Marzieh

$\theta : A \xrightarrow[\text{map}]{\text{linear}} B$ is called nuclear if

$$\begin{tikzcd} A \arrow[rr, "\theta"] \arrow[dr, dashed, "\exists \varphi_\alpha"', "\text{c.c.p.}"] & & B \\ & M_{n_\alpha}(\mathbb{C}) \arrow[ur, dashed, "\exists \psi_\alpha \text{ c.c.p}"'] & \end{tikzcd}$$

s.t. $\| \psi_\alpha \circ \varphi_\alpha(a) - \theta(a) \| \to 0$ for $a \in A$

and C^*-alg A is called nuclear

if $id : A \to A$ is nuclear.

Example: $C(X)$ for $X = T_2 + cpt$.

Remarks:

i) $\theta : A \xrightarrow{\text{nuclear}} B$ iff for any $\varepsilon > 0$ & $F \underset{\text{finite}}{\subseteq} A$

$\exists \varphi$ and ψ c.c.p map s.t.

$\| \psi \circ \varphi(a) - a \| < \varepsilon$ for $a \in F$.

ii) $\theta : A \xrightarrow{\text{nuclear}} B$ iff $\exists \varphi_\alpha : A \xrightarrow{\text{c.c.p}} C_{n_\alpha}$, $\psi : C_{n_\alpha} \xrightarrow{\text{c.c.p}} B$

in which C_{n_α} is f.d C^*-alg s.t.

$\| \psi_\alpha \circ \varphi_\alpha(a) - \theta(a) \| \to 0$ for $a \in A$.

Pf of example: Using Partition of unity and 1^{st} part of remark

$F \underset{\text{finite}}{\subseteq} A$, $\varepsilon > 0$ $\exists \{U_1 \dots U_n\}$ = open cover of X s.t. $\| f(x) - f(y) \| < \varepsilon$

for all $x, y \in U_i$ & $f \in F$

Consider $\{\sigma_i\}$ as a Partition of unity of $\{U_i\}$.

$\varphi : A \to \mathbb{C}^n$

$\varphi(f) := (f(y_1), \dots, f(y_n))$

$\psi : \mathbb{C}^n \to A$

$(d_1, \dots, d_n) \mapsto \sum_1^n d_i \sigma_i$

Then

$\| \psi \circ \varphi(f) - f \| < \varepsilon$ for $f \in F$.

Cor. Every abelian vN-alg is semi-discrete.

Aim:

Type I vN-alg are nuclear.

Remark:

Let $\theta : A \xrightarrow{\text{linear}} M$. Then θ is called weakly nuclear if

$$\begin{tikzcd} A \arrow[rr, "\theta"] \arrow[dr, "\varphi_\alpha"'] & & M \\ & M_{n_\alpha} \arrow[ur, "\psi_\alpha"'] & \end{tikzcd} \quad \text{s.t.}$$

$\langle \xi, \psi_\alpha \circ \varphi_\alpha(a) \rangle \longrightarrow \langle \xi, \theta(a) \rangle \quad \xi \in M_*$

- A is nuclear iff A^{**} is semidiscrete.
- A is type I C^*-alg iff A^{**} is type I vN-alg.

" Type I vN-alg $:= \prod^{s} A_i \bar{\otimes} B(\mathcal{H}_i)$

A_i: abelian vN-alg

Lemma: M vN-alg acting on $\mathcal{H}$ and (P_λ) be a net of Proj s.t. $P_\lambda \longrightarrow 1$ on sot-top in $B(\mathcal{H})$. if $P_\lambda M P_\lambda$ is semi-discrete then M is semi-discrete.

pf/ $\exists \lambda_0$ s.t. $\| P_{\lambda_0} \xi - \xi \| < \varepsilon \quad \forall \xi \in \Omega \underset{\text{finite}}{\subseteq} \mathcal{H}$

$\exists \tilde{\varphi} : P_{\lambda_0} M P_{\lambda_0} \longrightarrow M_n(\mathbb{C})$ and $\tilde{\psi} : M_n(\mathbb{C}) \longrightarrow P_{\lambda_0} M P_{\lambda_0}$ s.t

$\| \langle \psi \circ \varphi (P_{\lambda_0} m P_{\lambda_0}) \xi, \eta \rangle - \langle P_{\lambda_0} m P_{\lambda_0} \xi, \eta \rangle \| < \varepsilon$

$m \in F \underset{\text{finite}}{\subseteq} M$

$\tilde{\varphi} := \varphi(P_{\lambda_0} m P_{\lambda_0})$.

Thm. $M =$ Type I vN-alg $\Rightarrow M =$ Semi-dis.

pf/ we construct the above proj mentening in previous lemma:

Let $\Omega \underset{\text{finite}}{\subseteq} \mathcal{H}$ and $Q_\Omega : \mathcal{H} \xrightarrow{\text{Proj}} [\Omega] \leq \mathcal{H}$

$P_\Omega := 1 \otimes Q_\Omega \xrightarrow{\text{Sot.}} 1$

$P_\Omega (A \bar{\otimes} B(\mathcal{H})) P_\Omega = A \otimes M_n(\mathbb{C})$

Semi-discrete

A nuclear C^*-alg may have a non-nuclear C^*-subalg but:

Thm: A sep. C^*-alg is type I iff every its C^*-subalg is nuclear.

Example: $C^*_r(F) \neq$ type I C^*-alg but is nuclear
