The 12th Seminar on
Commutative Algebra and Related Topics
IPM, November 11-12, 2015

Bi-Cohen-Macaulay graphs

Jürgen Herzog & Ahad Rahimi

Razi University

BACKGROUND

The graphs considered here will all be finite, simple graphs, that is, they will have no double edges and no loops. Furthermore we assume that G has no isolated vertices. The vertex set of G will be denoted $V(G)$ and will be the set $[n] = \{1, 2, \ldots, n\}$, unless otherwise stated. The set of edges of G we denote by $E(G)$.

A subset $F \subset [n]$ is called a *clique* of G, if $\{i, j\} \in E(G)$ for all $i, j \in F$ with $i \neq j$. The set of all cliques of G is a simplicial complex, denoted $\Delta(G)$.

A subset $C \subset [n]$ is called a *vertex cover* of G if $C \cap \{i, j\} \neq \emptyset$ for all edges $\{i, j\}$ of G.

The graph G is called *unmixed* if all minimal vertex covers of G have the same cardinality.

A subset $D \subset [n]$ is called an *independent set* of G if D contains no set $\{i, j\}$ which is an edge of G. Note that D is an independent set of G if and only if $[n] \setminus D$ is a vertex cover. Thus the minimal vertex covers of G correspond to the maximal independent sets of G. The cardinality of a maximal independent set is called the *independence number* of G.

The graph G is called *bipartite* if $V(G)$ is the disjoint union of V_1 and V_2 such that V_1 and V_2 are independent sets.

The graph G is called *chordal* if each cycle of G of length ≥ 4 has a chord. A graph which has no cycle and which is connected is called a *tree*.

Let $I \subset S$ be a squarefree monomial ideal. Then $I = \bigcap_{j=1}^{m} P_j$ where each of the P_j is a monomial prime ideal of I. The ideal $I^\vee$ which is minimally generated by the monomials $u_j = \prod_{x_i \in P_j} x_i$ is called the *Alexander dual* of I. One has $(I^\vee)^\vee = I$.

2. Various characterizations of Bi-Cohen-Macaulay graphs

DEFINITION 2.1.

A simplicial complex Δ is called *bi-Cohen-Macaulay* (bi-CM), if Δ and its Alexander dual $\Delta^\vee$ are Cohen-Macaulay. This concept was introduced by Fløystad and Vatne.

Given a field K and a simple graph on the vertex set $[n] = \{1, 2, \ldots, n\}$, one associates with G the edge ideal I_G of G, whose generators are the monomials $x_i x_j$ with $\{i, j\}$ an edge of G. We say that G is bi-CM if the simplicial complex whose Stanley-Reisner ideal coincides with I_G is bi-CM, that is, I_G as well as the Alexander dual $(I_G)^\vee$ of I_G is a Cohen-Macaulay ideal.

RECALL:

An ideal I in a polynomial ring S over a field K have a linear resolution if S/I has a minimal free resolution such that for all $j > 1$ the nonzero entries of the matrices of the maps $S^{\beta_j} \to S^{\beta_{j-1}}$ are of degree 1.

EAGON-REINER THEOREM:

I is a Cohen-Macaulay ideal if and only if $I^\vee$ has a linear resolution. Thus I is bi-CM if and only if I is a Cohen-Macaulay ideal with linear resolution.

PROPOSITION 2.1 *Let K be an infinite field and G a graph on the vertex set $[n]$ with independence number c. The following conditions are equivalent:*

(a) *G is a bi-CM graph over K;*

(b) *G is a CM graph over K, and S/I_G modulo a maximal regular sequence of linear forms is isomorphic to $T/\mathfrak{m}_T^2$ where T is the polynomial ring over K in $n-c$ variables and $\mathfrak{m}_T$ is the graded maximal ideal of T.*

COROLLARY 2.2

Let G be a graph on the vertex set $[n]$ with independence number c.

