Linear Resolution, Chordality and Ascent of Clutters

Ashkan Nikseresht
ashkan_nikseresht@yahoo.com
Rashid Zaare-Nahandi
rashidzn@iasbs.ac.ir

Department of Mathematics, Institute for Advanced Studies in Basic Sciences, Zanjan, Iran

12th Seminar on Commutative Algebra and Related Topics, IPM, Tehran, November 11 & 12, 2015

some notations

- $\mathcal{C} \longrightarrow$ a uniform d-dimensional clutter on $[n] = \{1, \ldots, n\}$, that is, a family of $(d+1)$-subsets of $[n]$ called circuits of $\mathcal{C}$.
- $I = I(\mathcal{C}) \longrightarrow$ circuit ideal of $\mathcal{C} = \langle x_F | F \in \mathcal{C} \rangle$ in the ring $S = k[x_1, \ldots, x_n]$, where $x_F = \prod_{i \in F} x_i$.
  Note: I is a square free monomial ideal and every sq. free monomial ideal is $I(\mathcal{C})$ for some $\mathcal{C}$ (not necessarily uniform).

some notations

- $\mathcal{C} \longrightarrow$ a uniform d-dimensional clutter on $[n] = \{1, \ldots, n\}$, that is, a family of $(d+1)$-subsets of $[n]$ called circuits of $\mathcal{C}$.
- $I = I(\mathcal{C}) \longrightarrow$ circuit ideal of $\mathcal{C} = \langle x_F | F \in \mathcal{C} \rangle$ in the ring $S = k[x_1, \ldots, x_n]$, where $x_F = \prod_{i \in F} x_i$.
  Note: I is a square free monomial ideal and every sq. free monomial ideal is $I(\mathcal{C})$ for some $\mathcal{C}$ (not necessarily uniform).
- $\overline{\mathcal{C}} \longrightarrow$ d-complement of $\mathcal{C}$ = family of $(d+1)$-subsets of $[n]$ not in $\mathcal{C}$.

some notations

- $\mathcal{C} \longrightarrow$ a uniform d-dimensional clutter on $[n] = \{1, \dots, n\}$, that is, a family of $(d+1)$-subsets of $[n]$ called circuits of $\mathcal{C}$.
- $I = I(\mathcal{C}) \longrightarrow$ circuit ideal of $\mathcal{C} = \langle x_F | F \in \mathcal{C} \rangle$ in the ring $S = k[x_1, \dots, x_n]$, where $x_F = \prod_{i \in F} x_i$.
  Note: I is a square free monomial ideal and every sq. free monomial ideal is $I(\mathcal{C})$ for some $\mathcal{C}$ (not necessarily uniform).
- $\overline{\mathcal{C}} \longrightarrow$ d-complement of $\mathcal{C}$ = family of $(d+1)$-subsets of $[n]$ not in $\mathcal{C}$.
- simplicial complex on $[n] \longrightarrow$ a family Δ of subsets of $[n]$ with: $A \subseteq B$ & $B \in \Delta \Rightarrow A \in \Delta$.

some notations

- $\mathcal{C} \longrightarrow$ a uniform d-dimensional clutter on $[n] = \{1, \ldots, n\}$, that is, a family of $(d+1)$-subsets of $[n]$ called circuits of $\mathcal{C}$.
- $I = I(\mathcal{C}) \longrightarrow$ circuit ideal of $\mathcal{C} = \langle x_F | F \in \mathcal{C} \rangle$ in the ring $S = k[x_1, \ldots, x_n]$, where $x_F = \prod_{i \in F} x_i$.
  Note: I is a square free monomial ideal and every sq. free monomial ideal is $I(\mathcal{C})$ for some $\mathcal{C}$ (not necessarily uniform).
- $\overline{\mathcal{C}} \longrightarrow d$-complement of $\mathcal{C}$ = family of $(d+1)$-subsets of $[n]$ not in $\mathcal{C}$.
- simplicial complex on $[n] \longrightarrow$ a family Δ of subsets of $[n]$ with: $A \subseteq B$ & $B \in \Delta \Rightarrow A \in \Delta$.
- A clique of $\mathcal{C} \longrightarrow$ a subset of $[n]$, all $(d+1)$-subsets of which is in $\mathcal{C}$.
- $\Delta(\mathcal{C}) =$ clique complex of the clutter $\mathcal{C}$= the family of all cliques of $\mathcal{C}$.

