Vadim:

Def: A group G is called finite if it has an invariant prob. measure.

$\ell^1_{+,1}(G) = \{ \lambda(g) ;\ \lambda(g) \geq 0 \text{ and } \sum \lambda(g) = 1 \}$

we can an action on prob. meas: $g\delta_x = \delta_{gx}$ (another way to define $g\lambda(A) = \lambda(g^{-1}A)$.).

$\lambda(g) = 1/|G|$ ← Haar measure when G is finite

What's happen when G is infinite

G = finite $\iff \exists \lambda \in \ell^1_{1,+}(G) : g\lambda = \lambda$ for any $g \in G$

If we don't have invariant measure in $\ell^1_{1,+}(G)$ we have to consider bigger space than $\ell^1_{1,+}(G)$. We consider $\ell^\infty(G)^*_{+,1}$ → Positive functional, $\chi_{(\Omega)} = 1$

finitely additivity measures

$A \subset G$ finite, $1_A \in \ell^\infty$

Modifing Conditions — means: finitly add. prob measure.

G = finite $\iff \exists \lambda \in \ell^\infty(G)^*_{+,1} : g\lambda = \lambda$ for any $g \in G$.

Now keep the space $\ell^1_{+,1}(G)$ and relax the second condition "$g\lambda = \lambda$".

Consider $(\lambda_n) \subseteq \ell^1(G)_{+,1}$ s.t. $\|g\lambda_n - \lambda_n\| \longrightarrow 0$ for any $g \in G$

total variation in $\ell^1(G)$. "Asymptotic invariants"

~~Ces~~ Cesaro averaging: $a_1, a_2, \ldots$

$a_1, \frac{a_1+a_2}{2}, \ldots$

$\delta_1, \delta_1 + \delta_2/2, \ldots, \delta_1+\delta_2+\ldots+\delta_n/n$ has ~~the~~ the approximation property what's a relation between these two definitions:

vN-def. $\iff$ ~~approximate~~ invariance (Day Reiter 1950)

Krylov – Bogolyubov Theorem: (1933)

$T: X \xrightarrow{\text{homo}} X$, T has an invariant measure.
cpt

λ on X, $\lambda_1 = T(\lambda)$, ..., $\lambda_n = T^n(\lambda)$

Now consider $\theta_n = \dfrac{\lambda_0 + \cdots + \lambda_n}{n+1}$ θ_n is not invariant but

it's approximate invariant. we have $\|\theta_n - T\theta_n\| \Rightarrow \left\|\dfrac{\lambda_0 - \lambda_{n+1}}{n+1}\right\| \leq 2/_{n+1} \to 0$

By compactness there is a weakly limit point of the sequence (θ_n) which is invariant measure.

One characterization of amenability

Thm (Bogolyubov):

G = amenable $\iff$ for any continuous action of G on a cpt space, there is a G-inv prob. measure on X.

Take $\{\lambda_n\}$ = app. inv ~~seq~~ seq. "$\|g\lambda_n - \lambda_n\| \to 0$" and θ on X.

Then $\|g\lambda_n\theta - \lambda_n\theta\| \to 0$. Like previous proof consider weakly limit point m of the seq. $\lambda_n\theta$.

Examples:

0, finite groups

1, $\mathbb{Z}$

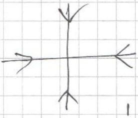
2, Nilpotent groups $\begin{pmatrix} 1 & & \\ & 1 & \\ 0 & & 1 \end{pmatrix}$

3, Solvable groups $\begin{pmatrix} * & & \\ & * & \\ 0 & & * \end{pmatrix}$

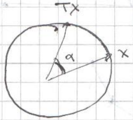
Non-Example: Free groups

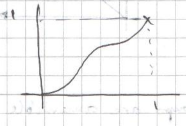
It's much easier to consider the latter, than to show that Free groups are not amenable.

F_2: Free group with two generators



For irrational number α $\quad T: x \mapsto x+\alpha$ (mod 1) $\qquad S^1 = \mathbb{R}/\mathbb{Z}$





Cor. $G' \leq G$, $G \to G'$

if G = amenable Then G' = amenable

1, Extension:

$$e \to H \to G \to G/H \to e$$

let assume that H & G/H are amenable. Then G is.

2, Inductive limits:

$$G_1 \subset G_2 \subset \cdots \subset G_n \subseteq \cdots \subseteq G$$

$\forall i$ G_i = amenable $\implies \lim G_i$ = amenable

Building Blocks: | Finite group
| abelian group

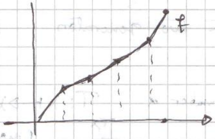
called elementary amenable groups.

Question:
~~Most~~ Elementary amenable groups ⟶

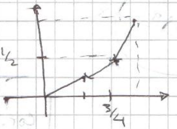
{groups with a free subgroups} $\subsetneq$ non-amenable

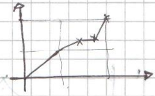
Thomson Group:

Group of dyatic-rational homeomorphisms of [0,1]
(Piecewise affin)



The slope is a power of 2.





Open Question: These groups are amenable or not?