Model structures on the category of complexes of quiver representations

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(IPM)

(A joint work with Rasool Hafezi)

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Let $\mathfrak{A}$ be an additive category

$\mathbb{C}(\mathfrak{A})$= The category of complexes over $\mathfrak{A}$.

$\mathbb{K}(\mathfrak{A})$=The classical homotopy category of $\mathfrak{A}$

- $\mathrm{Obj}(\mathbb{K}(\mathfrak{A}))= \mathrm{Obj}(\mathbb{C}(\mathfrak{A}))$
- $\mathrm{Hom}_{\mathbb{K}(\mathfrak{A})}(X^\bullet, Y^\bullet) = \mathrm{Hom}_{\mathbb{C}(\mathfrak{A})}(X^\bullet, Y^\bullet)/\sim$
- $f, g : X^\bullet \to Y^\bullet$ are homotopic if there exists a s such that $f - g = d_Y s + s d_X$.

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$\mathbb{D}(\mathfrak{A})$=The derived category of $\mathfrak{A}$

- $\mathrm{Obj}(\mathbb{D}(\mathfrak{A}))= \mathrm{Obj}(\mathbb{C}(\mathfrak{A}))$
- $\mathrm{Hom}_{\mathbb{D}(\mathfrak{A})}(X^\bullet, Y^\bullet)$ = The equivalence classes of diagrams

$$X^\bullet \xrightarrow{r} Y^\bullet \xleftarrow{s} Z^\bullet$$

where s is a quasi-isomorphism.

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- E: the set of arrows
- $s, t : E \to V$ two maps such that $\forall a \in E$, $s(a)$ is the source of a and $t(a)$ is the target of a

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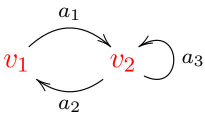

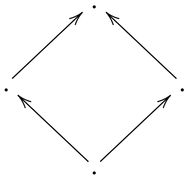

- A quiver $\mathcal{Q}$ is said to be finite if V and E are finite sets.
- A path p of a quiver $\mathcal{Q}$ is a sequence of arrows $a_n \cdots a_2 a_1$ with $t(a_i) = s(a_{i+1})$.
- A path of length $l \geq 1$ is called cycle whenever its source and target coincide.
- A quiver is called acyclic if it contains no cycles.

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We denoted the category of all representations of $\mathcal{Q}$ in $\mathcal{G}$ by $\mathrm{Rep}(\mathcal{Q},\mathcal{G})$.
In particular if R is an associative ring with identity we denoted by $\mathrm{Rep}(\mathcal{Q},R)$(resp. $\mathrm{rep}(\mathcal{Q},R)$) the category of all representations by (resp. finitely generated) R-modules

1) Model structures and Hovey pair

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A model structure on $\mathfrak{C}$ is a triple $(\mathrm{Cof}, \mathrm{W}, \mathrm{Fib})$ of classes of morphisms, called cofibrations, weak equivalences and fibrations, respectively, such that satisfying certain axioms.

An object $W \in \mathfrak{C}$ is said to be a trivial object if $\emptyset \to W$ is a weak equivalence.

An object $A \in \mathfrak{C}$ is said to be a cofibrant if $\emptyset \to A$ is a cofibration

Dually $B \in \mathfrak{C}$ is fibrant if $B \to *$ is fibration .

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A pair $(\mathcal{F},\mathcal{C})$ of classes of object of $\mathfrak{A}$ is said to be a cotorsion pair if $\mathcal{F}^{\perp}=\mathcal{C}$ and $\mathcal{F}={}^{\perp}\mathcal{C}$, where the left and right orthogonals are defined as follows

$$^{\perp}\mathcal{C} := \{A\in\mathfrak{A} \mid \mathrm{Ext}^1_{\mathfrak{A}}(A,Y)=0, \text{ for all } Y\in\mathcal{C}\}$$

and

$$\mathcal{F}^{\perp} := \{A\in\mathfrak{A} \mid \mathrm{Ext}^1_{\mathfrak{A}}(W,A)=0, \text{ for all } W\in\mathcal{F}\}.$$

Let $\mathfrak{A}$ be an abelian category

A cotorsion pair $(\mathcal{F}, \mathcal{C})$ is called complete if for every $A \in \mathfrak{A}$ there exist exact sequences

$$0 \to Y \to W \to A \to 0 \quad \text{and} \quad 0 \to A \to Y' \to W' \to 0,$$

where $W, W' \in \mathcal{F}$ and $Y, Y' \in \mathcal{C}$.

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where $W, W' \in \mathcal{F}$ and $Y, Y' \in \mathcal{C}$.

Definition

A thick subcategory of an abelian category $\mathfrak{A}$ is a class of objects $\mathcal{W}$ which is closed under direct summands and such that if two out of three of the terms in a short exact sequence are in $\mathcal{W}$, then so is the third.

Abelian model structure:

An abelian model category is an complete and cocomplete abelian category $\mathfrak{A}$ equipped with a model structure such that

(1) A map is a cofibration if and only if it is a monomorphism with cofibrant cokernel.

(2) A map is a fibration if and only if it is an epimorphism with fibrant kernel.