The following conditions are equivalent:

(a) *G is a bi-CM graph over K;*

(b) *G is a CM graph over K and $|E(G)| = \binom{n-c+1}{2}$;*

(c) *G is a CM graph over K and the number of minimal vertex covers of G is equal to $n-c+1$;*

(d) *$\beta_i(I_G) = (i+1)\binom{n-c+1}{i+2}$ for $i = 0, \ldots, n-c-1$.*

FACT 2.3

G is a bi-CM graph over K if and only if $(I_G)^\vee$ the vertex cover ideal of G is a codimension 2 Cohen-Macaulay ideal with linear relations.

3. The classification of bipartite and chordal bi-CM graphs

THEOREM 3.1 *Let G be a bipartite graph on the vertex set V with bipartition $V = V_1 \cup V_2$ where $V_1 = \{v_1, \ldots, v_n\}$ and $V_2 = \{w_1, \ldots, w_m\}$. Then the following conditions are equivalent:*

(a) *G is a bi-CM graph;*
(b) *$n = m$ and $E(G) = \{\{v_i, w_j\}: 1 \leq i \leq j \leq n\}$.*

The following picture shows a bi-CM bipartite graph for $n = 4$.

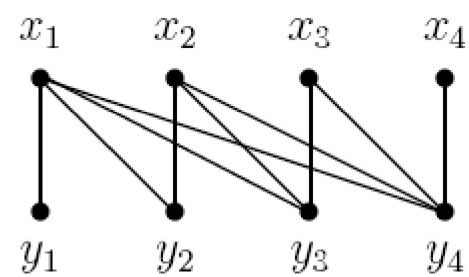


FIGURE 1. A bi-CM bipartite graph.

THEOREM 3.2. *Let G be a chordal graph on the vertex set $[n]$. The following conditions are equivalent:*

(a) *G is a bi-CM graph;*

(b) *Let $F_1, \ldots, F_m$ be the facets of the clique complex of G. Then $m = 1$, or $m > 1$ and*

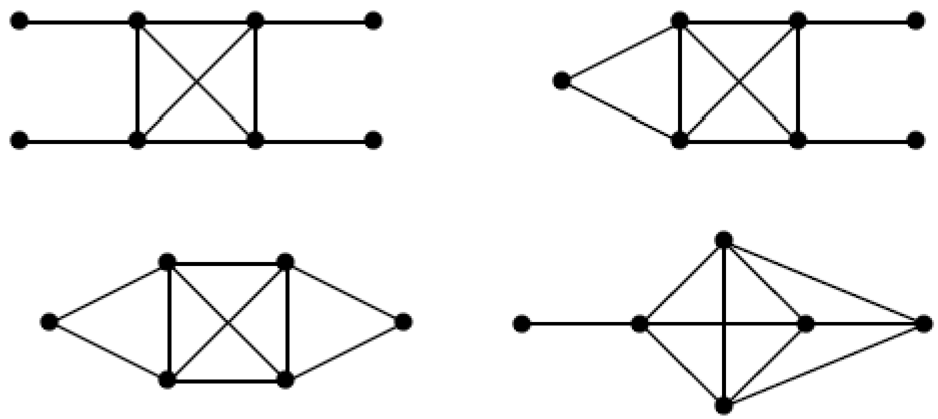
  (i) *$V(G) = V(F_1) \cup V(F_2) \cup \ldots \cup V(F_m)$, and this union is disjoint;*

  (ii) *each F_i has exactly one free vertex j_i;*

  (iii) *the restriction of G to $[n] \setminus \{j_1, \ldots, j_m\}$ is a clique.*

Let G be a chordal bi-CM graph as in Theorem 3.2(b) with $m > 1$. We call the complete graph G'' which is the restriction of G to $[n] \setminus \{j_1, \ldots, j_m\}$ the *center* of G.

