Higher iterated Hilbert coefficients of the graded components of bigraded modules

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Overview

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Preliminaries and notations

Definition (Graded Ring)

Let G be an abelian semigroup with identity element 0. A ring R is called G-*graded* if there exists a family of subgroups $\{R_i\}_{i\in G}$ of R such that

(i) $R = \bigoplus_{i\in G} R_i$ (as abelian groups);

(ii) $R_i R_j \subseteq R_{i+j}$ for all $i, j \in G$.

Note that if $R = \bigoplus_{i\in G} R_i$ is a G-graded ring, then R_0 is a subring of R, $1 \in R_0$ and R_i is an R_0-module for all i.

The abelian subgroup R_i of R is called the *i-th graded component* of R. A nonzero element $r \in R_i$ is called a *homogeneous element* of R of degree i. Any nonzero element $r \in R$ can be written as sum of nonzero homogeneous elements of R, called homogeneous components of r.

Preliminaries and notations

Definition (Graded Module)

Let G be an abelian semigroup with identity element 0. Let R be a G-graded ring and M an R-module. We say that M is a G-graded R-module if there exists a family of subgroups $\{M_i\}_{i\in G}$ of M such that

(i) $M = \bigoplus_{i\in G} M_i$ (as abelian groups);

(ii) $R_i M_j \subseteq M_{i+j}$ for all i, j.

As for graded rings, the abelian subgroup M_i of M is called the *i-th graded component* of M. A nonzero element $m \in M_i$ is called a *homogeneous* element of M of degree i. Any nonzero element $m \in M$ can be written as sum of nonzero homogeneous elements of M, called *homogeneous components* of m.

Preliminaries and notations

Let R is an Artinian local ring, and that R is finitely generated over R_0. Notice that for each finite graded R-module M, the homogeneous components M_n of M are finite R-modules, and hence have finite length.

Definition (The Hilbert functions)

Let M be a graded R-module whose graded components M_n have finite length for all n. The numerical function $H(M, -) : \mathbb{Z} \longrightarrow \mathbb{Z}$ with $H(M, n) = \lambda(M_n)$ for all $n \in \mathbb{Z}$ is the Hilbert function.

Theorem (D. Hilbert (1890))

Let R be an Artin ring and M be a finitely generated, graded R-module of dimension d. Then for all $k \gg 0$, the Hilbert function $H(k, M)$ is equal to polynomial in k of degree $d - 1$.

Introduction

1. Our work is motivated by Kodiyalam's work [6], the papers by Theodorescu [11], by Katz and Theodorescu [8], [9] and the paper [3].
2. In these papers it was shown that for finitely generated R-modules M and N over a Noetherian (local) ring R, and for an ideal $I \subset R$ such that the length of $\mathrm{Tor}_i^R(M, N/I^kN)$ is finite for all k, it follows that the length of $\mathrm{Tor}_R^i(M, N/I^kN)$ and is eventually a polynomial function in k.
3. In these papers bounds are given for the degree of these polynomials.
4. In some cases also the leading coefficient is determined. Similar results have been proved for the Ext-modules.

Introduction

1. Let R be a local Noetherian ring, I an ideal and M finitely generated R-modules where $\lambda(M/IM)$ is finite. Then $\lambda(M/I^nM)$ is given by a polynomial in n for $n \gg 0$. (see [1])
2. (Kirby, 1989) Let R be a commutative ring with identity, I be an ideal of R, and M be a finitely generated R-module. Let $r = grad_R(I; M)$ be finite and $Ext^r_R(R/I, M)$ has finite length. Then, for n large, $\lambda_R(Ext^r_R(R/I^n, M))$ is equal to a polynomial in n of degree at most r.
3. (Kodiyalam, 1993) Let R be a Noetherian ring, I be an ideal of R, and M, Q are finitely generated R-modules. If $\lambda_R(M \bigotimes_R Q) < \infty$, then, for all large n, each of the functions $\lambda_R(Tor_i^R(M/I^nM, Q))$ and $\lambda_R(Ext^i_R(Q, M/I^nM))$ is a polynomial in n of degree at most $\max\{0, \dim_R(M \bigotimes_R Q) - 1\}$.
4. (Theodorescu, 2002) Let R be Noetherian, I an ideal, M, N finitely generated R-modules such that $Var(I) \cap Supp(M) \cap Supp(N)$ be a finite set of maximal ideals of R. Then, for all $i \geq 0$, $\lambda_R(Ext^i_R(N/I^nN; M)$ has polynomial growth for $n \gg 0$.