some notations

- $\mathcal{C} \longrightarrow$ a uniform d-dimensional clutter on $[n] = \{1, \ldots, n\}$, that is, a family of $(d+1)$-subsets of $[n]$ called circuits of $\mathcal{C}$.
- $I = I(\mathcal{C}) \longrightarrow$ circuit ideal of $\mathcal{C} = \langle x_F | F \in \mathcal{C} \rangle$ in the ring $S = k[x_1, \ldots, x_n]$, where $x_F = \prod_{i \in F} x_i$.
  Note: I is a square free monomial ideal and every sq. free monomial ideal is $I(\mathcal{C})$ for some $\mathcal{C}$ (not necessarily uniform).
- $\overline{\mathcal{C}} \longrightarrow$ d-complement of $\mathcal{C}$ = family of $(d+1)$-subsets of $[n]$ not in $\mathcal{C}$.
- simplicial complex on $[n] \longrightarrow$ a family Δ of subsets of $[n]$ with: $A \subseteq B$ & $B \in \Delta \Rightarrow A \in \Delta$.
- A clique of $\mathcal{C} \longrightarrow$ a subset of $[n]$, all $(d+1)$-subsets of which is in $\mathcal{C}$.
- $\Delta(\mathcal{C}) =$ clique complex of the clutter $\mathcal{C}$= the family of all cliques of $\mathcal{C}$.
- $\Delta|_L = \{F \in \Delta | F \subseteq L\}$.

introduction

A question which has gained attention recently by many is:

When a graded ideal I of S has a linear resolution?

Polarization $\Rightarrow$ for monomial I reduces to sq. free monomial I

For more on this question and related concepts see [Herzog, Hibi (2011)].

introduction

A question which has gained attention recently by many is:

When a graded ideal I of S has a linear resolution?

Polarization $\Rightarrow$ for monomial I reduces to sq. free monomial I

For more on this question and related concepts see [Herzog, Hibi (2011)].

In the case of sq. free's, a theorem of [Fröberg, 1990]: if $d = 1$ (that is, when $\mathcal{C}$ is a graph), $I(\overline{\mathcal{C}})$ has a linear resolution $\Leftrightarrow$ $\mathcal{C}$ is a chordal graph (that is, a graph with no non-complete induced cycle).

introduction

A question which has gained attention recently by many is:

When a graded ideal I of S has a linear resolution?

Polarization $\Rightarrow$ for monomial I reduces to sq. free monomial I

For more on this question and related concepts see [Herzog, Hibi (2011)].

In the case of sq. free's, a theorem of [Fröberg, 1990]: if $d = 1$ (that is, when $\mathcal{C}$ is a graph), $I(\bar{\mathcal{C}})$ has a linear resolution $\Leftrightarrow$ $\mathcal{C}$ is a chordal graph (that is, a graph with no non-complete induced cycle).

Many have tried to generalize the concept of chordal graphs to clutters of arbitrary dimension in a way that Fröberg's theorem remains true for $d > 1$.

chordal clutters

- submaximal circuits $\longrightarrow \mathcal{SC}(\mathcal{C}) = d$-subsets of circuits of $\mathcal{C}$ (correspond to vertices in graphs). In the following $e \in \mathcal{SC}(\mathcal{C})$.
- $\deg(e) =$ number of circuits containing e.
- $\mathcal{C} - e \longrightarrow$ delete all circuits of $\mathcal{C}$ containing e.
- $\mathrm{N}[e] = e \cup \{v \in [n] | e \cup \{v\} \in \mathcal{C}\}$.

chordal clutters

- submaximal circuits $\longrightarrow \mathcal{SC}(\mathcal{C}) = d$-subsets of circuits of $\mathcal{C}$ (correspond to vertices in graphs). In the following $e \in \mathcal{SC}(\mathcal{C})$.
- $\deg(e) =$ number of circuits containing e.
- $\mathcal{C} - e \longrightarrow$ delete all circuits of $\mathcal{C}$ containing e.
- $\mathrm{N}[e] = e \cup \{v \in [n] | e \cup \{v\} \in \mathcal{C}\}$.
- simplicial submaximal circuit ($\mathcal{SSC}$) $\longrightarrow$ an $e \in \mathcal{SC}(\mathcal{C})$ for which $\mathrm{N}[e]$ is a clique.
- chordal clutter (see [Morales, et al (2014)]) $\longrightarrow$ a clutter $\mathcal{C}$ with a sequence of $\mathcal{SC}$'s $e_1, \ldots, e_t$ such that $e_i \in \mathcal{SSC}(\mathcal{C} - e_1 - \cdots - e_{i-1})$ and $\mathcal{C} - e_1 - \cdots - e_t = \emptyset$.

