets of vertices of G, i.e., sets of vertices of G where no two of them are adjacent. These complexes are called *flag* *complexes*, and their Stanley-Reisner ideal are generated by quadratic square-free monomials. By *dimension* of a graph G we mean the dimension of $\mathrm{Ind}(G)$. A graph G is said to be *unmixed* if $\mathrm{Ind}(G)$ is pure. A graph G is called CM$_t$ if $\mathrm{Ind}(G)$ is CM$_t$.

Cook and Nagel showed that the only non-Cohen-Macaulay unmixed Buchsbaum bipartite graphs are complete bipartite graphs [25, Theorem 4.10] and [26, Theorem 1.3]. We give a generalization of this fact to unmixed bipartite graphs which are CM$_t$.

A basic tool for checking CM$_t$ property of complexes is the following lemma.

Lemma 3.1. *Let $t \geq 1$ and let G be a graph. Then the following are equivalent:*

(i) *G is a* CM$_t$ *graph.*
(ii) *G is unmixed and $G \setminus N[v]$ is a* CM$_{t-1}$ *graph for every vertex $v \in G$.*

Let $G \sqcup G'$ denote the disjoint union of graphs G and G'. By the fact that $\mathrm{Ind}(G \sqcup G') = \mathrm{Ind}(G) * \mathrm{Ind}(G')$, the counter-part of Theorem 2.9 for graphs will be the following.

Theorem 3.2. *Let G and G' be two graphs on disjoint sets of vertices and of dimensions $d-1$ and $d'-1$, respectively. Then*

(i) *The graph $G \sqcup G'$ is Cohen-Macaulay if and only if both G and G' are so.*
(ii) *If G is Cohen-Macaulay and G' is $CM_{r'}$ for some $r' \geq 1$, then $G \sqcup G'$ is $CM_{d+r'}$. If G' is not $CM_{r'-1}$, then $G \sqcup G'$ is not $CM_{d+r'-1}$.*
(iii) *If G is CM_r and G' is $CM_{r'}$ for some $r, r' \geq 1$, then $G \sqcup G'$ is CM_t where $t = \max\{d + r', d' + r\}$. Conversely, if $G \sqcup G'$ is CM_t, then G is $CM_{t-d'}$ and G' is CM_{t-d}.*

A graph $G = (V, E)$ is called *bipartite* if V is a disjoint union of a partition V_1 and V_2 such that $E \subset V_1 \times V_2$. If $\#(V_1) = m$ and $\#(V_2) = n$ and $E = V_1 \times V_2$, then G is the *complete* bipartite graph $K_{m,n}$. We will be interested in unmixed complete bipartite graphs $K_{n,n}$.

Cohen-Macaulay bipartite graphs are characterized by Herzog and Hibi in the following result.

Theorem 3.3. [29, Theorem 3.4] *Let G be a bipartite graph without an isolated vertex. Then G is unmixed if and only if there is a partition $V_1 = \{x_1, \cdots, x_n\}$ and $V_2 = \{y_1, \cdots, y_n\}$ of vertices of G such that*

(1) *$x_i y_i$ is an edge in G for $1 \leq i \leq n$ and*
(2) *If $x_i y_j$ is an edge in G, then $i \leq j$.*
(2) *If $x_i y_j$ and $x_j y_k$ are edges in G, for some distinct i, j and k, then $x_i y_k$ is an edge in G.*

The order in the above result is called a *Macaualy* order on the given Cohen-Macaulay bipartite graph. In fact, by a result of Villarreal, validity of the conditions (1) and (2) is equivalent to unmixed-ness of G [35, Theorem 1.1].

Recall that a complex is Buchsbaum if and only if it is pure and the link of each vertex is Cohen-Macaulay [33]. Thus, a graph is Buchsbaum if and only if G is unmixed and for every vertex $v \in G$, $G \setminus N[v]$ is Cohen-Macaulay. For bipartite graphs there is a sharper result by Cook and Nagel.

Theorem 3.4. *(see [25, Theorem 4.10] or [26, Theorem1.3]) Let G be a bipartite graph. Then G is Buchsbaum if and only if G is a complete bipartite graph $K_{n,n}$ for some $n \geq 2$, or G is Cohen-Macaulay.*

We now generalize the results of Cook and Nagel in light of the result of Herzog and Hibi.

