ON CELLULAR RESOLUTION OF MONOMIAL IDEALS

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Abstract. This is a general talk in front of specialists on computational differential algebra. Monomials play a significant role in most computational methods culminating in Gröbner bases. But the topic of this talk aims to encode the data of a monomial ideal on an object in algebraic topology which has some finite and geometric nature. Although these objects could be more general, we will restrict to the so called "polytopal complexes" only.

1. Cell complexes and their chain complexes, cellular resolution of monomial ideals

We First recall the concept of a polyhedral cell complex.

Definitions. A *polytope* $\mathcal{P}$ is the convex hull of a finite set of points in some $\mathbb{R}^m$. A *polyhedral cell complex* X is a finite collection of polytopes (in $\mathbb{R}^m$) called *faces* of X, satisfying the following two properties:

- If $\mathcal{P}$ is a polytope in X and F is a face of $\mathcal{P}$, then F is in X.
- If $\mathcal{P}$ and $\mathcal{Q}$ are in X, the $\mathcal{P} \cap \mathcal{Q}$ is a face of both $\mathcal{P}$ and $\mathcal{Q}$.

For example, the set of all faces of a polytope is a polyhedral cell complex. Any simplicial complex may be considered as a polyhedral cell complex via its geometric realization.

Since we will only deal with polyhedral cell complexes, we will simply call them *cell complexes.*

The dimension of a polytope (or a face) is its Euclidian dimension. The dimension of a cell complex is the maximum of the dimensions of its polytopes. The zero dimensional faces of Γ are called the vertices of Γ.

The set of q-dimensional faces of a cell complex X will be denoted by X_q.

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For any cell complex X with vertex set $V = \{v_0, v_1, \cdots, v_n\}$, there is a chain complex similar to the reduced chain complex for a simplicial complex. Let $C_q(X)$ be the free $\mathbb{Z}$-module with basis consisting of the faces in X_q. The *chain complex* of X is the complex

$$C_\bullet : 0 \longrightarrow C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \tag{1}$$

where for $F \in X_q$,

$$\partial_q(F) = \sum_G \epsilon(F, G).G \tag{2}$$

Here $\epsilon(F, G)$ is the *orientation* or *incidence function*

$$\epsilon : X \times X \longrightarrow \{-1, 0, 1\}$$

which satisfies

- $\epsilon(F, G) = 0$ unless $F \in X_q$ and $G \in X_{q-1}$ is a face of F for some q.
- For all F and H, $\sum_G \epsilon(F, G)\epsilon(G, H) = 0$.

The orientation function $\epsilon(F, G)$ indicates whether G appears with positive or negative orientation in the boundary of F.

Let $R = k[x_1, \cdots, x_n]$ be the polynomial ring in n variables over a field k. Let $I \subset R$ be a homogeneous ideal and let

$$L_\bullet : 0 \longrightarrow R^{\beta_n} \xrightarrow{d_n} R^{\beta_{n-1}} \xrightarrow{d_{n-2}} \cdots \xrightarrow{d_2} R^{\beta_1} \xrightarrow{d_1} R^{\beta_0} \xrightarrow{d_0} I \longrightarrow 0 \tag{3}$$

be a minimal free resolution of I. Set $L_q = R^{\beta_q}$.

Roughly speaking, a resolution $L_\bullet$ is said to be supported on a cell complex X if the faces of X can be labeled with monomials in a way which allows one to recover the resolution $L_\bullet$ from the chain complex associated to X. Below we will see a more precise definition.

