Ideals and Algebras defined by Isotone Maps between Posets

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Outline

Hibi rings

The category of posets and ideals attached to graphs of isotone maps

Alexander duality for such ideals

The K-algebra $K[P, Q]$ given by the posets P and Q

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Fix a field K and let L be a distributive lattice. The Hibi ring $K[L]$ is the K-algebra generated over K by the elements $\alpha \in L$ with defining relations

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Hibi: $K[L]$ is an ASL and a normal Cohen–Macaulay domain.

Furthermore, the defining ideal of a Hibi ring has a quadratic Gröbner basis and hence is a Koszul algebra.

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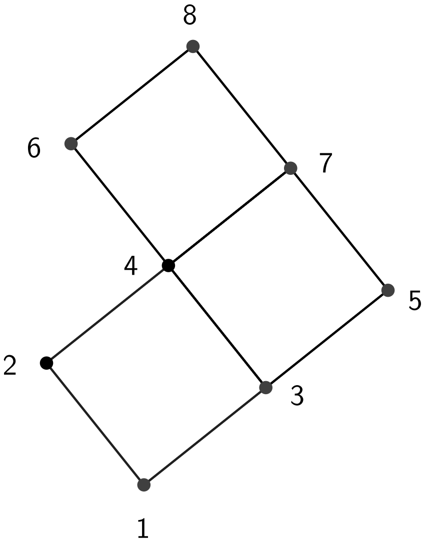
Birkhoff: $L \simeq I(P)$.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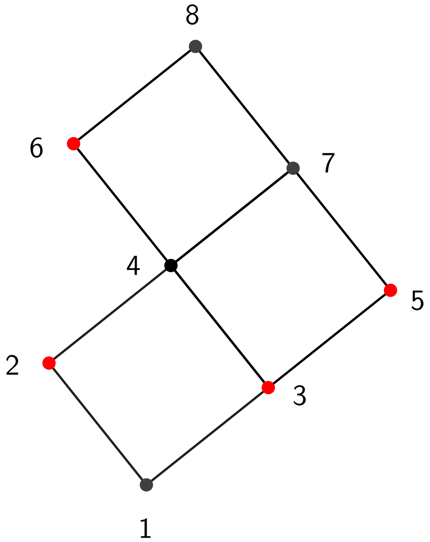
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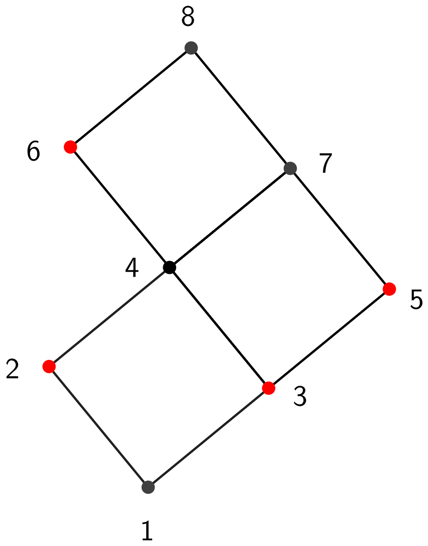
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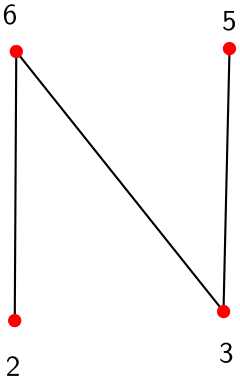
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$K[L]$ is Gorenstein if and only if P is pure (that is, all maximal chains in P have the same length).

Alternatively, the Hibi ring of L has a presentation

$$K[L] \simeq K[\{s \prod_{p \in \alpha} t_p : \alpha \in L\}] \subset T,$$

where $T = K[s, \{t_p \mid p \in P\}]$ is the polynomial ring in the variables s and t_p.

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Let $\hat{P}$ be the poset obtained from P by adding the elements $-\infty$ and ∞ with $\infty > p$ and $-\infty < p$ for all $p \in P$.

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Let $\hat{P}$ be the poset obtained from P by adding the elements $-\infty$ and ∞ with $\infty > p$ and $-\infty < p$ for all $p \in P$.

We denote by $\mathcal{T}(\hat{P})$ the set of integer valued functions

$$v : \hat{P} \to \mathbb{N}$$

with $v(\infty) = 0$ and $v(p) < v(q)$ for all $p > q$.

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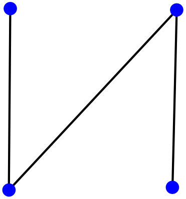
These are the strictly order reversing functions on $\hat{P}$.