Theorem 1.1 ([Morales, et al (2014), Remark 3.10])

$\mathcal{C}$ chordal $\Rightarrow I(\bar{\mathcal{C}})$ has a linear resolution over every field.

the converse?

The converse is not know to be true or not. Converse $\Leftrightarrow$: $I(\overline{\mathfrak{C}})$ has linear resolution, then $\mathcal{SSC}(\mathfrak{C}) \neq \emptyset$.

the converse?

The converse is not know to be true or not. Converse $\Leftrightarrow$: $I(\overline{\mathcal{C}})$ has linear resolution, then $\mathcal{SSC}(\mathcal{C}) \neq \emptyset$.
In [Bigdeli, et al (2015)] and [Nikseresht, Zaare-Nahandi], is proved that:

the converse?

The converse is not know to be true or not. Converse $\Leftrightarrow$: $I(\bar{\mathcal{C}})$ has linear resolution, then $\mathcal{SSC}(\mathcal{C}) \neq \emptyset$.
In [Bigdeli, et al (2015)] and [Nikseresht, Zaare-Nahandi], is proved that:

- If $\mathcal{C}$ is chordal in the sense of [Woodroofe, 2011] or [Emtander, 2010], or if $I(\bar{\mathcal{C}})$ is sq. free stable, then it is chordal.

the converse?

The converse is not know to be true or not. Converse $\Leftrightarrow$: $I(\overline{\mathcal{C}})$ has linear resolution, then $\mathcal{SSC}(\mathcal{C}) \neq \emptyset$.
In [Bigdeli, et al (2015)] and [Nikseresht, Zaare-Nahandi], is proved that:

- If $\mathcal{C}$ is chordal in the sense of [Woodroofe, 2011] or [Emtander, 2010], or if $I(\overline{\mathcal{C}})$ is sq. free stable, then it is chordal.
- If $I(\overline{\mathcal{C}})$ is polymatroidal, or if $I(\overline{\mathcal{C}})$ is the vertex cover ideal of a Cohen-Macaulay graph, then $\mathcal{SSC}(\mathcal{C}) \neq \emptyset$.

the converse?

The converse is not know to be true or not. Converse $\Leftrightarrow$: $I(\overline{\mathcal{C}})$ has linear resolution, then $\mathcal{SSC}(\mathcal{C}) \neq \emptyset$.
In [Bigdeli, et al (2015)] and [Nikseresht, Zaare-Nahandi], is proved that:

- If $\mathcal{C}$ is chordal in the sense of [Woodroofe, 2011] or [Emtander, 2010], or if $I(\overline{\mathcal{C}})$ is sq. free stable, then it is chordal.
- If $I(\overline{\mathcal{C}})$ is polymatroidal, or if $I(\overline{\mathcal{C}})$ is the vertex cover ideal of a Cohen-Macaulay graph, then $\mathcal{SSC}(\mathcal{C}) \neq \emptyset$.

So it's reasonable to guess:
$I(\overline{\mathcal{C}})$ has a linear resolution over every field $\Rightarrow$ $\mathcal{C}$ is chordal?
or at least:
$I(\overline{\mathcal{C}})$ has linear quotients $\Rightarrow$ $\mathcal{C}$ is chordal?

aims of this research

In general the above two questions seem not to be easy. So we try to reduce the questions to simpler cases. Indeed, our final goal in this research is to reduce these questions to the case that $\mathcal{C}$ has no cliques on more than $d+1$ vertices.

aims of this research

In general the above two questions seem not to be easy. So we try to reduce the questions to simpler cases. Indeed, our final goal in this research is to reduce these questions to the case that $\mathcal{C}$ has no cliques on more than $d+1$ vertices.
To this end, we study the following clutter

$$\mathcal{C}^+ = \mathcal{F}(\Delta(\mathcal{C})^{[d+1]}) = \text{all cliques of } \mathcal{C} \text{ on } d+2 \text{ vertices},$$

which we call the ***ascent*** of $\mathcal{C}$.

aims of this research

In general the above two questions seem not to be easy. So we try to reduce the questions to simpler cases. Indeed, our final goal in this research is to reduce these questions to the case that $\mathcal{C}$ has no cliques on more than $d+1$ vertices.

To this end, we study the following clutter

$$\mathcal{C}^+ = \mathcal{F}(\Delta(\mathcal{C})^{[d+1]}) = \text{all cliques of } \mathcal{C} \text{ on } d+2 \text{ vertices},$$

which we call the ***ascent*** of $\mathcal{C}$.