The idea to describe a resolution of a monomial ideal by means of combinatorial chain complexes was initiated by Bayer, Peeva and Sturmfels [1], and was extended by Bayer and Sturmfels [2], and further extension was made by Jöllembeck and Welker [6]. Sinefakoupols [8] showed that the Borel-fixed monomial ideals which are Borel-fixed generated in one degree are cellular supported on a union of convex polytopes [8, Theorem 20]. Mermin constructed a regular cell complex which supports the Eliahou-Kervaire resolution of a stable monomial ideal [7]. This resolution is not in general polytopal [7, Theorem 4.20]. Dochtermann and Engström gave a cellular resolutions for the ideals of cointerval hypergraphs supported by polyhedral complexes and extended their construction to more general hypergraphs by decomposing them into cointerval hypergraphs [3, Theorems 4.4 and 6.1]. Recently, Dochtermann and Engström constructed cellular resolutions for powers of ideals of a bipartite graph on n vertices and using Morse theory, they provided explicit minimal cellular resolution for powers of the edge ideals of paths [4, Proposition 4.4 and Theorem 7.2]. Recently, Goodarzi [5] gave a more general result proving the minimal free resolution of a monomial ideal with linear quotients, the so-called Herzog-Takayama resolution is also cellular, covering most of previous results.

We consider the square-free Veronese ideal of degree t in n variables and illustrate the method of showing that such a monomial ideal has a polytopal minimal free resolution.

Determination of a canonical minimal free resolution for an arbitrary monomial ideal I is a major open problem in combinatorial commutative algebra which was indeed posed by Kaplansky in the early 1960's. If a monomial ideal has a cellular resolution its minimal free resolution is characteristic-free. If the cell complex providing the resolution is a simplicial complex, then, the minimal free resolution has the structure of an associative commutative graded algebra. Furthermore, the finely-graded Hilbert series of of the quotient ring by the ideal equals the finely-graded Euler characteristic of the cell-complex.

Let $I \subset R = \mathbf{k}[y_1, \cdots, y_n]$ be a monomial ideal in the polynomial ring over a field $\mathbf{k}$ and let $G(I)$ be the unique minimal monomial generating set of I. Let X be a polytopal cell complex with $G(I)$ as its vertices. Let ϵ_X be an incidence function on X. Any face of X will be labeled by $\mathbf{m}_F$, the least common multiple of the monomials in $G(I)$ which correspond to the vertices of F. If $\mathbf{m}_F = y_1^{a_1} \cdots y_n^{a_n}$, then the *degree* $\mathbf{a}_F$ is defined to be the exponent vector $e(\mathbf{m}_F) = (a_1, \cdots, a_n)$. Let RF be the free R-module with one generator in degree $\mathbf{a}_F$. The *cellular complex* $\mathbf{F}_X$ is the chain complex of $\mathbb{Z}^n$-graded R-modules $(\mathbf{F}_X)_i = \bigoplus_{F \in X, \dim F = i} RF$ with differentials

$$\partial(F) = \sum_{\emptyset \neq F' \in X} \epsilon(F, F') \frac{\mathbf{m}_F}{\mathbf{m}_{F'}} F'.$$

If the complex $\mathbf{F}_X$ is exact, then $\mathbf{F}_X$ is called a cellular resolution of I. Alternatively, we say that I has a cellular resolution supported on the labeled cell complex X. If X is a polytope or a simplicial complex, then $\mathbf{F}_X$ is called *polytopal*, and *simplicial*, respectively. A cellular resolution $\mathbf{F}_X$ is minimal if and only if any two comparable faces $F' \subset F$ of the same degree coincide. For more on cellular resolutions and polytopal complexes we refer to [10], [11] and [13].

As an example, the minimal free resolution of an ideal generated by a regular sequence of monomials $m_1, \ldots, m_r$ of length r, $r \leq n$, in $R = k[x_1, \cdots, x_n]$ is the Koszul complex associated to the regular sequence and it is supported on the $(r-1)$-simplex $\Delta(m_1, \ldots, m_r)$ in $\mathbb{R}^{r-1}$.

The following two lemmas will be essential for our constructions.