The following picture shows, up to isomorphism, all bi-CM chordal graphs whose center is the complete graph K_4 on 4 vertices:



4. Generic Bi-CM graphs

EXAMPLE 4.1. Consider the bi-CM graph G on the vertex set $[5]$ and edges $\{1,2\}$ $\{2,3\}$, $\{3,1\}$, $\{2,4\}$, $\{3,4\}$, $\{4,5\}$ as displayed in Figure 3.

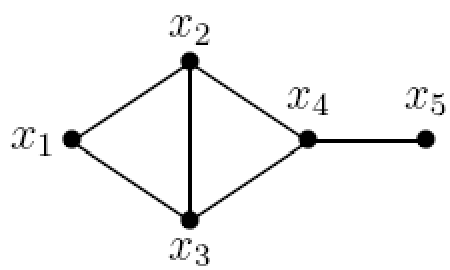


Figure 3

The ideal $J = I_G^\vee$ is generated by $u_1 = x_2x_3x_4$, $u_2 = x_1x_3x_4$, $u_3 = x_2x_3x_5$ and $u_4 = x_1x_2x_4$. Because J has a linear resolution, the generating relations of J may be chosen all of the form $x_ku_i - x_lu_j = 0$. This implies that in each row of the relation matrix there are exactly two non-zero entries (which are variables with different signs). We call such relations, *relations of binomial type.*

The relation matrices with respect to u_1, u_2, u_3 and u_4 are the matrices

$$A_1 = \begin{pmatrix} x_1 & -x_2 & 0 & 0 \\ x_5 & 0 & -x_4 & 0 \\ x_1 & 0 & 0 & -x_3 \end{pmatrix},$$

and

$$A_2 = \begin{pmatrix} x_1 & -x_2 & 0 & 0 \\ x_5 & 0 & -x_4 & 0 \\ 0 & x_2 & 0 & -x_3 \end{pmatrix}.$$

One assigns to the relation matrix A the following graph Γ: $\{i, j\}$ is said to be an edge of Γ if and only if some row of A has non-zero entries for the ith- and jth-component.

Note that Γ is a tree. This tree is in general not uniquely determined by G.

The relation tree of A_1 is

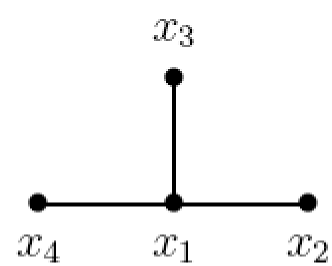


FIGURE 4

while the relation tree of A_2 is

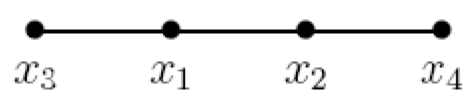


FIGURE 5

For any given tree T on the vertex set $[m] = \{1, \ldots, m\}$ with edges $e_1, \ldots, e_{m-1}$ the $(m-1) \times m$-matrix A_T whose entries a_{kl} are defined as follows: we assign to the kth edge $e_k = \{i, j\}$ of T with $i < j$ the kth row of A_T by setting

$$a_{kl} = \begin{cases} x_{ij}, & \text{if } l = i, \\ -x_{ji}, & \text{if } l = j, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

The matrix A_T is called the *generic matrix* attached to the tree T.

Let T_1 and T_2 be the relation trees of A_1 and A_2, respectively. Then the generic matrices corresponding to these trees are

$$B_1 = \begin{pmatrix} x_{12} & -x_{21} & 0 & 0 \\ x_{13} & 0 & -x_{31} & 0 \\ x_{14} & 0 & 0 & -x_{41} \end{pmatrix},$$

and

$$B_2 = \begin{pmatrix} x_{12} & -x_{21} & 0 & 0 \\ x_{13} & 0 & -x_{31} & 0 \\ 0 & x_{24} & 0 & -x_{42} \end{pmatrix}.$$

