Random Walks on groups, Poisson boundary, entropy.

- Random Walks on groups
- Poisson Boundaries
- Tail Boundaries
- Comparison of Poisson bdd and Tail bdd
- Triviality of " "
- Conditional random walk and entropy.

Ref:

1, Random Walks on discrete groups, Kaimanovich – Vershik, 1983

2, The Poisson formula for groups with hyperbolic properties, Kaimanovich, 2000

3, The " boundary of discrete group. Kaimanovich

Random Walks on group:

Let G be a countable group and μ = Prob. measure.

$\forall g \in G \quad g \longmapsto \pi_g$ is a probability measure on G. $\quad \pi_g(h) := \mu(g^{-1}h)$

$\{\pi_g\}$ are transition probabilities $\quad g \xrightarrow{\mu(g^{-1}h)} h$

Fix θ a prob. measure on G (θ is initial distribution),

$$(G, \theta) \longleftarrow (G \times G, \mathbb{P}_\theta^{[1]})$$

$$\mathbb{P}_\theta^{[1]}(g_0, g_1) = \theta(g_0)\pi_{g_0}(g_1) = \theta(g_0)\mu(g_0^{-1}g_1)$$

$(G\times G\times G, \mathbb{P}_\theta^{[2]})$ $\quad \mathbb{P}_\theta^{[2]}(g_0,g_1,g_2) = \theta(g_0)\,\pi_{g_0}(g_1)\,\pi_{g_1}(g_2)$

$$(G,\theta) \leftarrow (G^2, \mathbb{P}_\theta^{[1]}) \leftarrow (G^3, \mathbb{P}_\theta^{[2]}) \leftarrow \cdots$$

by kolomogrov consistency theorem $\exists! \, (G^{\mathbb{Z}_+}, \mathbb{P}_\theta)$ is called space of sample paths.

$\left\{ C_{y_0,y_1,\dots,y_m} = \{x \in G^{\mathbb{Z}_+} : x_0 = y_0, \dots, x_m = y_m\} \right.$ ← Basis for measure space

Another way to define $(G^{\mathbb{Z}_+}, \mathbb{P}_\theta)$

$$(G^{\mathbb{Z}_+}, \theta\otimes\mu^\infty) \xrightarrow{\varphi} G^{\mathbb{Z}_+}$$

$$(g_0,g_1,g_2,\dots) \mapsto (g_0, g_0g_1, g_0g_1g_2, \dots)$$

$A \subseteq G^{\mathbb{Z}_+}$ $\quad \varphi^{-1}(A)$ is measurable in $(G^{\mathbb{Z}_+}, \theta\otimes\mu^\infty)$, $\quad \mathbb{P}_\theta(A) := \theta\otimes\mu^\infty(\varphi^{-1}(A))$

X_n = Position of random walks at time n : $g_0g_1\cdots g_n$

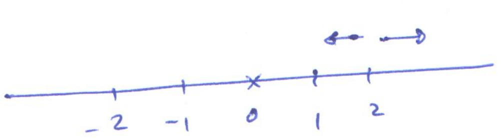
$X_0 = g_0 \sim \theta$ $\qquad$ Example: $G = \mathbb{Z}_2$, $\mu = \delta_1$

$X_1 = g_0g_1 \sim \theta * \mu$

$\vdots$

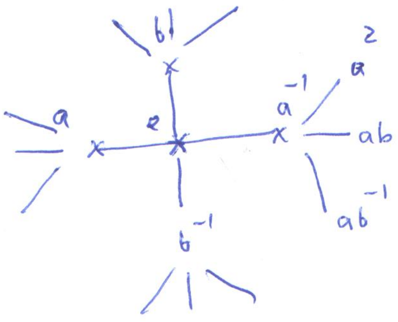
$X_n = g_0\cdots g_n \sim \theta * \mu^{*n}$

Example: $G = \mathbb{Z}$ $\quad \mu = \frac{1}{2}(\delta_1 + \delta_{-1})$, $\theta = \delta_0$



$\mu(0+1) = \mu(1) = 1/2$

Example: $G = \mathbb{F}_2 = \langle a, b\rangle$, $\mu = \frac{1}{4}(\delta_a + \delta_{a^{-1}} + \delta_b + \delta_{b^{-1}})$, $\theta = \delta_e$



Poisson Boundary: we have not top but we are looking for sth similar to limit!