Lemma 1.1. *(The gluing lemma)* [8, Lemma 6]. *Let I and J be two ideals in R such that $G(I+J) = G(I) \cup G(J)$. Suppose that*
(i) X and Y are labeled regular cell complexes in some $\mathbb{R}^N$ that supports a minimal free resolution $\mathbf{F}_X$ and $\mathbf{F}_Y$ of I and J, respectively, and
(ii) $X \cap Y$ is a labeled regular cell complex that supports a minimal free resolution $\mathbf{F}_{X \cap Y}$ of $I \cap J$.
Then $X \cup Y$ is a labeled regular cell complex that supports a minimal free resolution of $I + J$.

Remark 1.2. [8, Remark 7]. *For any two monomial ideals I and J, we have*

$$G(I+J) \subseteq G(I) \cup G(J).$$

A case where equality holds is when all elements of $G(I) \cup G(J)$ are of the same degree.

Lemma 1.3. [8, Lemma 8]. *Let $I \subset \mathbf{k}[y_1, \cdots, y_k]$ and $J \subset \mathbf{k}[y_{k+1}, \cdots, y_n]$ be two monomial ideals. Suppose that X and Y are labeled regular cell complexes in some $\mathbb{R}^N$ of dimensions $k-1$ and $n-k-1$, respectively, that support minimal free resolutions $\mathbf{F}_X$ and $\mathbf{F}_Y$ of I and J, respectively. Then the labeled cell complex $X \times Y$ supports a minimal free resolution $\mathbf{F}_{X\times Y}$ of IJ.*

2. A cellular resolution for square-free Veronese ideals

We now consider construction of a cellular resolution for square-free Veronese ideals. Let $S = \mathbf{k}[x_1, \cdots, x_n]$. To facilitate the notations in the proofs, the square-free Veronese ideal in degree t in S, i.e., the the ideal generated by all square-free monomials of degree t in n indeterminates $x_1, \cdots, x_n$, will be denoted by $I_{n,t}$. To illustrate the method for the construction of the cellular resolution for $I_{n,t}$, we will first consider two examples. We use $\Gamma_{n,t}$ to denote a polytopal cell complex which is supposed to support a minimal free resolution of $I_{n,t}$.

Example 2.1. *A cellular resolution of $I_{5,2}$.*
Consider the following decomposition of $I_{5,2}$:

$$I_{5,2} = (x_1).(x_2,x_3,x_4,x_5) + (x_1,x_2).(x_3,x_4,x_5) + (x_1,x_2,x_3).(x_4,x_5)+$$
$$(x_1,x_2,x_3,x_4).(x_5).$$

We show that

$$\Gamma_{5,2} = [\Delta(x_1) \times \Delta(x_2,x_3,x_4,x_5)] \cup [\Delta(x_1,x_2) \times \Delta(x_3,x_4,x_5)] \cup$$
$$[\Delta(x_1,x_2,x_3) \times \Delta(x_4,x_5)] \cup [\Delta(x_1,x_2,x_3,x_4) \times \Delta(x_5)],$$

and this is a subdivision of the tetrahedron $\Delta(x_1x_2, x_2x_3, x_3x_4, x_4x_5)$. We will use the gluing lemma. Observe that

$$[\Delta(x_1) \times \Delta(x_2,x_3,x_4,x_5)] \cap [\Delta(x_1,x_2) \times \Delta(x_3,x_4,x_5)] = \Delta(x_1) \times \Delta(x_3,x_4,x_5),$$

which supports a minimal free resolution of

$$[(x_1).(x_2,x_3,x_4,x_5)] \cap [(x_1,x_2).(x_3,x_4,x_5)] = (x_1).(x_3,x_4,x_5).$$

