Suitable chains of semidualizing modules

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Semidualizing modules

Throughout $(R, \mathfrak{m}, k)$ is a commutative Noetherian local ring.

Definition

An R–module C is called *semidualizing*, if

- C is finite (i.e. finitely generated)
- The natural homothety map $\chi_C^R : R \longrightarrow \mathrm{Hom}_R(C, C)$ is an isomorphism
- For all $i > 0$, $\mathrm{Ext}_R^i(C, C) = 0$

Example

Examples of semidualizing modules include

- R
- The dualizing module of R if it exists (dualizing module is a semidualizing module with finite injective dimension).

Semidualizing modules

Throughout C assumed to be a semidualizing R–module.

Basic properties

- $\mathrm{Ann}_R(C) = 0$ and $\mathrm{Supp}_R(C) = \mathrm{Spec}(R)$.
- $\dim_R(C) = \dim(R)$ and $\mathrm{Ass}_R(C) = \mathrm{Ass}_R(R)$.
- If R is local, then $\mathrm{depth}_R(C) = \mathrm{depth}(R)$.

If R is Gorenstein and local, then R is the only semidualizing R–module. Conversely, if the dualizing R–module is just the only semidualizing R–module, then R is Gorenstein.

Totally C–reflexive modules

Definition

A finite R–module M is *totally C–reflexive* when it satisfies the following conditions.

- The natural homomorphism $\delta_M^C : M \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(M, C), C)$ is an isomorphism.
- For all $i > 0$, $\mathrm{Ext}^i_R(M, C) = 0 = \mathrm{Ext}^i_R(\mathrm{Hom}_R(M, C), C)$.

- Every finite projective R–module is totally C–reflexive.

The set $\mathfrak{G}_0(R)$

The set of all isomorphism classes of semidualizing R–modules is denoted by $\mathfrak{G}_0(R)$, and the isomorphism class of a semidualizing R–module C is denoted $[C]$.

- Write $[C] \trianglelefteq [B]$ when B is totally C–reflexive.
- Write $[C] \triangleleft [B]$ when $[C] \trianglelefteq [B]$ and $[C] \neq [B]$.
- If $[C] \trianglelefteq [B]$, then
  - *(1)* $\mathrm{Hom}_R(B, C)$ is a semidualizing, and
  - *(2)* $[C] \trianglelefteq [\mathrm{Hom}_R(B, C)]$.

Chain in $\mathfrak{G}_0(R)$

A chain in $\mathfrak{G}_0(R)$ is a sequence $[C_n] \trianglelefteq \cdots \trianglelefteq [C_1] \trianglelefteq [C_0]$, and such a chain has length n if $[C_i] \neq [C_{i-1}]$ whenever $1 \leqslant i \leqslant n$.

Chain in $\mathfrak{G}_0(R)$

Assume that $[C_n] \lhd \cdots \lhd [C_1] \lhd [C_0]$ is a chain in $\mathfrak{G}_0(R)$.

- For each $i \in [n] = \{1, \cdots, n\}$ set $B_i = \mathrm{Hom}_R(C_{i-1}, C_i)$.
- For each sequence of integers $\mathbf{i} = \{i_1, \cdots, i_j\}$ with $j \geqslant 1$ and $1 \leqslant i_1 < \cdots < i_j \leqslant n$, set $B_{\mathbf{i}} = B_{i_1} \otimes_R \cdots \otimes_R B_{i_j}$.
  ($B_{\{i_1\}} = B_{i_1}$ and set $B_{\emptyset} = C_0$.)

Chain in $\mathfrak{G}_0(R)$

For a semidualizing R–module C, set $(-)^{\dagger_C} = \mathrm{Hom}_R(-, C)$.

Definition

Let $[C_n] \lhd \cdots \lhd [C_1] \lhd [C_0]$ be a chain in $\mathfrak{G}_0(R)$ of length n. For each sequence of integers $\mathbf{i} = \{i_1, \cdots, i_j\}$ such that $j \geqslant 0$ and $1 \leqslant i_1 < \cdots < i_j \leqslant n$, set $C_{\mathbf{i}} = C_0^{\dagger_{C_{i_1}} \dagger_{C_{i_2}} \cdots \dagger_{C_{i_j}}}$.
(When $j = 0$, set $C_{\mathbf{i}} = C_{\emptyset} = C_0$).
We say that the above chain is *suitable* if $C_0 = R$ and $C_{\mathbf{i}}$ is totally C_t–reflexive, for all $\mathbf{i}$ and t with $i_j \leqslant t \leqslant n$.