$\bar{X} = (x_0, x_1, x_2, \dots)$, $T: G^{\mathbb{Z}_+} \longrightarrow G^{\mathbb{Z}_+}$

$T(\bar{x}) = (x_{n+1}) = (x_1, x_2, \dots)$

$\bar{X} \sim \bar{Y}$ $\exists n, m \geq 0$ s.t. $T^n x = T^m y$. $A_T = \sigma$-alg T-inv ($T^{-1}A = A$ for $A \in A_T$)

" Lebesgue Space: $(X, A, m) \overset{\sim}{=} (a_0, a_1) \cup \{a_n : \substack{n \in \mathbb{N} \\ a_n \geq 0}\} \subseteq (0,1)$ "

" Rokhlin Thm: (X, A, m) = leb. space then there is a 1-1-corresponding between

1, morphism 2, complete sub-σ-alg of A

3, measurable partition, ξ of X. "

By Rokhlin's Thm $\exists$ bnd : $(G^{\mathbb{Z}_+}, P_\theta) \longrightarrow \Gamma$

$\gamma_\theta = \text{bnd} \circ P_\theta$ ($\forall A \subseteq \Gamma$ $P_\theta(\text{bnd}^{-1} A) =: \gamma_\theta(A)$)

$\theta \sim$ Counting measure on G

$[\gamma_\theta]$ is called harmonic measure type.

$g \in G$ $g.(x_1, x_2, \dots) = (gx_1, gx_2, \dots)$

So $g.T(\bar{x}) = T(g.\bar{x})$

$\gamma_\theta = \text{bnd}(P_\theta) = \text{bnd}(TP_\theta)$ " $TP_\theta = P_{\theta * \mu}$ $\begin{array}{cc} (x_0, x_1 \dots) & \longrightarrow (x_1, x_2 \dots) \\ \downarrow \ \downarrow & \downarrow \\ \theta \ \ \theta * \mu & \theta * \mu \ \ \theta * \mu^2 \ \dots \end{array}$ "

$\gamma_\theta = \text{bnd}(TP_\theta) = \text{bnd}(P_{\theta*\mu}) = \gamma_{\theta*\mu}$ (*)

$\gamma_\theta = \text{bnd}(P_\theta) = \text{bnd}(\theta * P) = \theta * \text{bnd}(P)$ $(P := P_{\delta_e})$

$\mathbb{P} := P_{\delta_e}$, $\nu := \nu_{\delta_e} = \nu_e$ $\quad \nu \overset{?}{=} \mu * \nu$

$$\nu_{\delta_e} = \nu_{\delta_e * \mu} = \nu_\mu \to \nu = \nu_\mu = \mu * \mathrm{bnd}(\mathbb{P}) = \mu * \nu$$

$(\Gamma, \nu) =$ Poisson Boundary.

Example: $\mathbb{Z}_2 = \{0, 1\}$ $\quad \mu = \delta_1$ $\quad (0, 1, 0, 1, \ldots)$

$(1, 0, 1, 0, \ldots)$

$$\Gamma(\mathbb{Z}_2, \delta_1) = \{\text{Single point}\}$$

Harmonic functions:

Let f be a bdd function on G. $P : L^\infty(G) \to L^\infty(G)$

$$Pf(g) = \langle f, \pi_g \rangle = \sum_h f(h)\mu(g^{-1}h)$$

The function f is called μ-harmonic if $Pf = f$.

$$H^\infty(G, \mu) = \{ f : \text{bdd} \ \& \ P_f = f \}$$

Poisson Formula: $\quad H^\infty(G, \mu) \cong L^\infty(\Gamma, [\nu])$ $\quad$ counting measure

$f \in H^\infty(G, \mu)$ $\quad \lim f(x_n)$ exists P_θ-a.e. and it's denoted by

$\hat{f}(\mathrm{bnd}(\bar{x}))$. $\quad f(g) = \langle \hat{f}, g\nu \rangle$. " $g\nu(A) := \nu(g^{-1}A)$ "

Definition: $(X, \mathcal{A}, m) =$ Leb. space & $\mathcal{J} \subseteq \mathcal{A}$ and $f = \mathcal{A}$-meas.
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \sigma$-sub-alg