Hence

$$[\Delta(x_1) \times \Delta(x_2,x_3,x_4,x_5)] \cup [\Delta(x_1,x_2) \times \Delta(x_3,x_4,x_5)]$$

supports a minimal free resolution of $(x_1).(x_2,x_3,x_4,x_5)+(x_1,x_2).(x_3,x_4,x_5)$. Furthermore,

$$[\Delta(x_1) \times \Delta(x_2,x_3,x_4,x_5) \cup \Delta(x_1,x_2) \times \Delta(x_3,x_4,x_5)] \cap [\Delta(x_1,x_2,x_3) \times \Delta(x_4,x_5)]$$
$$= \Delta(x_1,x_2) \times \Delta(x_4,x_5)$$

and it supports a minimal free resolution of

$$[(x_1).(x_2,x_3,x_4,x_5)+(x_1,x_2).(x_3,x_4,x_5)] \cap [(x_1,x_2,x_3).(x_4,x_5)] = (x_1,x_2).(x_4,x_5).$$

Therefore,

$$[\Delta(x_1) \times \Delta(x_2,x_3,x_4,x_5)] \cup [\Delta(x_1,x_2) \times \Delta(x_3,x_4,x_5)] \cup [\Delta(x_1,x_2,x_3) \times \Delta(x_4,x_5)]$$

supports a minimal free resolution of

$$(x_1).(x_2, x_3, x_4, x_5) + (x_1, x_2).(x_3, x_4, x_5) + (x_1, x_2, x_3).(x_4, x_5).$$

Finally,

$$[\Delta(x_1) \times \Delta(x_2, x_3, x_4, x_5) \cup \Delta(x_1, x_2) \times \Delta(x_3, x_4, x_5) \cup \Delta(x_1, x_2, x_3) \times \Delta(x_4, x_5)] \cap [\Delta(x_1, x_2, x_3, x_4) \times \Delta(x_5)] = \Delta(x_1, x_2, x_3) \times \Delta(x_5),$$

and it supports a minimal free resolution of

$$[(x_1).(x_2, x_3, x_4, x_5)+(x_1, x_2).(x_3, x_4, x_5)+(x_1, x_2, x_3).(x_4, x_5)] \cap [(x_1, x_2, x_3, x_4).(x_5)] = (x_1, x_2, x_3).(x_5).$$

Therefore, $\Gamma_{5,2}$ *supports a minimal free resolution of* $I_{5,2}$ *and it is a regular subdivision of* $\Delta(x_1x_2, x_2x_3, x_3x_4, x_4x_5)$ *(see Fig. 1).*

Example 2.2. *A cellular resolution of* $I_{6,3}$*.*
The appropriate decomposition for the ideal $I_{6,3}$ *is:*

$$I_{6,3} = I_{2,2}.(x_3, x_4, x_5, x_6) + I_{3,2}.(x_4, x_5, x_6) + I_{4,2}.(x_5, x_6) + I_{5,2}.(x_6).$$

We claim that

$$\Gamma_{6,3} = [\Gamma_{2,2} \times \Delta(x_3, x_4, x_5, x_6)] \cup [\Gamma_{3,2} \times \Delta(x_4, x_5, x_6)] \cup [\Gamma_{4,2} \times \Delta(x_5, x_6)] \cup [\Gamma_{5,2} \times \Delta(x_6)],$$

