Groupoids & C^*-algebras

Recall notion of Category $\mathcal{C}$;

- $\mathcal{C}_0 :=$ Object$(\mathcal{C}) =$ Set of objects $\cdot$.
- $\mathcal{C}_1 :=$ Hom$(\mathcal{C}) =$ Set of homs. $\cdot \xrightarrow{f} \cdot$
- $d_0 : \mathcal{C}_1 \to \mathcal{C}_0$ domain/Sources $d_0(a \leftarrow b) = b$
- $c_0 : \mathcal{C}_1 \to \mathcal{C}_0$ codomain/target $c_0(a \leftarrow b) = a$
- $\mathcal{C}_2 :=$ Set of Composable morphism $:=$ pullback of
$$\begin{tikzcd} \mathcal{C}_2 \arrow[r] \arrow[d] & \mathcal{C}_1 \arrow[d, "c_0"] \\ \mathcal{C}_1 \arrow[r, "d_0"'] & \mathcal{C}_0 \end{tikzcd}$$

(Pull back:
$$\begin{tikzcd} & X \arrow[d, "f"] \\ Z \arrow[r, "g"'] & Y \end{tikzcd} \qquad Z \times_Y X := \{(z, x) : g(z) = f(x)\}$$
)

- $\square : \mathcal{C}_2 \to \mathcal{C}_1$

$\square(a \leftarrow b \leftarrow c) = a \leftarrow c$ such that:

1) $(f \square g) \square h = f \square (g \square h)$

2) $d_0 \circ u = id_{\mathcal{C}_1}$ & $u \circ id = id_{\mathcal{C}_2}$

in which $u : \mathcal{C}_0 \to \mathcal{C}_1$ is unit.

3) $f \square u(a) = f = u(b) \square f$ if

$d_0(f) = a$ and $c_0(f) = b$.

Notations: $Hom_{\mathcal{C}}(a,b) :=$ Pull back of the following diagram:

$$\begin{tikzcd} Hom_{\mathcal{C}}(a,b) \arrow[r, dashed] \arrow[d, dashed] & \mathcal{C}_1 \arrow[d, "d_0 \times c_0"] \\ \{(a,b)\} \arrow[r] & \mathcal{C}_0 \times \mathcal{C}_0 \end{tikzcd} \qquad Hom(\mathcal{C}) := \coprod_{a,b \in \mathcal{C}_0} Hom_{\mathcal{C}}(a,b)$$

Examples: 1) Category of Sets: $\mathcal{C}_0 = \{\text{Sets}\}$ $\mathcal{C}_1 = \{\text{map between sets}\}$

$\square =$ ordinary composition

2) Category of Hilbert space:

$\mathcal{C}_0 = \{\text{Hilbert spaces}\}$ $\mathcal{C}_1 = \{\text{unitary map}\}$

3, Category of Continuous map $C_0 : \begin{array}{c} E \\ \downarrow f \\ B \end{array}$ $\quad C_1$ $\begin{tikzcd} E \arrow[r, "G"] \arrow[d, "f"'] & E' \arrow[d, "f'"] \\ B \arrow[r, "G'"'] & B' \end{tikzcd}$.

4, Fix a top space B define $\begin{array}{c} B \\ \downdownarrows \\ B \end{array}$ $\quad obj\left(\begin{array}{c} B \\ \downdownarrows \\ B \end{array}\right) = B$, $Hom\left(\begin{array}{c} B \\ \downdownarrows \\ B \end{array}\right) = B$

$d_0 = c_0 = u = id$, $\square\left(\begin{array}{c} B \\ \downdownarrows \\ B \end{array}\right) = B \underset{B}{\times} B \cong B$

5, G a group, define $\begin{array}{c} G \\ \downdownarrows \\ * \end{array}$

$obj\left(\begin{array}{c} G \\ \downdownarrows \\ * \end{array}\right) = \{1\}$, $Hom\left(\begin{array}{c} G \\ \downdownarrows \\ * \end{array}\right) = G$

$d_0 = c_0 =$ constant $\quad \square :=$ group multiplication.