Then $\exists\ E(f \mid \mathcal{J})$ is a $\mathcal{J}$-meas. function and $\int_A E(f \mid \mathcal{J})\, dm = \int_A f\, dm$

and it's called Conditional Expectation w.r.t. $\mathcal{J}$. $\quad A \in \mathcal{J}$

Def. the sequence $\{f_n, \mathcal{J}_n\}$ is called martingeles if

- $\mathcal{J}_n \subseteq \mathcal{J}_{n+1}$
- $f_n \in L^1(X, m)$
- $|f_n| \leq C \quad , n \geq 1$

Martingales' Thm: $\exists f \in L^1(X,m)$ s.t. $f_n \xrightarrow[L^1]{a.e} f$ and $E(f|\mathcal{J}_n) = f$.

Proof of Poisson formula:

$A_0^n \quad \bar{X} \overset{n}{\underset{0}{\sim}} \bar{Y} \iff x_i = y_i \quad i=0, \dots, n$

$A_0^n \subseteq A_0^{n+1}$

Suppose that $f \in H^\infty(G, \mu)$. Define $F_n(\bar{X}) = f(x_n)$. Then $E(F_{n+1}|A_0^n) = F_n$.

~~...~~ By Martingales' Thm we have $F_n(\bar{X}) \to F_\infty \in L^\infty(\Gamma, [\gamma])$.

$\hat{f} \in L^\infty(\Gamma, [\gamma]) \quad f(g) = \int \hat{f}(\gamma)\, dg\gamma(\gamma) \Rightarrow Pf(g) = f(g)$.

$\Delta f = 0 \quad$ " $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ " $\Rightarrow f(x) = \int_{B_r(x)} f(y)\,dy \Big/ m(B(x,r))$

Ex: $G = \mathbb{Z} \quad \mu = \frac{1}{2}(\delta_0 + \delta_1)$

$f \in H^\infty(G, \mu) \to Pf = f \Rightarrow f(n) = \frac{1}{2} f(n) + \frac{1}{2} f(n+1) \to \underline{\underline{f(n) = f(n+1)}}$

Tail Boundary:

let $f(\cdot)$ be a sequence such that $Pf_{n+1} = f_n$

$$HS^{\infty} = \{(f_n) : Pf_{n+1} = f_n \,,\ |f_n| < c\}_{n \in \mathbb{N}}$$

$$\bar{x} \approx \bar{y} \iff \exists n \ \ T^n x = T^n y$$

$$\bar{x} \overset{\alpha_n}{\sim} \bar{y} \iff x_m = y_m \quad m \geq n .$$

infact

$$\begin{tikzcd} \text{tail} : (G, \mathbb{Z}^+) \arrow[r] \arrow[dr, dashed, "\text{bnd}"'] & (E, \mathcal{E}_\theta) \arrow[d] \\ & (\Gamma, [\gamma]) \end{tikzcd}$$

Poisson bdd is a quotient of tail boundary.

Ex, $G = \mathbb{Z}_2 \,,\ \mu = \delta_1$ $\Gamma = \{\text{singleton}\}$ but $E = \{0,1\}$

$(0,1,0,1,0,1,\dots)$

$(1,0,1,0,1,0,\dots)$

Ex, $G = \mathbb{Z} \,,\ \mu = \frac{1}{2}(\delta_0 + \delta_1) \longrightarrow \Gamma = \text{singleton}$

By induction we have $\mu^{*n} = \frac{1}{2^n} \sum_0^n \binom{n}{k} \delta_k .$

$$\| g\mu^{*n} - \mu^{*n} \| = \frac{1}{2^n} \sum_0^{n+1} \left| \binom{n}{k} - \binom{n}{k-1} \right| \longrightarrow 0 \qquad \text{"} g\mu^{*n}(q) = \mu^{*n}(g^{-1}q) \text{"}$$

$$\mu^{*(n+1)} = \tfrac{1}{2}\, g\mu^{*n} + \mu^{*n} \implies \| \mu^{*(n+1)} - \mu^{*n} \| \longrightarrow 0$$

$f_n = P f_{n+1}$ $\qquad f_0,\ P f_1,\ P f_2,\ \dots$

$$f_0(g) = P^n f_n(g)$$

$$f_k(g) = P^{n-k} f_n(g)$$

$$|f_0(g) - f_k(g)| \leq \sum_n f_n(g)\, \| \mu^n(g h^{-1}) - \mu^{n-k}(g h^{-1}) \| \leq C$$

$$\| \mu^n - \mu^{n-k} \| \longrightarrow 0 \qquad \therefore\ f_0 \equiv f_k$$

So $\quad \Gamma = \{\cdot\} = (E, \mathcal{E}_\sigma)$.