Here we present some results on how the concepts of linear quot., linear res. and chordality behave under passing from $\mathcal{C}$ to $\mathcal{C}^+$.

linear resolution under ascension

Here $\widetilde{H}_i(\Gamma; k)$ denotes the i'th homology of the augmented oriented chain complex of a simplicial complex Γ over a field k.

linear resolution under ascension

Here $\widetilde{H}_i(\Gamma; k)$ denotes the i'th homology of the augmented oriented chain complex of a simplicial complex Γ over a field k.

Proposition 2.1

The ideal $I(\overline{\mathcal{C}})$ has a linear resolution over a field k, $\Leftrightarrow$ $I(\overline{\mathcal{C}^+})$ has a linear resolution over k and $\widetilde{H}_d(\Delta(\mathcal{C})|_W; k) = 0$ for all $W \subseteq [n]$.

linear resolution under ascension

Here $\widetilde{H}_i(\Gamma; k)$ denotes the i'th homology of the augmented oriented chain complex of a simplicial complex Γ over a field k.

Proposition 2.1

The ideal $I(\overline{\mathcal{C}})$ has a linear resolution over a field k, $\Leftrightarrow$ $I(\overline{\mathcal{C}^+})$ has a linear resolution over k and $\widetilde{H}_d(\Delta(\mathcal{C})|_W; k) = 0$ for all $W \subseteq [n]$.

This result could be proved using the following theorem of Fröberg [Fröberg, 1985] or could be proved independently and used as a proof of Fröberg's theorem.

Theorem 2.2 (Fröberg)

Suppose that $\Delta = \Delta(\mathcal{C})$. Then I_Δ has a linear resolution over k, $\Leftrightarrow$ $\widetilde{H}_i(\Gamma; k) = 0$ for every induced subcomplex Γ of Δ and $i \geq d$.

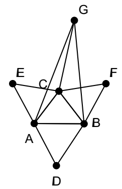
passing chordality to the ascent

Lemma 2.3

$e \in \mathcal{SSC}(\mathcal{C})$ *with* $\deg(e) > 1$, $v \in \mathrm{N}_{\mathcal{C}}[e] \setminus e \Rightarrow ev \in \mathcal{SSC}(\mathcal{C}^+)$.

Example 2.4

$\mathcal{C} \longrightarrow$ *triangles in the following figure.* $\mathcal{C}^+ = \{ABCG\}$.
$ABC \in \mathcal{SSC}(\mathcal{C}^+)$ *but* $AB, AC, BC \notin \mathcal{SSC}(\mathcal{C})$.



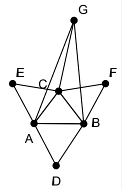
passing chordality to the ascent

Lemma 2.3

$e \in \mathcal{SSC}(\mathcal{C})$ *with* $\deg(e) > 1$, $v \in \mathrm{N}_{\mathcal{C}}[e] \setminus e \Rightarrow ev \in \mathcal{SSC}(\mathcal{C}^+)$.

Example 2.4

$\mathcal{C} \longrightarrow$ *triangles in the following figure.* $\mathcal{C}^+ = \{ABCG\}$.
$ABC \in \mathcal{SSC}(\mathcal{C}^+)$ *but* $AB, AC, BC \notin \mathcal{SSC}(\mathcal{C})$.



Theorem 2.5

If $\mathcal{C}$ *is chordal, then* $\mathcal{C}^+$ *is chordal.*

d-chorded clutters

In [Connon, Faridi (2013)] a combinatorial condition (*d*-chorded) equivalent to $\widetilde{H}_d(\Delta(\mathcal{C})|_W; \mathbb{Z}_2) = 0$ for all $W \subseteq [n]$, is presented.

Lemma 2.6

The clutter $\mathcal{C}$ is d-chorded $\Leftrightarrow$ for each $\mathcal{D} \subseteq \mathcal{C}$ with the property that $\deg_{\mathcal{D}}(e)$ is even for all $e \in \mathcal{SC}(\mathcal{D})$, there is a family $\mathcal{D}_1, \ldots, \mathcal{D}_k$ of cliques on $(d+2)$-subsets of $V(\mathcal{D})$ such that $\mathcal{D} = \triangle_{i=1}^{k} \mathcal{D}_i$.

d-chorded clutters

In [Connon, Faridi (2013)] a combinatorial condition (d-chorded) equivalent to $\widetilde{H}_d(\Delta(\mathcal{C})|_W; \mathbb{Z}_2) = 0$ for all $W \subseteq [n]$, is presented.