- If $[C_n] \lhd \cdots \lhd [C_1] \lhd [R]$ is a suitable chain, then $C_{\mathbf{i}}$ is a semidualizing R–module for each $\mathbf{i} \subseteq [n]$.
- For each sequence of integers $\{x_1, \cdots, x_m\}$ with $1 \leqslant x_1 < \cdots < x_m \leqslant n$, the sequence $[C_{x_m}] \lhd \cdots \lhd [C_{x_1}] \lhd [R]$ is a suitable chain in $\mathfrak{G}_0(R)$.

Chains in $\mathfrak{G}_0(R)$

Theorem (Gerko)

If $[C_n] \trianglelefteq \cdots \trianglelefteq [C_1] \trianglelefteq [C_0]$ is a chain in $\mathfrak{G}_0(R)$, then, for $i \in [n]$,

$$C_i \cong C_0 \otimes_R \operatorname{Hom}_R(C_0, C_1) \otimes_R \cdots \otimes_R \operatorname{Hom}_R(C_{i-1}, C_i).$$

Proposition

Assume that $[C_n] \triangleleft \cdots \triangleleft [C_1] \triangleleft [C_0]$ is a suitable chain in $\mathfrak{G}_0(R)$.

(1) For each sequence $\mathbf{i} \subseteq [n]$, the R–module $B_{\mathbf{i}}$ is a semidualizing.

(2) If $\mathbf{i}, \mathbf{s} \subseteq [n]$ are two sequences with $\mathbf{s} \subseteq \mathbf{i}$, then $[B_{\mathbf{i}}] \trianglelefteq [B_{\mathbf{s}}]$ and $\operatorname{Hom}_R(B_{\mathbf{s}}, B_{\mathbf{i}}) \cong B_{\mathbf{i} \setminus \mathbf{s}}$.

(3) $|\mathfrak{G}_0(R)| \geqslant |\{[C_{\mathbf{i}}] \mid \mathbf{i} \subseteq [n]\}| = 2^n$.

(4) $\{[B_{\mathbf{u}}] \mid \mathbf{u} \subseteq [n]\} = \{[C_{\mathbf{i}}] \mid \mathbf{i} \subseteq [n]\}$.

Suitable chains modulo regular sequences

Proposition

Assume that $\mathbf{x} = x_1, \cdots, x_d$ is an R-regular sequence in $\mathfrak{m}$ and $[C_n] \lhd \cdots \lhd [C_1] \lhd [C_0]$ is a suitable chain in $\mathfrak{G}_0(R)$ of length n.

Then $[\overline{C}_n] \lhd \cdots \lhd [\overline{C}_1] \lhd [\overline{C}_0]$ is also a suitable chain in $\mathfrak{G}_0(\overline{R})$ of length n, where $\overline{R} = R/\mathbf{x}R$ and $\overline{C}_i = \overline{R} \otimes_R C_i$ for $i = 0, 1, \cdots, n$.

Tor-*independent modules*

For the remaining part of this talk we assume that R is an Artinian local ring and that all modules are finite.

Definition (Gerko)

- The modules $K_1, K_2, \cdots, K_n$ are said to be weakly Tor-independent if $\mathrm{amp}(\otimes^{L}_{1\leqslant i\leqslant n} K_i) = 0$.
- These modules are said to be strongly Tor-independent if for any subset $\Lambda \subseteq [n]$ we have $\mathrm{amp}(\otimes^{L}_{i\in\Lambda} K_i) = 0$.

Theorem (Gerko)

If the modules $K_1, K_2, \cdots, K_n$ are non-free and strongly Tor-independent, then $\mathfrak{m}^n \neq 0$. If, under the same conditions, $\mathfrak{m}^{n+1} = 0$, then the Poincaré series of k has the form $1/\prod_{i=1}^{n}(1 - d_i t)$ for some positive integers d_i.