and $\Gamma_{6,3}$ *is a subdivision of the tetrahedron*

$$\Delta(x_1x_2x_3, x_2x_3x_4, x_3x_4x_5, x_4x_5x_6).$$

Note that each product in the above union is the polytopal cell complex that supports a minimal free resolution of the corresponding summand in the expression of $I_{6,3}$*. The products in the union, consecutively, satisfy the hypotheses of the gluing lemma. Indeed, the first two product polytopes in the union are in common on* $\Gamma_{2,2} \times \Delta(x_4, x_5, x_6)$ *which supports a minimal free resolution of* $I_{2,2}.(x_4, x_5, x_6) = [I_{2,2}.(x_3, x_4, x_5, x_6)] \cap [I_{3,2}.(x_4, x_5, x_6)]$*. Hence their union supports a minimal free resolution of the sum of the first two summands in the expression of* $I_{6,3}$*. This union intersects the third polytope in the expression of* $\Gamma_{6,3}$ *along* $\Gamma_{3,2} \times \Delta(x_5, x_6)$ *which supports a minimal free resolution of* $I_{3,2}.(x_5, x_6) = [I_{2,2}.(x_3, x_4, x_5, x_6) + I_{3,2}.(x_4, x_5, x_6)] \cap [I_{4,2}.(x_5, x_6)]$*. Hence the union of the first three polytopes is a polytopal cell complex which supports a minimal free resolution of the sum of the first three ideals in the expression of* $I_{6,3}$*. Finally, by Example 2.1,* $\Gamma_{5,2} \times \Delta(x_6)$ *supports a minimal free resolution of* $I_{5,2}.(x_6)$ *and the union of the first three polytopes intersects the last polytope in the expression of* $\Gamma_{6,3}$ *along* $\Gamma_{4,2} \times \Delta(x_6)$ *which supports a minimal free resolution of* $I_{4,2}.(x_6) = [I_{2,2}.(x_3, x_4, x_5, x_6) + I_{3,2}.(x_4, x_5, x_6) + I_{4,2}.(x_5, x_6)] \cap [I_{5,2}.(x_6)]$*. Therefore, the union of the four polytopes supports a minimal free resolution of* $I_{6,3}$*. This cell complex is clearly a regular subdivision of the tetrahedron* $\Delta(x_1x_2x_3, x_2x_3x_4, x_3x_4x_5, x_4x_5x_6)$ *(see Fig. 3).*

The following lemma is a key item for the construction of the polytopal cell complex to support a minimal free resolution of $I_{n.t}$.

Lemma 2.3. *For all* t*,* $2 \leq t \leq n-1$ *we have the following decomposition*

$$I_{n,t} = I_{t-1,t-1}.(x_t, \cdots, x_n) + I_{t,t-1}.(x_{t+1}, \cdots, x_n) + \cdots + I_{n-1,t-1}.(x_n),$$

and for all k*,* $t \leq k \leq n-1$*,*

$$[I_{t-1,t-1}.(x_t, \cdots, x_n) + I_{t,t-1}.(x_{t+1}, \cdots, x_n) + \cdots + I_{k-1,t-1}.(x_k, \cdots, x_n)] \cap$$

$$[I_{k,t-1}.(x_{k+1},\cdots,x_n)] = I_{k-1,t-1}.(x_{k+1},\cdots,x_n).$$

Now we can state the main theorem.

Theorem 2.4. *There exists a polytopal cell complex $\Gamma_{n,t} \subset \mathbb{R}^{n-t}$ that supports a minimal free resolution of $I_{n,t}$. Moreover, $\Gamma_{k,t} \subset \Gamma_{k+1,t}$, for all $t \leq k \leq n-1$, and*

$$\Gamma_{n,t} = [\Gamma_{t-1,t-1}\times\Delta(x_t,\cdots,x_n)]\cup[\Gamma_{t,t-1}\times\Delta(x_{t+1},\cdots,x_n)[\cup\cdots\cup[\Gamma_{n-1,t-1}\times\Delta(x_n)].$$

Proposition 2.5. *The polytopal cell complex $\Gamma_{n,t}$ is a regular polytopal subdivision of the $(n-t)$-simplex*

$$\Delta(x_1\cdots x_t, x_2\cdots x_{t+1},\cdots,x_{n-t+1}\cdots x_n).$$

In particular $\Gamma_{n,t}$ is shellable.

Remark 2.6. *Reiner and Welker* [12] *have expressed linear syzigies of any Stanley-Reisner ideal I_Δ in terms of homologies of the Alexander dual complex Δ^*. When I_Δ is a matroidal ideal, they were able to go further and provide a rather explicit free resolution for this ideal* [12, §6]. *It would be tempting to ask whether their resolution is cellular.*

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