Introduction

We consider a related problem. Here $I \subset S$ is graded ideal and S is the polynomial ring.

1. It is shown in Corollary 8 that for any finitely generated graded S-module M, the modules $\mathrm{Tor}_i^S(M, I^k)$ are finitely graded S-modules which for $k \gg 0$ have constant Krull dimension, and furthermore in Corollary 10 it is shown that the higher iterated Hilbert coefficients (which appear as the coefficients of the higher iterated Hilbert polynomials) are all polynomials functions.
2. A related result has been shown in [4] for the case M/I^kM and in [5] for the case $\mathrm{Tor}_i^s(S/\mathfrak{m}, I^k)$, where $\mathfrak{m}$ denotes the graded maximal ideal of S.

Introduction

Let K be a field, $S = K[x_1, \ldots, x_n]$ the polynomial ring in n variables with the standard grading. Let $A = K[x_1, \ldots, x_n, y_1, \ldots, y_m]$ with bigrading defined by $\deg x_i = (1, 0)$ and $\deg y_j = (p_j, 1)$, for some integers $p_j \geq 0$.
For a finitely generated bigraded A-module $M = \bigoplus_{i,j \in \mathbb{Z}} M_{(i,j)}$, we define M_k to be the graded S-module $\bigoplus_{i \in \mathbb{Z}} M_{(i,k)}$.
For $a, b \in \mathbb{Z}$, the twisted module A-module $M(-a, -b)$ is defined to be the bigraded A-module with components $M(-a, -b)_{(i,j)} = M(i - a, j - b)$.

Introduction

Definition (The higher iterated Hilbert functions)

For a finite graded S-module M and all $k \gg 0$, the numerical function $H(M,k) = \dim_K M_k$ is called the Hilbert function of M. For $i \in \mathbb{N}$, the higher iterated Hilbert functions $H_i(M,k)$ are defined recursively as follows:

$$H_0(M,k) = H(M,k), \quad \text{and} \quad H_i(M,k) = \sum_{j \leq k} H_{i-1}(M,j).$$

By Hilbert it is known that $H_i(M,k)$ is of polynomial type of degree $d+i-1$, where d is the Krull dimension of M. In other words, there exists a polynomial $P^i_M(x) \in \mathbb{Q}[x]$ of degree $d+i-1$ such that $H_i(M,k) = P^i_M(k)$ for all $k \gg 0$. This unique polynomial is called the *ith Hilbert polynomial* of M.

The graded components of a bigraded module and their higher iterated Hilbert coefficients

A graded free S-resolution of M_k can be obtained by the graded components of the bigraded free A-module resolution of M.
Let

$$0 \longrightarrow F_t \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

be a bigraded free resolution of M. Then

$$0 \longrightarrow (F_t)_k \longrightarrow \cdots \longrightarrow (F_1)_k \longrightarrow (F_0)_k \longrightarrow M_k \longrightarrow 0$$

is a graded free resolution of M_k.
let $F_i = \bigoplus_j A(-a_{i_j}, -b_{i_j})$. Then $(F_i)_k = \bigoplus_j (A(-a_{i_j}, -b_{i_j}))_k$, where $A(-a,-b)_k = \bigoplus_j A(-a,-b)_{(j,k)} = \bigoplus_j A(j-a, k-b)$.

The graded components of a bigraded module and their higher iterated Hilbert coefficients

These resolutions are then used to compute the higher iterated Hilbert polynomials of the graded S-modules M.
The first (and important step) is to show that the higher iterated Hilbert coefficients of the components $A(-a,-b)_k$ of the bi-shifted free A-module $A(-a,-b)$ are polynomial functions in k for $k \gg 0$. Note that

$$\begin{aligned} A(-a,-b)_k &= (A_{k-b})(-a) \\ &\cong \bigoplus_{\beta_1+\cdots+\beta_m=k-b} S(-(p_1\beta_1+\cdots+p_m\beta_m)-a)y_1^{\beta_1}\cdots y_m^{\beta_m}. \end{aligned}$$

Hence, in a first step, we have to determine the Hilbert coefficients of $S(-c)$ for some $c \in \mathbb{Z}$.