Theorem 3.5. *Let G be a Cohen-Macaulay bipartite graph with a Macaulay order on the vertex set $V(G) = V \cup W$ where $V = \{x_1, \cdots, x_d\}$ and $W = \{y_1, \cdots, y_d\}$. Let $n_1, \cdots, n_d$ be any positive integers with $n_i \geq 2$ for at list one i. Let $G' = G(n_1, \cdots, n_d)$ be the graph obtained by replacing each edge $x_i y_i$ with the complete bipartite graph K_{n_i,n_i} for all $i = 1, \ldots, d$. Let $n_{i_0} = \min\{n_i > 1 : i = 1, \cdots, d\}$, $n = \sum_{i=1}^{d} n_i$. Then G' is exclusively a CM$_{n-n_{i_0}+1}$ graph. Furthermore, any bipartite CM$_t$ graph is obtained by such a replacement of complete bipartite graphs in a unique bipartite Cohen-Macaulay graph.*

We now provide some examples:

Example 3.6. *(Bipartite CM$_2$ graphs which are not Buchsbaum):*
There are just two non-isomorphic bipartite Cohen-Macaulay graphs of dimension one. By the replacing process, they produce two types of bipartite CM$_2$ graphs which are not Buchsbaum. They are of arbitrary dimensions. More precisely, one such graph is the disjoint union of an edge $x_1 y_1$ with $K_{n_2,n_2} = \{x_{21}, \cdots, x_{2n_2}\} \times \{y_{21}, \cdots, y_{2n_2}\}$, $n_2 \geq 2$, and the other one consists of the first graph together with the edges $x_1 y_{2i}$ for all $i = 1, \cdots, n_2$. The second graph with $n_2 = 3$ could be depicted in Figure 1.

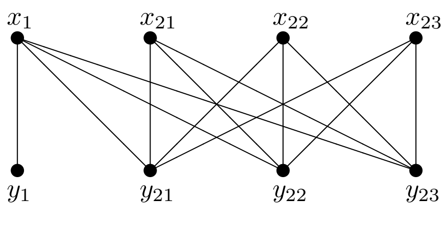

Figure 1

Example 3.7. *(Bipartite CM$_3$ graphs which are not CM$_2$):*
These arise only in the following cases:

(a) There are just two bipartite CM$_3$ graphs obtained by replacing two edges of a perfect matching by $K_{2,2}$'s. In this case, $\dim G = 3$ *(see Figure 2 and Figure 3).*

(b) There are 4 *non-isomorphic bipartite Cohen-Macaulay graphs of dimension* 2*. By replacing one perfect matching edge with $K_{n,n}$ of arbitrary size in each Cohen-Macaulay graph, they produce* 7 *types of bipartite CM$_3$ graphs which are not CM$_2$. Note that depending on the choice of the edge to be replaced in each case, we may get non-isomorphic bipartite graphs. In this case* $\dim G = n + 1$.

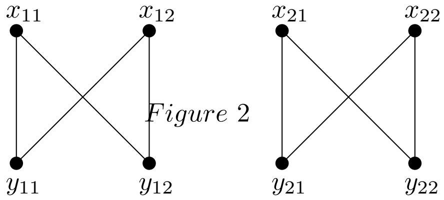

Figure 2

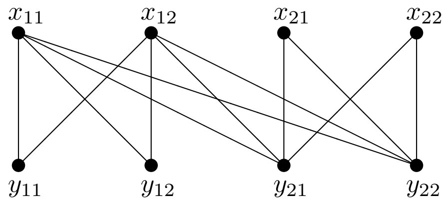

Figure 3

Example 3.8. *(Bipartite CM$_4$ graphs which are not CM$_3$):*
These arise in the following cases:

(a) There are two bipartite CM$_4$ graphs obtained by replacing two edges of a perfect matching by $K_{3,3}$'s. And, similarly, there are two others obtained by replacing one edge with $K_{2,2}$ and another edge with $K_{3,3}$. In both cases, $\dim G = 5$.

(b) While there are 4 *non-isomorphic bipartite Cohen-Macaulay graphs of dimension* 2*, by replacing two perfect matching edges with $K_{2,2}$'s in each Cohen-Macaulay graph, they produce* 7 *bipartite CM$_4$ graphs which are not CM$_3$. They all have dimension* 4.

(c) There are 10 *non-isomorphic bipartite Cohen-Macaulay graphs of dimension* 3*. Replacing one perfect matching edge with $K_{n,n}$, $n \geq 2$, in each Cohen-Macaulay graph, they produce* 25 *bipartite CM$_4$ graphs which are not CM$_3$. They all have dimension $n+2$. Out of all 36 bipartite CM$_4$ graphs, 21 graphs are connected.*

Final word. The CM$_t$ property for simplicial complexes and graphs has opened a variety of interesting questions which are already considered for Cohen-Macaualy and Buchsbaum ones. For example, Pournaki, Seyed Fakhari and Yassemi has considered the h-vector of a CM$_t$ simplicial complex [15] extending a result of Terai [23]. We have been considering CM$_t$ squar-efree lexsegment ideals generalizing results of Bonanzinga, Serrenti [7] and Bonanzinga, Serrenti and Terai [8]. Therefore, it is legitimate to invite interested people to work on this topic.

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