6, $X =$ space, $G =$ group $\quad \overset{\text{action}}{\cdot : X \times G \to X}$, define $\begin{array}{c} X \times G \\ \downdownarrows \\ X \end{array}$

$obj\left(\begin{array}{c} X \times G \\ \downdownarrows \\ X \end{array}\right) = X$, $Hom\left(\begin{array}{c} X \times G \\ \downdownarrows \\ X \end{array}\right) = X \times G$

$$x \xleftarrow{(x,g)} x\cdot g \xleftarrow{(x\cdot g,h)} x\cdot g\cdot h$$

7: Fix a top space B:

$obj\left(Cont \begin{array}{c} E \\ \downdownarrows \\ B \end{array}, \alpha \in H^3(E, \mathbb{Z})\right)$ $\quad mor\left(p\begin{array}{c} E \\ \downarrow \\ B \end{array}, \alpha\right) \longrightarrow \left(\begin{array}{c} E' \\ \downarrow p' \\ B \end{array}, \alpha'\right)$

such that $\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & B & \end{tikzcd}$, $\alpha = f^* \alpha' + p^* x$

Def: A groupoid is a category where all morphisms are isomorphism

Def: A topological groupoid is a groupoid such That all structure maps are continuous between $l.cpt + T_2 +$ 2nd Count. top spaces C_1, C_0.

Example 4, 5, 6 are top. groupoids.

7,

Def. Let $G_1 \rightrightarrows G_0^*$ be a top groupoid. A <u>Haar system</u> on $G_1 \rightrightarrows G_0$ is a family of (Borel) measures $\{\lambda^a\}_{a \in G_0}$ on G_1 such that

a) $\operatorname{supp} \lambda^a \subset d_0^{-1}(\{a\}) \subset G_1$

b) $(\gamma^*)_! \lambda^a = \lambda^b$ where $\gamma : b \to a$ (In term of integrals)

$\gamma^* : d_0^{-1}(a) \to d_0^{-1}(b)$

$\varepsilon \mapsto \varepsilon \circ \gamma$

$$\int_{d_0^{-1}(\{b\})} F(\varepsilon\gamma)\, d\lambda^b(\varepsilon) \overset{\text{Def}}{=} \int_{d_0^{-1}(\{a\})} F(\varepsilon)\, d\gamma^*_! \lambda^b(\varepsilon) \overset{(b)}{=} \int_{d_0^{-1}(\{a\})} F(\varepsilon)\, d\lambda^a(\varepsilon)$$

c) if $f \in C_c(G_1)$, then $G_0 \ni a \mapsto \int_{G_1} f\, d\lambda^a \in \mathbb{C}$ is continuous

" (X, μ) = Borel measurable space, (a) $\operatorname{supp} \mu = X \setminus \bigcup_{\substack{U \subseteq X \\ \text{open} \\ \mu(U)=0}} U$, (b) $\varphi : X \to Y$ measurable map $\varphi_! \mu$ = Push forward of μ along φ by $\varphi_! \mu(A) := \mu(\varphi^{-1}(A))$. 4

Construct Haar systems in examples:

4, $\begin{matrix} B \\ \downarrow\downarrow \\ B \end{matrix}$ $\quad d_0^{-1}(\{a\}) = \{a\}$ $\quad \lambda^a(\{A\}) = \begin{cases} 1 & a \in A \\ 0 & a \notin A \end{cases}$ $\quad A \subset G_1$

5, Special case of example 6.

6, $\begin{matrix} X \times G \\ \downarrow\downarrow \\ X \end{matrix}$ $\quad x \cdot g^{-1} \xleftarrow{(xg^{-1}, g)} x \xrightarrow{(x, h)} x \cdot h$

- $d_0^{-1}(x) = \{(x \cdot g^{-1}, g) : g \in G\} \overset{\varphi_x}{\underset{\cong}{\longleftarrow}} G$
- Let λ be a Haar meas. of G then $(h^*)_! \lambda = \lambda$ in which $h^* : G \to G$, $g \mapsto g \cdot h$

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