The graded components of a bigraded module and their higher iterated Hilbert coefficients

Let $P_M^i(x)$ be the ith Hilbert polynomial of M. It can be written in the form

$$P_M^i(x) = \sum_{j=0}^{d+i-1} (-1)^j e_j^i(M) \binom{x+d+i-j-1}{d+i-j-1}$$

with integer coefficients $e_j^i(M)$, called the *higher iterated Hilbert coefficients* of M, where by definition

$$\binom{i}{j} = \frac{i(i-1)\cdots(i-j+1)}{j(j-1)\cdots 2\cdot 1} \quad \text{if} \quad j>0 \quad \text{and} \quad \binom{i}{0} = 1.$$

The graded components of a bigraded module and their higher iterated Hilbert coefficients

Lemma

In the important special case when $M = S$ we have

$$P_S^i(x) = \binom{x+n+i-1}{n+i-1}.$$

More generally, if $c \in \mathbb{Z}$, then

$$P_{S(-c)}^i(x) = \binom{x-c+n+i-1}{n+i-1} \tag{1}$$

In particular, $\deg P_{S(-c)}^i(x) = n+i-1$.

The graded components of a bigraded module and their higher iterated Hilbert coefficients

Lemma

Let Δ be difference operator ($(\Delta P)(a) = P(a) - P(a-1)$ for all $a \in \mathbb{Z}$), and the d times iterated Δ operator will be denoted by Δ^d. Then

$$e_j^i(M) = (-1)^j \Delta^{d+i-j-1} P_M^i(-1) \quad \text{for} \quad j = 0, \ldots, d+i-1, \tag{2}$$

where $d = \dim M$.

Proposition

Let $c \in \mathbb{Z}$. Then the higher iterated Hilbert coefficients of $S(-c)$ are

$$e_j^i(S(-c)) = \binom{c}{j} \quad \text{for all } i \geq 0 \text{ and all } j \text{ with } 0 \leq j \leq n+i-1.$$

In particular, $e_j^i(S(-c)) = 0$ if and only if $0 \leq c < j \leq n+i-1$.

The graded components of a bigraded module and their higher iterated Hilbert coefficients

Proposition

Let K be a field, $S = K[x_1, \ldots, x_n]$ the polynomial ring in n variables with the standard grading. Let $A = K[x_1, \ldots, x_n, y_1, \ldots, y_m]$ with bigrading defined by $\deg x_i = (1, 0)$ and $\deg y_j = (p_j, 1)$, for some integers $p_j \geq 0$. For $k \gg 0$, the higher iterated Hilbert coefficients $e_j^i(A(-a, -b)_k)$ are polynomial functions of degree $m + j - 1$ with

$$e_j^i(A(-a,-b)_k) \leq \binom{k-b+m-1}{m-1}\binom{p_m(k-b)+a}{j}.$$

Equality holds, if and only if $p_1 = p_2 = \cdots = p_m$ for all j.

The higher iterated Hilbert coefficients of the graded components of a bigraded A-module

Lemma

Let K be a field, $S = K[x_1, \ldots, x_n]$ the polynomial ring in n variables with the standard grading. Let $A = K[x_1, \ldots, x_n, y_1, \ldots, y_m]$ with bigrading defined by $\deg x_i = (1, 0)$ and $\deg y_j = (p_j, 1)$, for some integers $p_j \geq 0$. We set $\mathfrak{m} = (x_1, \ldots, x_n)$ and $\mathfrak{n} = (y_1, \ldots, y_m)$. Then $A/\mathfrak{n} = S$ and $A/\mathfrak{m} = S'$ where is the polynomial ring $K[y_1, \ldots, y_m]$. Let M be a finitely generated bigraded A-module. Then the following holds:

- *(a) There exists an integer s such that $M_{k+1} = \mathfrak{n}M_k$ for $k \geq s$.*
- *(b) The Krull dimension $\dim M_k$ of M_k is constant for all $k \gg 0$. We set $\operatorname{ldim} M = \lim_{k\to\infty} \dim M_k$.*
- *(c) Let $M' = \bigoplus_{k \geq k_0} M_k$ where k_0 is chosen such that $\dim M_k = \operatorname{ldim} M$ and $M_{k+1} = \mathfrak{n}M_k$ for all $k \geq k_0$. Then*
  - *(i) $\dim M'/\mathfrak{n}M' = \operatorname{ldim} M' = \operatorname{ldim} M$;*
  - *(ii) $\dim M'/\mathfrak{m}M' = \dim M/\mathfrak{m}M$.*