Tor-*independent semidualizings*

Conjecture (Gerko)

If $[C_n] \lhd \cdots \lhd [C_1] \lhd [C_0]$ is a chain in $\mathfrak{G}_0(R)$ of length n, then $\mathfrak{m}^n \neq 0$. If, under the same conditions, $\mathfrak{m}^{n+1} = 0$, then the Poincaré series of k has the form $1/\prod_{i=1}^{n}(1 - d_i t)$ for some positive integers d_i.

Theorem

Let $[C_n] \lhd \cdots \lhd [C_1] \lhd [C_0]$ be a suitable chain in $\mathfrak{G}_0(R)$ of length n, then $\mathfrak{m}^n \neq 0$. If, under the same conditions, $\mathfrak{m}^{n+1} = 0$, then the Poincaré series of k has the form $1/\prod_{i=1}^{n}(1 - d_i t)$ for some positive integers d_i.

Poincaré series of residue field

Remark

When $(S, \mathfrak{n})$ is a Cohen-Macaulay local ring with dimension d and $[C_n] \lhd \cdots \lhd [C_1] \lhd [C_0]$ is a suitable chain in $\mathfrak{G}_0(S)$ of length n, then $[\overline{C}_n] \lhd \cdots \lhd [\overline{C}_1] \lhd [\overline{C}_0]$ is also a suitable chain in $\mathfrak{G}_0(\overline{S})$ of length n, where $\overline{S} = S/\mathbf{x}S$, $\overline{C}_i = \overline{S} \otimes_S C_i$ and $\mathbf{x} = x_1, \cdots, x_d$ is an S-regular sequence which is contained in $\mathfrak{n}$.

- If $\overline{\mathfrak{n}}^{n+1} = 0$, then the Poincaré series of $S/\mathfrak{n} \cong \overline{S}/\overline{\mathfrak{n}}$ has the form $1/\prod_{i=1}^{n}(1 - d_i t)$ for some positive integers d_i.

- The length of a suitable chain in $\mathfrak{G}_0(S)$ is less than the number $l = \min\{\, i > 0 \mid \overline{\mathfrak{n}}^i = 0 \,\}$.

$SD(n)$-full rings

Definition (Gerko)

An Artinian ring R is called $SD(n)$-full if the following conditions are satisfied.

(i) $\mathfrak{m}^{n+1} = 0$.

(ii) There are strongly Tor-independent non-free semidualizing modules $K_1, K_2, \cdots, K_n$ such that for any subset $\Lambda \subseteq [n]$ the module $\otimes_{i\in\Lambda} K_i$ is semidualizing.

Theorem

An Artinian ring R is $SD(n)$-full if and only if there is a suitable chain in $\mathfrak{G}_0(R)$ of length n and $\mathfrak{m}^{n+1} = 0$.

Example of suitable chains

Let F be a field. Set $S_i = F \ltimes F^{a_i}$ for all $1 \leqslant i \leqslant n$, where $a_i > 1$. Then S_i is an Artinian non-Gorenstein ring with dualizing module $D_i = \mathrm{Hom}_F(S_i, F)$.

Set $S = \otimes_F^{1 \leqslant i \leqslant n} S_i$.

Set $C_0 = S$ and $C_j = (\otimes_F^{1 \leqslant i \leqslant j} D_i) \otimes_F (\otimes_F^{j < i \leqslant n} S_i)$ for all $1 \leqslant j \leqslant n$.

Then the sequence $[C_n] \lhd \cdots \lhd [C_1] \lhd [C_0]$ is a suitable chain in $\mathfrak{G}_0(S)$.

Bass series of semidualizing modules

Lemma

Let R be an Artinian ring with $\mathfrak{m}^{n+1} = 0$. Assume that $[C_n] \lhd \cdots \lhd [C_1] \lhd [C_0]$ is a suitable chain in $\mathfrak{G}_0(R)$ of length n. Then for each $i \in [n]$ the Poincaré series of B_i is

$$P^R_{B_i}(t) = \frac{\beta^R_0(B_i) - t}{1 - \beta^R_0(B_i)t}$$

and the Bass series of $B_{[n]\setminus i}$ is

$$I_R^{B_{[n]\setminus i}}(t) = \frac{\mu^0_R(B_{[n]\setminus i}) - t}{1 - \mu^0_R(B_{[n]\setminus i})t}.$$

