Depth and regularity of powers of sum of ideals

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Depth and regularity

Let R be a polynomial ring over a field k.
Let M be a f.g. graded R-module.

Let $0 \to F_s \to \cdots F_0 \to M$ be a graded minimal free resolution.

$$\text{depth}\, M = \dim R - s,$$

$$\text{reg}\, M = \max\{d(F_i) - i \mid i = 0, ..., s\},$$

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In general, depth M and reg M can be defined in terms of the local cohomology modules of M.

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Brodmann: $\operatorname{depth} R/Q^n = \text{const}$ for $n \gg 0$.

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In general, it is a hard problem. There are partial results, e.g. by Herzog-Hibi, Herzog-Vladiou: depth for monomial ideals, Eisenbud-Harris, Eisenbud-Ulrich: regularity for zero-dimensional ideas.

Sum of ideals

Let $A = k[x_1, \ldots, x_r]$ and $B = k[y_1, \ldots, y_s]$.
Let $I \subset A$ and $J \subset B$ be nonzero proper ideals.
Let $R = k[x_1, \ldots, x_r, y_1, \ldots, y_s]$.
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Proposition:
$\operatorname{depth} A[x]/(I,x)^n = \min_{i \le n} \operatorname{depth} A/I^i$,
$\operatorname{reg} A[x]/(I,x)^n = \max_{i \le n}\{\operatorname{reg} A/I^i - i\} + n$.

Motivation

Geometry: Fiber product of two varieties $X \times_k Y$

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Combinatoric: Edge ideal of a graph (or hypergraph)

$$I(G) := (x_i x_j | \{i, j\} \in G).$$

If $G = G_1 \sqcup G_2$, then $I(G) = I(G_1) + I(G_2)$

Estimation by approximation

Set $Q_i := I^n + I^{n-1}J + \cdots + I^{n-i}J^i$, $i = 0, \ldots, n$. Then

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$$0 \to Q_i/Q_{i-1} \to R/Q_{i-1} \to R/Q_i \to 0.$$

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$$0 \to Q_i/Q_{i-1} \to R/I^{n-i+1}J^i \to R/I^{n-i}J^i \to 0.$$

Hoa-Tam:

depth $R/IJ =$ depth $A/I +$ depth $B/J + 1$,

reg $R/IJ =$ reg $A/I +$ reg $B/J + 1$.

First bounds

Theorem:

$$\operatorname{depth} R/(I+J)^n \geq \min_{i\in[1,n-1],\ j\in[1,n]} \{\operatorname{depth} A/I^{n-i} + \operatorname{depth} B/J^i + 1, \operatorname{depth} A/I^{n-j+1} + \operatorname{depth} B/J^j\},$$

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No hope for exact formulas.

Estimation by decomposition

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Goto-Watanabe: Formula for the local cohomology modules of $M \otimes_k N$, where M and N are f.g. graded R-modules.

From this it follows:

$$\operatorname{depth} M \otimes_k N = \operatorname{depth} M + \operatorname{depth} N,$$
$$\operatorname{reg} M \otimes_k N = \operatorname{reg} M + \operatorname{reg} N.$$

Second bounds

Theorem:

$$\operatorname{depth}(I+J)^n/(I+J)^{n+1} = \min_{i+j=n}\{\operatorname{depth} I^i/I^{i+1} + \operatorname{depth} J^j/J^{j+1}\},$$

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Corollary:

$$\operatorname{depth} R/(I+J)^n \ge \min_{i+j\le n-1}\{\operatorname{depth} I^i/I^{i+1} + \operatorname{depth} J^j/J^{j+1}\},$$
$$\operatorname{reg} R/(I+J)^n \le \max_{i+j\le n-1}\{\operatorname{reg} I^i/I^{i+1} + \operatorname{reg} J^j/J^{j+1}\}.$$

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These bounds are not related to the first given bounds.

Asymptotic depth

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depth $Q^{n-1}/Q^n = \text{const}$ for $n \gg 0$,

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$$\lim_{n\to\infty} \text{depth } R/(I+J)^n =$$

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$$\min_{i\geq 1} \text{depth } A/I^i + \lim_{j\to\infty} \text{depth } B/J^n \Big\}.$$

Asymptotic regularity

Lemma: Let $s(Q)$ denote the least integer m such that $\operatorname{reg} Q^n = dn + e$ for $n \geq m$. Then

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Theorem:

Assume that $\operatorname{reg} I^n = dn + e$ and $\operatorname{reg} J^n = cn + f$ for $n \gg 0$. Set

$$e^* := \max_{i \leq s(I)}\{\operatorname{reg} I^i - ci\},$$
$$f^* := \max_{j \leq s(J)}\{\operatorname{reg} J^j - dj\}.$$

For $n \gg 0$, we have

$$\operatorname{reg}(I+J)^n = \begin{cases} c(n+1) + f + e^* - 1 & \text{if } c > d, \\ d(n+1) + \max\{f + e^*, e + f^*\} - 1 & \text{if } c = d. \end{cases}$$

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One can give upper bound for $s(I+J)$ in terms of $s(I)$ and $s(J)$.

Cohen-Macaulayness of powers

Theorem: The following conditions are equivalent:

(i) $R/(I+J)^t$ is Cohen-Macaulay for all $t \leq n$,

(ii) $(I+J)^{n-1}/(I+J)^n$ is Cohen-Macaulay,

(iii) A/I^t and B/J^t are Cohen-Macaulay for all $t \leq n$,

(iv) I^t/I^{t+1} and $J^t/J^{t+1)}$ are Cohen-Macaulay for all $t \leq n-1$.

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This result does not hold for an arbitrary ideal Q in R.

Constant depth function

Herzog-Takayama-Terai: depth $R/Q^n \leq$ depth R/Q if Q is a squarefee monomial ideal.

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Not true if I, J are not squarefree monomial ideals

Non-increasing depth function

Herzog-Hibi: Given a convergent non-negative integer valued function f, does there exist a monomial ideal Q such that depth $R/Q^n = f(n)$ for all $n \geq 1$.

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Theorem: Yes for all non-increasing functions.

Construct Q such that depth $Q^n/Q^{n+1} = f(n)$ by taking sums of ideals having depth functions of the form 1,...,1,0,0,....

References

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