Random Walks on groups, Poisson boundary, entropy.

- Random Walks on groups
- Poisson Boundaries
- Tail Boundaries
- Comparison of Poisson bdd and Tail bdd
- Triviality of " "
- Conditional random walk and entropy.

Ref:

1, Random Walks on discrete groups, Kaimanovich – Vershik, 1983

2, The Poisson formula for groups with hyperbolic properties, Kaimanovich, 2000

3, The " boundary of discrete group. Kaimanovich

Random Walks on group:

Let G be a countable group and μ = Prob. measure.

$\forall g \in G$ $g \longmapsto \pi_g$ is a probability measure on G. $\pi_g(h) := \mu(g^{-1}h)$

$\{\pi_g\}$ are transition probabilities $g \xrightarrow{\mu(g^{-1}h)} h$

Fix θ a prob. measure on G (θ is initial distribution),

$$(G, \theta) \longleftarrow (G \times G, P_\theta^{[1]})$$

$$P_\theta^{[1]}(g_0, g_1) = \theta(g_0)\pi_{g_0}(g_1) = \theta(g_0)\mu(g_0^{-1}g_1)$$

$(G\times G\times G,\ \mathbb{P}_\theta^{[2]})$ $\quad \mathbb{P}_\theta^{[2]}(g_0,g_1,g_2) = \theta(g_0)\,\pi_{g_0}(g_1)\,\pi_{g_1}(g_2)$

$(G,\theta) \leftarrow (G^2, \mathbb{P}_\theta^{[1]}) \leftarrow (G^3, \mathbb{P}_\theta^{[2]}) \leftarrow \cdots$

by kolomogrov consistency theorem $\exists!\,(G^{\mathbb{Z}_+}, \mathbb{P}_\theta)$ is called space of sample paths.

$\{ C_{y_0,y_1,\dots,y_m} = \{x\in G^{\mathbb{Z}_+} : x_0=y_0,\dots,x_m=y_m\}\}$ ← Basis for measure space

Another way to define $(G^{\mathbb{Z}_+}, \mathbb{P}_\theta)$

$(G^{\mathbb{Z}_+},\ \theta\otimes\mu^\infty) \xrightarrow{\varphi} G^{\mathbb{Z}_+}$

$(g_0,g_1,g_2,\dots)\mapsto(g_0,\ g_0g_1,\ g_0g_1g_2,\dots)$

$A\subseteq G^{\mathbb{Z}_+}$ $\quad \varphi^{-1}(A)$ is measurable in $(G^{\mathbb{Z}_+},\ \theta\otimes\mu^\infty)$, $\quad \mathbb{P}_\theta(A) := \theta\otimes\mu^\infty(\varphi^{-1}(A))$

X_n = Position of random walks at time n : $g_0g_1\cdots g_n$

$X_0=g_0\sim\theta$

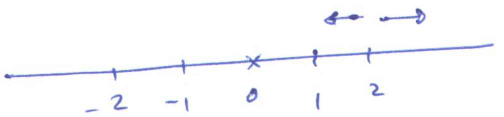
$X_1=g_0g_1\sim\theta*\mu$

$\vdots$

$X_n=g_0\cdots g_n\sim\theta*\mu^{*n}$

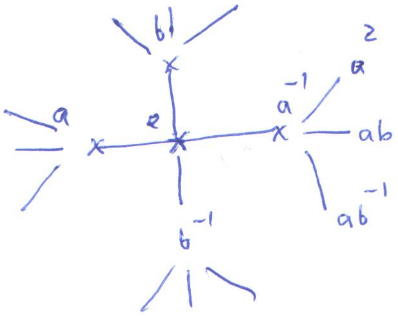
Example : $G=\mathbb{Z}_2$, $\mu=\delta_1$

Example: $G=\mathbb{Z}$ $\quad \mu=\frac{1}{2}(\delta_1+\delta_{-1})$, $\theta=\delta_0$



$\mu(0+1)=\mu(1)=1/2$

Example : $G=\mathbb{F}_2=\langle a,b\rangle$, $\quad \mu=\frac{1}{4}(\delta_a+\delta_{a^{-1}}+\delta_b+\delta_{b^{-1}})$, $\theta=\delta_e$



Poisson Boundary: we have not top but we are looking for sth similar to limit!

$\bar{X} = (x_0, x_1, x_2, \dots)$, $T : G^{\mathbb{Z}_+} \longrightarrow G^{\mathbb{Z}_+}$

$T(\bar{x}) = (x_{n+1}) = (x_1, x_2, \dots)$

$\bar{X} \sim \bar{Y}$ $\exists n, m \geq 0$ s.t. $T^n x = T^m y$. A_T = σ-alg T-inv ($T^{-1}A = A$ for $A \in A_T$)

" Lebesgue Space: $(X, A, m) \cong (a_0, a_1) \cup \{a_n : \substack{n \in \mathbb{N} \\ a_n \geq 0}\} \subseteq (0,1)$ "

" Rokhlin Thm: (X, A, m) = leb. space then there is a 1-1-corresponding between "

1, morphism 2, complete sub-σ-alg of A

3, measurable partition, ξ of X.

By Rokhlin's Thm $\exists$ bnd : $(G^{\mathbb{Z}_+}, \mathbb{P}_\theta) \longrightarrow \Gamma$

$\nu_\theta = \text{bnd} \circ \mathbb{P}_\theta$ ($\forall A \subseteq \Gamma$ $\mathbb{P}_\theta(\text{bnd}^{-1} A) =: \nu_\theta(A)$)

$\theta \sim$ counting measure on G

$[\nu_\theta]$ is called harmonic measure type.

$g \in G$ $g.(x_1, x_2, \dots) = (gx_1, gx_2, \dots)$

so $g. T(\bar{x}) = T(g.\bar{x})$

$\nu_\theta = \text{bnd}(\mathbb{P}_\theta) = \text{bnd}(T\mathbb{P}_\theta)$ " $T\mathbb{P}_\theta = \mathbb{P}_{\theta * \mu}$ $\begin{matrix} (x_0, & x_1 & \dots) \\ \downarrow & \downarrow & \\ \theta & \theta*\mu & \end{matrix} \rightarrow \begin{matrix} (x_1, & x_2 & \dots) \\ \downarrow & & \\ \theta*\mu & \theta*\mu^2 & \dots \end{matrix}$ "

$\nu_\theta = \text{bnd}(T\mathbb{P}_\theta) = \text{bnd}(\mathbb{P}_{\theta*\mu}) = \nu_{\theta * \mu}$ (*)

$\nu_\theta = \text{bnd}(\mathbb{P}_\theta) = \text{bnd}(\theta * \mathbb{P}) = \theta * \text{bnd}(\mathbb{P})$ ($\mathbb{P} := \mathbb{P}_{\delta_e}$)