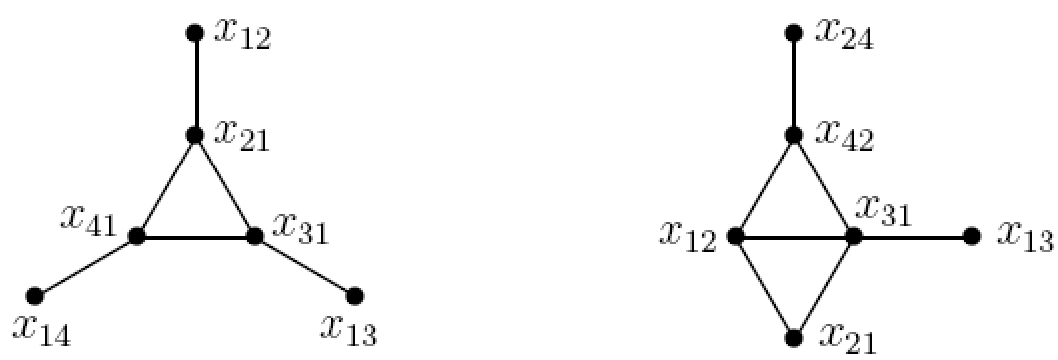
In order to describe the vertices and edges of G_T, let i and j be any two vertices of the tree T. There exists a unique path $P: i = i_0, i_1, \ldots, i_r = j$ from i to j. We set $b(i,j) = i_1$ and call $b(i,j)$ the *begin* of P, and set $e(i,j) = i_{r-1}$ and call $e(i,j)$ the *end* of P.

Thus the vertex set of the graph G_T is given as

$$V(G_T) = \{(i,j), (j,i) \colon \{i,j\} \text{ is an edge of } T\}.$$

In particular, $\{(i,k),(j,l)\}$ is an edge of G_T if and only if there exists a path P from i to j such that $k = b(i,j)$ and $l = e(i,j)$.

The generic graphs corresponding to the trees T_1 and T_2 are displayed in Figure 6.



G_{T_1} G_{T_2}

FIGURE 6

By using Hilbert-Burch theorem we have

PROPOSITION 4.3. *For any tree T, the graph G_T is bi-CM.*

5. Inseparable models of Bi-CM graphs

Our aim is to give a classification of all bi-CM graphs up to separation.

Recall the concept of inseparability introduced by Fløystad, Greve and Herzog.

Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring over the field K and $I \subset S$ a squarefree monomial ideal minimally generated by the monomials $u_1, \ldots, u_m$. Let y be an indeterminate over S. A monomial ideal $J \subset S[y]$ is called a *separation* of I for the variable x_i if the following holds:

(i) the ideal I is the image of J under the K-algebra homomorphism $S[y] \to S$ with $y \mapsto x_i$ and $x_j \mapsto x_j$ for all j;

(ii) x_i as well as y divide some minimal generator of J;

(iii) $y - x_i$ is a non-zero divisor of $S[y]/J$.

We now apply these concepts to edge ideals. A *separation* of the graph G with respect to the vertex i is a graph G' whose vertex set is $[n] \cup \{i'\}$ having the property that G is obtained from G' by identifying i with i' and such that $x_i - x_{i'}$ is a non-zerodivisor modulo $I_{G'}$. Algebraically, this identification amounts to say that $S/I_G \cong (S'/I_{G'})/(x_{i'} - x_i)(S'/I_{G'})$, where $S' = S[x_{i'}]$ and $x_{i'} - x_i$ is a non-zerodivisor of $S'/I_{G'}$. The algebraic condition on separation makes sure that the essential algebraic and homological invariants of I_G and $I_{G'}$ are the same. In particular, G is bi-CM if and only if G' is bi-CM. A graph which does not allow any separation is called *inseparable*, and a inseparable graph which is obtained by a finite number of separation steps from G is called a *separable model* of G.

EXAMPLE 5.1.

Let G be the triangle and G' be the line graph displayed in Figure 7.