The higher iterated Hilbert coefficients of the graded components of a bigraded A-module

Theorem

Let M be a finitely generated bigraded A-module. Then for $k \gg 0$, $e_j^i(M_k)$ is a polynomial in k, and

$$\deg e_j^i(M_k) \le m + j - 1 \quad \textit{for} \quad j = 0, \ldots, \operatorname{ldim} M + i - 1,$$

and $e_j^i(M_k) = 0$ for $j > \operatorname{ldim} M + i - 1$, where $\operatorname{ldim} M = \lim_{k \to \infty} \dim M_k$.

The higher iterated Hilbert coefficients of the graded components of a bigraded A-module

Example

Let $S = K[x_1, x_2], \mathfrak{m} = (x_1, x_2)$ and $\mathcal{R}(\mathfrak{m}) = \bigoplus_{k\geq 0} \mathfrak{m}^k$. Let $A = K[x_1, x_2, y_1, y_2]$ with bigrading defined by $\deg(x_i) = (1, 0)$ and $\deg(y_i) = (1, 1)$, for $i = 1, 2$. The natural map defined by $x_i \mapsto x_i$ and $y_i \mapsto x_i t$, for $i = 1, 2$, is then a surjective homomorphism. So $\mathcal{R}(\mathfrak{m})$ has a bigraded free resolution of the form

$$0 \to A(-2, -1) \to A \to \mathcal{R}(\mathfrak{m}) \to 0$$

Hence $e^i_j(\mathfrak{m}^k) = e^i_j(A_k) - e^i_j(A(-2, -1)_k)$. One has $e^i_j(A_k) = (k+1)\binom{k}{j}$ and $e^i_j(A(-2, -1)_k) = k\binom{k+1}{j}$. So $e^i_j(\mathfrak{m}^k) = \frac{k(k-1)\cdots(k-j+2)}{j}(1-j)(k+1)$. Therefore $\deg(e^i_j(\mathfrak{m}^k)) = j$, and by Theorem 3 our upper bound is $j+1$.

The higher iterated Hilbert coefficients of the graded components of a bigraded A-module

In the special case that all p_i are the same, we can improve the upper bound for the degree of the higher iterated Hilbert coefficients as follows:

Theorem

Assume that $p_1 = p_2 = \cdots = p_m = p$, and let M be a finitely generated bigraded A-module. Then for $k \gg 0$, $e_j^i(M_k)$ is a polynomial in k, and

$$\deg e_j^i(M_k) \leq \dim M/\mathfrak{m}M + j - 1 \quad \textit{for} \quad j = 0, \ldots, \operatorname{ldim} M + i - 1,$$

and $e_j^i(M_k) = 0$ for $j > \operatorname{ldim} M + i - 1$.

The given upper bound for the degree of the higher iterared Hilbert coefficients of a bigraded A-module as given in Theorem 5 is in general sharp, for example for $M = A$. In more special cases it may not be sharp. Indeed, let $I \subset S$ be a graded ideal generated by m homogeneous polynomials of degree p, and let $\mathcal{R}(I) = \bigoplus_{k \geq 0} I^k$ the Rees ring of I. Then $\mathcal{R}(I)$ is a bigraded A-algebra with $\mathcal{R}(I)_k \cong I^k$ and $\dim \mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I) = \ell(I)$, which by definition is the analytic spread of I. Thus we have

Corollary

Let $I \subset S$ be a graded ideal generated in a single degree. Then for all $k \gg 0$, $e_j^i(I^k)$ is a polynomial function of degree $\leq \ell(I) + j - 1$.

In case that I is $\mathfrak{m}$-primary, one has $e_0^i(I^k) = 1$ for all i and k so that $\deg e_0^i(I^k) = 0$, while the formula in Corollary 6 gives the degree bound $n - 1$, since $\ell(I) = n$.

The higher iterated Hilbert coefficients of the graded components of Tor and Ext

Let M be a graded S-module and $N = \bigoplus_{i,j \in \mathbb{N}} N_{(i,j)}$ bigraded A-module. We will see that $\mathrm{Tor}_i^S(M, N)$ and $\mathrm{Ext}_S^i(M, N)$ are naturally bigraded A-modules. Thus we may then study the higher iterated Hilbert coefficients of the graded components of these modules.