Lemma 2.6

The clutter $\mathcal{C}$ is d-chorded $\Leftrightarrow$ for each $\mathcal{D} \subseteq \mathcal{C}$ with the property that $\deg_{\mathcal{D}}(e)$ is even for all $e \in \mathcal{SC}(\mathcal{D})$, there is a family $\mathcal{D}_1, \ldots, \mathcal{D}_k$ of cliques on $(d+2)$-subsets of $V(\mathcal{D})$ such that $\mathcal{D} = \triangle_{i=1}^{k} \mathcal{D}_i$.

In [Connon, Faridi (2015), Theorem 18], an equivalent combinatorial condition for having linear resolution over field of char 2 is given. (2.1) provides another proof of this Theorem.

deletion of simplicial circuits

Theorem 2.7

Suppose that $\mathcal{C}$ is d-chorded and $F \in \mathcal{SSC}(\mathcal{C}^+)$. Then $\mathcal{C} - F$ is d-chorded.

deletion of simplicial circuits

Theorem 2.7

Suppose that $\mathcal{C}$ is d-chorded and $F \in \mathcal{SSC}(\mathcal{C}^+)$. Then $\mathcal{C} - F$ is d-chorded.

Simplicial edge of $G \longrightarrow v_1 v_2 \in \mathrm{E}(G)$ such that for $D = \{v_1, v_2\} \cup (\mathrm{N}_G(v_1) \cap \mathrm{N}_G(v_2))$: $|D| \geq 3$ and $G[D]$ is a clique.

Corollary 2.8

If a graph G is chordal and $e_1, \ldots, e_t$ are a sequence of edges such that e_i is simplicial in $G_i = G - e_1 - \cdots - e_{i-1}$, then G_{t+1} is chordal and if G_{t+1} has no simplicial edge, then it is a tree.

linear quotients and ascension

Theorem 2.9

Assume that $I(\overline{\mathcal{C}})$ has linear quotients. Then $I(\overline{\mathcal{C}^+})$ has linear quotients. Moreover, if $F \in \mathcal{SSC}(\mathcal{C}^+)$, then both of the ideals $I(\overline{\mathcal{C}^+ - F})$ and $I(\overline{\mathcal{C} - F})$ have linear quotients.

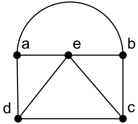
linear quotients and ascension

Theorem 2.9

Assume that $I(\overline{\mathcal{C}})$ has linear quotients. Then $I(\overline{\mathcal{C}^+})$ has linear quotients. Moreover, if $F \in \mathcal{SSC}(\mathcal{C}^+)$, then both of the ideals $I(\overline{\mathcal{C}^+ - F})$ and $I(\overline{\mathcal{C} - F})$ have linear quotients.

Example 2.10

$G \longrightarrow$ the following graph. Then G^+ has a non-circuit ideal with linear quotients and is chordal. $F \in \mathcal{SSC}(G^+) \Rightarrow G - F$ is chordal and has non-circuit ideal with linear quotients. But G is not chordal.



For Further Reading I

Herzog J. and Hibi T., 2011
Monomial Ideals, Springer-Verlag

R. Fröberg, 1990
On Stanely-Reisner rings
Banach Center Publications, 26 part 2: Topics in Algebra, 57–70.

M. Bigdeli, A. A. Yazdan Pour and R. Zaare-Nahandi, 2015
Stability of betti numbers under reduction processes: towards chordality of clutters
preprint, arXiv:1508.03799, 2015.

Morales, M., Nasrollah Nejad, A., Yazdan Pour, A. A. and Zaare-Nahanadi, Rashid, 2014
Monomial ideals with 3-linear resolutions,
Annales de la Faculté des Sciences de Toulouse 23(4): 877-891.

For Further Reading II

A. Nikseresht and R. Zaare-Nahandi
On generalization of cycles and chordal graphs to cluters of higher dimensions
preprint.

E. Emtander, 2010
A class of hypergraphs that generalizes chordal graphs,
Math. Scand. 106(1): 50-66.

R. Woodroofe, 2011
Chordal and sequentially Cohen-Macaulay clutters,
Elec. J. Combin. 18(1): #P208.

R. Fröberg, 1985.
Rings with monomial relations having linear resolutions
J. Pure Appl. Algebra 38, 235–241.

For Further Reading III

Connon, E. and Faridi, S., 2013
Chorded complexes and a necessary codition for a monomial ideal to have a linear resolution
J. Combin. Theory Ser. A 120: 1714–1731.

Connon, E. and Faridi, S., 2015
A criterion for a monomial ideal to have a linear resolution in characteristic 2
Elec. J. Combin., 22(1), #P1.63.

Thanks

Thanks for Your Attention