Exotic Group Algebras:

* Brown – Guentner 2012 – 13
* Kaliszewski – Landstad 2013
* Wiersma 2014

Let G be a LCG and $C_c(G) = \{ f : G \xrightarrow{\text{cont}} \mathbb{C} : \text{cpt support} \}$

$$f * g(t) = \int f(s) g(s^{-1}t) \, ds \qquad f^*(s) = \Delta(s^{-1}) \overline{f(s^{-1})}$$

Let $u : G \to B(\mathcal{H})$ be a group repn: $\tilde{u}(f) = \int_{s \in G} f(s) u_s \, ds \in B(\mathcal{H})$

we have $f \mapsto \tilde{u}(f)$ is a repn of $C_c(G)$ and

$$\| \tilde{u}(f^* * f) \| = \| \tilde{u}(f)^* u(f) \| = \| \tilde{u}(f) \|^2$$

$$\| f \|_{C^*} := \sup \| \tilde{u}(f) \| \implies \| f^* * f \|_{C^*} = \| f \|_{C^*}^2, \quad C^*(G) := \overline{C_c(G)}^{\|\cdot\|_{C^*}}.$$

Fact: There is a one-to-one correspondence $u \mapsto \tilde{u}$ between

- Unitary repn of G
- nondeg. repn of $C^*(G)$

$D \hookrightarrow B(\mathcal{H})$ $\quad M(D) = \{ T \in B(\mathcal{H}) : TD \cup DT \subseteq D \} \leq B(\mathcal{H})$.

maximal ideal having D

$u : G \to UM(D)$ strongly cont. $\{ (s \mapsto u_s d \text{ or } d u_s) \text{ is cont} \}$.

$u_G : G \to UM(C^*(G))$ $\quad u_G(s)(f)(t) := f(s^{-1}t)$, $f \in C_c(G)$.

Reduced Group C^*-alg: $\lambda : G \to \mathcal{U}(L^2(G))$ $(\lambda_t \xi)(s) = \xi(t^{-1}s)$

Then we have $\tilde{\lambda} : C_c(G) \to B(L^2(G))$ $\|f\|_r = \|\tilde{\lambda}(f)\|$

$$C^*_r(G) := \overline{C_c(G)}^{\|\cdot\|_r} \cong \tilde{\lambda}(C^*(G))$$

" $\tilde{\lambda} : C^*(G) \to C^*_r(G)$ is an isomorphism $\Leftrightarrow$ G amenable "

Exotic Group C^*-alg; G be a LCG. An exotic group C^*-alg is a C^*-alg completion $C^*_\mu(G) := \overline{C_c(G)}^{\|\cdot\|_\mu}$ s.t. $\|\cdot\|_r \leq \|\cdot\|_\mu \leq \|\cdot\|_{C^*}$

$$C^*(G) \twoheadrightarrow C^*_\mu(G) \twoheadrightarrow C^*_r(G)$$

Observation:

There are one-to-one correspondence between:

1) exotic group alg for G

2) Ideal $I \trianglelefteq C^*(G)$ s.t. $I \subseteq \ker \tilde{\lambda}$

3) weak equivalence classes of unitary repn u of G s.t. $\lambda \prec u$.

" $A = C^*$-alg, π, ρ = repn $\quad \pi \prec \rho \overset{\text{Def}}{\Leftrightarrow} \ker \pi \supseteq \ker \rho$ "

$$\pi \sim \rho \overset{\text{Def}}{\Leftrightarrow} \ker \pi = \ker \rho$$

G = non-amenable $\quad u = \lambda \oplus 1_G$

$$C^*_u(G) = \tilde{u}(C^*(G)).$$

Fourier - Stieltjes - Algebra $B(G)$:

$u : G \to U(\mathcal{H})$ $\xi, \eta \in \mathcal{H}$, $\varphi_{\xi,\eta} : s \mapsto \langle u_s \xi, \eta \rangle \in C_b(G)$

$$s \mapsto \langle u_s \xi_1, \eta_1 \rangle \langle v_s \xi_2, \eta_2 \rangle = \langle (u \otimes v)_s (\xi_1 \otimes \xi_2), (\eta_1 \otimes \eta_2) \rangle$$

we have $B(G) \simeq C^*(G)^*$ $\varphi \mapsto \tilde{\varphi}$ $\tilde{\varphi} : C^*(G) \to \mathbb{C}$

$\tilde{\varphi}(x) = \langle \tilde{u}(x)\xi, \eta \rangle$ if $\varphi(s) = \langle u_s \xi, \eta \rangle$

Consider $B(G)$ with w^*-top.

Thm : (Eymard)

There are one-to-one correspondence between:

1. Ideals in $C^*(G)$

2. Weak*-closed Translation invariant subspace $E \subseteq B(G)$

3. Weak-equivalence classes of unitary repn u of G.

Pf/

(3) $\Rightarrow$ (2) $u : G \to B(\mathcal{H})$

2//