Behrang (III & IV)

Recall: G: Countable Groups

μ: Prob. measure

bnd: $(G^{\mathbb{Z}_+}, P_\theta) \longrightarrow (\Gamma, \nu_\theta)$

tail: $(G^{\mathbb{Z}_+}, P_\theta) \longrightarrow (E, \varepsilon_\theta)$

$H^\infty(G,\mu) = \{ f \text{ bdd}: Pf = f \}$

$HS^\infty(G,\mu) = \{ (f_n): \begin{array}{l} |f_n| \le c \\ Pf_{n+1} = f_n \end{array} \}$

Thm (0–2 law):

Let G be a countable group and μ be a prob. measure on G:

$$\lim \|\mu^{n+1} - \mu^n\| = \begin{cases} 0 \\ 2 \end{cases}$$

Sketch of pf /

$\|\mu^{n+1} - \mu^n\| \le \|\mu\| \, \|\mu^n - \mu^{n-1}\|$.

if for any $n \in \mathbb{N}$, $\|\mu^n - \mu^{n+1}\| = 2$ there is not anything to prove.

So let $\|\mu^{n_0} - \mu^{n_0+1}\| \ne 2$ for $n_0 \in \mathbb{N}$

$\|\mu^{n_0} - \mu^{n_0+1}\| \ne 2 \iff \exists n_0$ s.t. $\mu^{n_0+1} \not\perp \mu^{n_0}$

$\exists \lambda \le \mu^{n_0}$ s.t. $\lambda * \mu \le \mu^{n_0}$

1) $\mu^{n_0} = \lambda + \tau_1 = \lambda + \mu + \tau_2 = \lambda * \frac{1}{2}(\delta_e + \mu) + \theta_1$

$(\theta_1 = \frac{1}{2}(\tau_1 + \tau_2))$

$\mu^{n_0} * \mu^{n_0} = \mu^{2n_0} = \lambda * \mu^{n_0} * \frac{1}{2}(\delta_e + \mu) + \theta_1 * \mu^{n_0}$

by induction

$$\mu^{k_0 n} = \lambda^{*k} * \left[\tfrac{1}{2}(\delta_e + \mu)\right]^k + \theta_k$$

for some θ_k, $\|\theta_k\| = 1 - |\lambda(G)|^k < 1$

$$\mu^{tkn_0} = \alpha_t * (\tfrac{1}{2}(\delta_e + \mu))^{*k} + \theta_k^{*t}$$

Some meas.

$$\|\mu^{tkn} * (\delta_e - \mu)\| \le \|\alpha_t\| \, \tfrac{1}{2^k} \left\| \sum \binom{k}{i} (\mu^i - \mu^{i+1}) + \alpha_0 \right\|$$

$$\sim \frac{c}{\sqrt{k}} + (1 - \lambda(G)^k)^t \longrightarrow 0.$$

$\boxtimes$

Lemma: $\frac{1}{2^{2n}} \sum_1^{2n} \left| \binom{2n}{k} - \binom{2n}{k-1} \right| \sim \frac{c}{\sqrt{n}}$

Thm: The Tail & Poisson boundary for R.W. (G,μ) coincide w.r.t. initial distribution concentrated at a point $(\theta = \delta_g)$

i) $\|\mu^n - \mu^{n+1}\| \to 0 \Rightarrow (f_n) \in H^\infty(G,\mu) \Rightarrow f_0 = f_i \quad i \ge 0$.

$f_0(g) = P^n f_n(g) = \sum f_n(gh) \mu^n(h)$

$f_k(g) = P^{n-k} f_n(g) = \sum f_n(gh) \mu^{*(n-k)}(h)$

$|f_0(g) - f_k(g)| \le C \|\mu^n - \mu^{n-k}\|$

ii) $\|\mu^{n+d} - \mu^n\| \to 0$ for some d

$\|\mu^{n+k} - \mu^n\| = 2 \quad d \nmid k$.