Bass series of semidualizing modules

Proposition

Let R be an Artinian ring with $\mathfrak{m}^{n+1} = 0$. Assume that $[C_n] \lhd \cdots \lhd [C_1] \lhd [C_0]$ is a suitable chain in $\mathfrak{G}_0(R)$ of length n. Then for each $i \in [n]$ the Poincaré series of C_i is

$$P^R_{C_i}(t) = \frac{\prod_{j=1}^{i}(\beta_0^R(B_j) - t)}{\prod_{j=1}^{i}(1 - \beta_0^R(B_j)t)} \ ,$$

$I_R^{C_n}(t) = 1$, and for $i \neq n$, the Bass series of C_i is

$$I_R^{C_i}(t) = \frac{\prod_{j=i+1}^{n}(\beta_0^R(B_j) - t)}{\prod_{j=i+1}^{n}(1 - \beta_0^R(B_j)t)} = \frac{\prod_{j=i+1}^{n}(\mu_R^0(B_{[n]\setminus j}) - t)}{\prod_{j=i+1}^{n}(1 - \mu_R^0(B_{[n]\setminus j})t)} \ .$$

Bass series of R

- The Bass series of R is

$$I_R(t) = P^R_{C_1}(t) I^{C_1}_R(t) = I_R^{B_{[n]\setminus 1}}(t) I_R^{B_{[n]\setminus 2}}(t) \cdots I_R^{B_{[n]\setminus n}}(t)$$

$$= \frac{\prod_{j=1}^n (\mu^0_R(B_{[n]\setminus j}) - t)}{\prod_{j=1}^n (1 - \mu^0_R(B_{[n]\setminus j}) t)}$$

- For each $i \in [n]$,
$I_R^{B_{[n]\setminus i}}(t) = \alpha_i + (\alpha_i^2 - 1)t + \alpha_i(\alpha_i^2 - 1)t^2 + \alpha_i^2(\alpha_i^2 - 1)t^3 + \cdots,$
where $\alpha_i = \mu^0_R(B_{[n]\setminus i})$. Thus $\{\mu^j_R(B_{[n]\setminus i})\}$ is strictly increasing.

$\therefore$ Therefore $\{\mu^j_R(R)\}$ is also strictly increasing whenever $n \geqslant 1$.

Questions of Huneke

Let R be a Cohen-Macaulay local ring.

(1) If the sequence $\{\mu_R^i(R)\}$ is bounded above by a polynomial in i, must R be Gorenstein?

(2) If R is not Gorenstein, must the sequence $\{\mu_R^i(R)\}$ grow exponentially?

Bass series of a C-M ring

Assume that $(S, \mathfrak{n})$ is a Cohen-Macaulay local ring with dimension d and $\mathbf{x} = x_1, \cdots, x_d$ is an S-regular sequence which is contained in $\mathfrak{n}$.

Set $\overline{S} = S/\mathbf{x}S$. We have $I_S(t) = t^d I_{\overline{S}}(t)$.

- If $[C_n] \lhd \cdots \lhd [C_1] \lhd [C_0]$ is a suitable chain in $\mathfrak{G}_0(S)$ of length n and $\overline{\mathfrak{n}}^{n+1} = 0$, then $I_S(t)$ has a very specific form and $\{\mu^i_S(S)\}$ is strictly increasing.

Thank You

References

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Definition

▶ For an R-module M, the ith *Bass number* of M is the integer $\mu_R^i(M) = \mathrm{rank}_k(\mathrm{Ext}_R^i(k, M))$, and the *Bass series* of M is the formal Laurent series $I_R^M(t) = \Sigma_{i\in\mathbb{Z}}\mu_R^i(M)t^i$.

▶The ith *Betti number* of M is the integer $\beta_i^R(M) = \mathrm{rank}_k(\mathrm{Ext}_R^i(M, k)) = \mathrm{rank}_k(\mathrm{Tor}_i^R(k, M))$, and the *Poincaré series* of M is the formal Laurent series $P_M^R(t) = \Sigma_{i\in\mathbb{Z}}\beta_i^R(M)t^i$..