FIGURE 7. A triangle and its inseparable model

Then $I_{G'} = (x_1x_2, x_1x_3, x_2x_4)$. Since $\mathrm{Ass}(I_{G'}) = \{(x_1, x_2), (x_1, x_4), (x_2, x_3)\}$, it follows that $x_3 - x_4$ is a non-zero divisor on $S'/I_{G'}$ where $S' = K[x_1, x_2, x_3, x_4]$. Moreover, $(S'/I_{G'})/(x_3 - x_4)(S'/I_{G'}) \cong S/I_G$. Therefore, the triangle in Figure 7 is obtained as a specialization from the line graph in Figure 7 by identifying the vertices x_3 and x_4.

We denote by $G^{(i)}$ the complementary graph of the restriction $G_{N(i)}$ of G to $N(i)$ where $N(i) = \{j\colon\ \{j, i\} \in E(G)\}$ is the neighborhood of i. In other words, $V(G^{(i)}) = N(i)$ and $E(G^{(i)}) = \{\{j, k\}\colon\ j, k \in N(i) \text{ and } \{j, k\} \notin E(G)\}$. Note that $G^{(i)}$ is disconnected if and only if $N(i) = A \cup B$, where $A, B \neq \emptyset$, $A \cap B = \emptyset$ and all vertices of A are adjacent to those of B.

Here we will need the following result of Altmann, Bigdeli, Herzog and Lu.

THEOREM 5.2 *The following conditions are equivalent:*

(a) *The graph G is inseparable;*
(b) *$G^{(i)}$ is connected for all i.*

Our main result is the following. In fact, we establish a bijection between the set of all trees and the set of inseparable bi-Cohen-Macaulay graphs.

THEOREM 5.3

(a) *Let T be a tree. Then G_T is an inseparable bi-CM graph.*

(b) *For any inseparable bi-CM graph G, there exists a unique tree T such that $G \cong G_T$.*

(c) *Let G be any bi-CM graph. Then there exists a tree T such that G_T is an inseparable model of G.*

REFERENCES

[1] K. Altmann, M. Bigdeli, J. Herzog, D. Lu, Algebraically rigid simplicial complexes and graphs, arXiv: 1503.08080[math.AC].

[2] W. Bruns and J. Herzog, "Cohen–Macaulay rings" (Revised edition), Cambridge Studies in Advanced Mathematics **39**, Cambridge University Press, 1998.

[3] W. Bruns and J. Herzog, On multigraded resolutions, Math. Proc. Camb. Phil. Soc. **118** (1995), 245–257.

[4] W. Bruns and U. Vetter, Determinantal rings, Springer Verlag, Graduate texts in Mathematics 150 (1995).

[5] J. A. Eagon, V. Reiner, Resolutions of Stanley-Reisner rings and Alexander duality, J. Pure Appl. Algebra **130** (1998), 265-275.

[6] D. Eisenbud, Commutative Algebra with a view to Algebraic geometry, Springer Verlag, 1995.

[7] G. Fløystad, B. M. Greve, J. Herzog, Letterplace and co-letterplace ideals of posets, arXiv:1501.04523[math.AC].

[8] G. Fløystad, J. E. Vatne, (Bi-)Cohen–Macaulay simplicial complexes and their associated coherent sheaves, Comm. Algebra **33** (2005), 3121–3136.

[9] J. Herzog and T. Hibi, Monomial Ideals. GTM 260. Springer 2010.

[10] J. Herzog and T. Hibi, Distributive lattices, Bipartite graphs and Alexander duality, J. Algebraic Combin., **22**(2005), 289–302.

[11] J. Herzog and T. Hibi, X. Zheng, Cohen–Macaulay chordal graphs, J. Combin. Theory Ser. A, **113**(2006), 911–916.

[12] H. Lohne, The many polarizations of powers of maximal ideals, arXiv:1303.5780 (2013).

[13] M. Naeem, Cohen–Macaulay monomial ideals of codimension 2, Manuscripta math, **127**(2008), 533–545.
[14] R. H. Villarreal, Monomial Algebras, Second Edition, Monographs and Research Notes in Mathematics, CRC Press, 2015.