$E = \Gamma \times \{1, 2, \dots, d\}$

iii) $\mu^n \perp \mu^{n+k} \quad \forall k, n$

$\operatorname{supp} \mu^n \cap \operatorname{supp} \mu^{n+k} = \phi \rightarrow (T^n \bar{x} = T^m \bar{y} \Leftrightarrow n = m)$

Thm: (Kaimonovich)

Poisson bdy = Trivial $\Leftrightarrow \|g\mu^n - \mu^{n+k}\| \to 0$ $\quad g \in \operatorname{supp}\mu^k$

Sketch of Pf:

$$C_g^k = \{\bar{x} \in G^{\mathbb{Z}_+} : x_k = g\}$$

1) $\mathbb{P}[C_g^k \mid \alpha_n^n] \xrightarrow{n\to\infty} \mathbb{P}[C_g^k \mid \alpha^\infty]$ (By Martingales)

$$\mathbb{P}[C_g^k \mid \alpha_n^n](\bar{g}) = \mu^k(g)\,\mu^{n-k}(g^{-1}g_n) \Big/ \mu^n(g_n)$$

2) α^∞ is trivial.

$$\mu^{k}(g)\,\mu^{n-k}(g^{-1}g_n) \Big/ \mu^n(g_n) \longrightarrow \mu^k(g)$$

So $\mu^{n-k}(g^{-1}g_n) \big/ \mu^n(g_n) \longrightarrow 1$.

G is amenable if

$\exists \{\lambda_n\} \subseteq P(G)$

$\|g\lambda_n - \lambda_n\| \to 0 \quad \forall g \in G$

$g\lambda_n \sim \lambda_n$

$\exists \mu \in P(G)$, $\langle \operatorname{supp}(\mu) \rangle = G$

$\|g\mu^n - \mu^n\| \to 0 \Leftrightarrow P = \{\cdot\}$.

Thm: If G = abelian, then the Poisson boundary is trivial.

(*) Lemma: $H \trianglelefteq G$, $G \overset{\mu}{\curvearrowright}$, $G \to G/H \overset{\tilde{\mu}}{\curvearrowleft}$

$$\tilde{\mu}(gH) = \sum_{g \sim h} \mu(g)$$

$$H^\infty(G,\mu) \cong H^\infty(G/H, \tilde{\mu})$$

Entropy:

$$H(\mu) = -\sum_{g \in G} \mu(g) \log \mu(g) < \infty$$

$$H(\mu^{n+m}) \leq H(\mu^n) + H(\mu^m)$$

$$h(G,\mu) = \lim_n H(\mu^{*n}) / n$$

The transformation:

$T : (X, A, m) \circlearrowleft$

is called ergodic if

$\mu(A) = \mu(T^{-1}A) \Leftrightarrow A = \emptyset$ or X

Kingman's subadditive theorem (1968)

i) $(f_n) \subseteq L^1$

ii) $f_{n+m}(x) \leq f_n(x) + f_m(T^n x)$

$\Rightarrow f_n / n \xrightarrow[L^1]{a.e} \bar{f}$ & $\bar{f}$ is T-inv.

in particular if f is ergodic then $\bar{f}$ = const.

$$u : G^{\mathbb{Z}_+} \longrightarrow G^{\mathbb{Z}_+}$$

$$u(e, x_1, \dots) = (e, x_1^{-1}x_2, x_1^{-1}x_3, \dots)$$

$$F_n(\bar{x}) = -\log \mu^n(x_n)$$

By definition of Convolution

we have

$$\mu^{n+1}(x_n G) \geq \mu^n(x_n) \mu(G)$$

$$F_{n+1}(x) \leq F_n(x) + F_1(ux)$$

Shannon's Formula:

$$-\log \mu^n(x_n) / n \longrightarrow h(G, \mu) \quad \mathbb{P}\text{-a.e. and in } L^1$$

$$H(\alpha_1^n | \mathcal{Y}) = nH(\mu) - h(G, \mu)$$

Thm: $\Gamma = \{.\} \Leftrightarrow h(G, \mu) = 0$.