The higher iterated Hilbert coefficients of the graded components of Tor and Ext

Let U by a finitely generated graded S-module, and V be a finitely generated bigraded A-module. We first notice that

$$U \otimes_S V \quad \text{and} \quad \mathrm{Hom}_S(U, V)$$

are bigraded A-modules. Indeed,

$$(U \otimes_S V)_{(c,d)} = \bigoplus_k U_k \otimes_K V_{(c-k,d)},$$

and

$$\mathrm{Hom}_S(U, V)_{(c,d)} = \{f \in \mathrm{Hom}_S(U, V) \colon \ f(U_i) \subset V_{(i+c,d)} \text{ for all } i\}.$$

With this bigraded structure as described above we have

$$(U \otimes_S V)_k = U \otimes_S V_k \quad \text{and} \quad \mathrm{Hom}_S(U, V)_k = \mathrm{Hom}_S(U, V_k) \quad \text{for all} \quad k.$$

The higher iterated Hilbert coefficients of the graded components of Tor and Ext

Lemma

Let M be a finitely generated graded S-module and N finitely generated bigraded A-module. Then, for all i, $\operatorname{Tor}_i^S(M,N)$ and $\operatorname{Ext}_S^i(M,N)$ are finitely generated bigraded A-modules, and

$$\operatorname{Tor}_i^S(M,N)_k \cong \operatorname{Tor}_i^S(M,N_k) \quad \text{and} \quad \operatorname{Ext}_S^i(M,N)_k \cong \operatorname{Ext}_S^i(M,N_k) \quad \text{for all } i$$

Corollary

Let M be a finitely generated graded S-module and N finitely generated bigraded A-module. Then the Krull dimension of the finitely generated graded S-modules $\operatorname{Tor}_i^S(M,N)_k$ and $\operatorname{Ext}_S^i(M,N)_k$ are constant for $k \gg 0$.

The higher iterated Hilbert coefficients of the graded components of Tor and Ext

Next we want to study further the graded S-modules $\mathrm{Tor}_i^S(M,N)_k$ and $\mathrm{Ext}_S^i(M,N)_k$. By the preceding corollary, their Hilbert polynomials have constant degree for large k. For $\mathrm{Tor}_i^S(M,N)_k$, these degrees can be bounded as follows

Proposition

With the notation and assumptions as before, we have

$$\dim \mathrm{Tor}_{i+1}^S(M,N)_k \leq \dim \mathrm{Tor}_i^S(M,N)_k \quad \textit{for all } k.$$

In particular, $\dim \mathrm{Tor}_{i+1}^S(M,N)_k \leq \dim(M \otimes_S N_k)$ *for all* k*, and hence for* $k \gg 0$*, the degree of the* j*th iterated Hilbert polynomial of* $\mathrm{Tor}_i^S(M,N)_k$ *is less than or equal to* $\dim(M \otimes_S \operatorname{ldim} N) + j - 1$*.*

Corollary

Let M be a finitely generated graded S-module and N finitely generated bigraded A-module. Then for all $k \gg 0$, $e^i_j(\operatorname{Tor}^S_l(M, N_k))$ and $e^i_j(\operatorname{Ext}^l_S(M, N_k))$ are polynomials in k of degree at most $m - 1 + j$
In special the case that $p_i = p$ for all i, the degree of $e^i_j(\operatorname{Tor}^S_l(M, N_k))$ is bounded by $\dim \operatorname{Tor}^S_l(M, N)/\mathfrak{m} \operatorname{Tor}^S_l(M, N) + j - 1$ *and the degree of* $e^i_j(\operatorname{Ext}^l_S(M, N_k))$ *is bounded by* $\dim \operatorname{Ext}^l_S(M, N)/\mathfrak{m} \operatorname{Ext}^l_S(M, N)$.

Corollary

Let M be a graded S-module, and $I \subset S$ a graded ideal. Then for $l > 1$, $e^i_j(\operatorname{Tor}^S_l(M, S/I^k))$ is polynomial function in k of degree less than or equal to $v(I) + j - 1$ where $v(I)$ denotes the number of generators of I. If all generators of I have the same degree then $v(I)$ can be replaced by $\dim \mathcal{R}(I)/\operatorname{Ann}_S(M)\mathcal{R}(I)$.

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Thank you for attention