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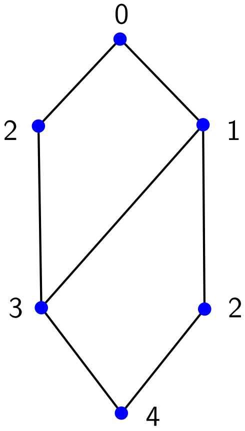

$\tilde{P}$

By using a result of Richard Stanley, Hibi showed that the monomials

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Let J_L denote the defining ideal of the Hibi ring $K[L]$.

Theorem. (Ene, H, Saeedi Madani) Let L be a finite distributive lattice and P the poset of join irreducible elements of L. Then

$$\operatorname{reg} J_L = |P| - \operatorname{rank} P.$$

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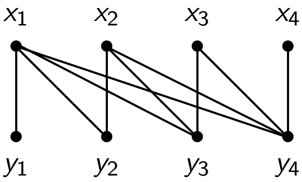

Let $\mathcal{P}$ be the category of finite posets.

- Objects: finite posets
- Morphisms: isotone maps (i.e. order preserving maps)

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$\mathrm{Hom}(P, Q)$, the set of isotone maps from P to Q, is itself a poset. We denote by $[n]$ the totally ordered poset $\{1 < 2 < \cdots < n\}$ on n elements. Then

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Now the theorem of Birkhoff, can be rephrased as follows: Let P be the subposet of join irreducible elements of the distributive lattice L. Then

$$L \simeq \mathrm{Hom}(P, [2])$$

2014 Fløystad, H, Greve introduced in "Letterplace and Co-letterplace ideals of posets" the ideals

$$L(P,Q) = (\prod_{p \in P} x_{p,\varphi(p)} : \varphi \in \mathrm{Hom}(P,Q))$$

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$L(P,[n])$ is the generalized Hibi ideal, introduced 2011 (European J.Comb.) by Ene, H, Mohammadi.

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This graph is the underlying graph of the so-called Hasse diagram of P which may also be viewed as a directed graph whose edges are (p_1, p_2) where p_2 covers p_1.

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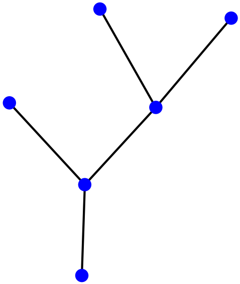
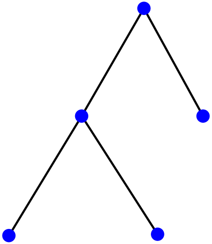
P is (co)-rooted if for all incomparable $p_1, p_2 \in P$ there is no $p \in P$ with $p > p_1, p_2$ ($p < p_1, p_2$).





Theorem. (H, Shikama, Qureshi) $L(P,Q)^{\vee} = L(Q,P)^{\tau}$ if and only if P or Q is connected and one of the following conditions hold:

(a) Both, P and Q are rooted;

(b) Both, P and Q are co-rooted;

(c) P is connected and Q is a disjoint union of chains;

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In the recent paper "Algebraic properties of ideals of poset homomorphisms" Juhnke-Kubitzke, Katthän and Saeedi Madani show for a large subclasses of the ideals $L(P, Q)$ when they are Buchsbaum, Cohen-Macaulay, Gorenstein and when they have a linear resolution.

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- Let I be the initial ideal of the ideal of s-minors of an $(n + s - 1) \times (m + s - 1)$-matrix of indeterminates. Then I is obtained from $L([s], [m] \times [n])$ by reduction modulo a regular sequence which identifies variables.

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- Strongly stable ideals.

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Theorem. (a) (Fløystad, H, Greve) Any monomial ideal I generated by a subset of the monomial generators of $L(P, Q)$ is inseparable.
(b) (Altmann, Bigdeli, H, Dancheng Lu) The ideals $L(P, Q)$ are rigid if and only if no two elements of P are comparable.

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An inseparable monomial ideal I which specializes to monomial ideal J is called a separated model of J. So the ideals $L(P,Q)$ are separated models of many monomial ideals.

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What is the Krull dimension of $K[P, Q]$?

Theorem. (Bigdeli, Hibi, H, Shikama, Qureshi) Let P and Q be finite posets. Then $\dim K[P, Q] = |P|(|Q| - s) + rs - r + 1$, where r is the number of connected components of P and s is the number of connected components of Q.

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Conjecture. For any poset P and Q, the defining ideal of the K-algebra $K[P, Q]$ has a squarefree initial ideal.

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Assuming the conjecture is true, the algebras $K[P, Q]$ are all normal by a theorem of Sturmfels, and then by a theorem of Hochster they are also Cohen-Macaulay.

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Theorem. (Bigdeli, Hibi, H, Shikama, Qureshi) Let P be the chain and suppose that each connected component of Q is either rooted or a co-rooted. Then the defining toric ideal of $K[P, Q]$ admits a quadratic Gröbner basis and a squarefree initial ideal.