Theorem[Hov02, Theorem 2.2]:

Let $\mathfrak{A}$ be an abelian category with an abelian model structure. Let $\mathcal{C}$ be the class of cofibrant objects, $\mathcal{F}$ the class of fibrant objects and $\mathcal{W}$ the class of trivial objects. Then $\mathcal{W}$ is a thick subcategory of $\mathfrak{A}$ and both $(\mathcal{C}, \mathcal{W}\cap\mathcal{F})$ and $(\mathcal{C}\cap\mathcal{W}, \mathcal{F})$ are complete cotorsion pairs in $\mathfrak{A}$.
Conversely, given a thick subcategory $\mathcal{W}$ and classes $\mathcal{C}$ and $\mathcal{F}$ making $(\mathcal{C}, \mathcal{W}\cap\mathcal{F})$ and $(\mathcal{C}\cap\mathcal{W}, \mathcal{F})$ each complete cotorsion pairs, then there is an abelian model structure on $\mathfrak{A}$ where $\mathcal{C}$ is the cofibrant objects, $\mathcal{F}$ is the fibrant objects and $\mathcal{W}$ is the trivial objects.

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A pair of cotorsion pairs $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ as in above theorem have been referred to as Hovey pair. We also call $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ a Hovey triple.

2) Homotopy category of model category

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Lemma,[Gil11,Proposition 4.4]

Let $\mathfrak{A}$ be an abelian model category and $f, g : X \to Y$ be two morphisms. If X is cofibrant and Y is fibrant, then f and g are homotopic (we denote by $f \sim g$) if and only if $f - g$ factor through a trivially fibrant and cofibrant object.

Fundamental theorem about model category:

Let $\mathfrak{C}$ be a model category.

Definition:

The axioms of model structure on $\mathfrak{C}$ implies that any object $X \in \mathfrak{C}$ has a cofibrant resolution consisting of cofibrant object $QX \in \mathfrak{C}$ equipped with a trivially fibration $QX \longrightarrow X$ in $\mathfrak{C}$. Dually, X has also a fibrant resolution consisting of a fibrant object $RX \in \mathfrak{C}$ equipped with a trivially cofibration $X \longrightarrow RX$. The object QX (resp. RX) is called cofibrant replacement (resp. fibrant replacement) of X.

Fundamental theorem about model category:

Theorem:

Let $\gamma : \mathfrak{C} \to \mathrm{Ho}\mathfrak{C}$ be the canonical localization functor, and denote by $\mathfrak{C}_{cf}$ the full subcategory given by the objects which are cofibrant and fibrant.

(1) The composition $\mathfrak{C}_{cf} \to \mathfrak{C} \to \mathrm{Ho}\mathfrak{C}$ induces a category equivalence $(\mathfrak{C}_{cf})/\sim \to \mathrm{Ho}\mathfrak{C}$, where $\mathfrak{C}_{cf}/\sim$ is defined by $(\mathfrak{C}_{cf}/\sim)(X,Y) = \mathfrak{C}_{cf}(X,Y)/\sim$.

(2) There are canonical isomorphism

$$\mathfrak{C}(QX,RY)/\sim \xrightarrow{\cong} \mathrm{Ho}\mathfrak{C}(\gamma X,\gamma Y) \text{ for arbitrary } X,Y \in \mathfrak{C},$$

whenever QX is a cofibrant replacement of X and RY is a fibrant replacement of Y.

3) Model structure on $\mathbb{C}(\mathrm{Rep}(\mathcal{Q},\mathcal{G}))$

Let $\mathcal{Q}$ be a quiver and $\mathcal{G}$ be a Grothendieck category.

$\mathbb{C}(\mathrm{Rep}(\mathcal{Q},\mathcal{G}))$ = The category of all complexes with entries in $\mathrm{Rep}(\mathcal{Q},\mathcal{G})$.

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Notation:

(a) Let $\mathcal{F}$ be a class of objects of $\mathcal{G}$. By $(\mathcal{Q},\mathcal{F})$ we mean the class of all representations $\mathcal{X} \in \mathrm{Rep}(\mathcal{Q},\mathcal{G})$ such that $\mathcal{X}_v$ belongs to $\mathcal{F}$ for each vertex $v \in V$.

(b) By $\mathbb{C}(\mathcal{Q},\mathcal{F})$ we mean the class of all complexes $\mathcal{X}^\bullet \in \mathbb{C}(\mathrm{Rep}(\mathcal{Q},\mathcal{G}))$ such that $\mathcal{X}^i$ belongs to $(\mathcal{Q},\mathcal{F})$ for each $i \in \mathbb{Z}$.

Hovey pairs in $\mathbb{C}(\mathrm{Rep}(\mathcal{Q},\mathcal{G}))$

Proposition:

Let $\mathcal{Q}$ be an acyclic finite quiver and $\mathcal{G}$ be a Grothendieck category. Suppose that $(\mathcal{A},\mathcal{B})$ and $(\mathcal{F},\mathcal{C})$ is a Hovey pair in $\mathbb{C}(\mathcal{G})$, then

(a) $(\mathbb{C}(\mathcal{Q},\mathcal{A}),\mathbb{C}(\mathcal{Q},\mathcal{A})^{\perp})$ and $(\mathbb{C}(\mathcal{Q},\mathcal{F}),\mathbb{C}(\mathcal{Q},\mathcal{F})^{\perp})$ is a Hovey pair in $\mathbb{C}(\mathrm{Rep}(\mathcal{Q},\mathcal{G}))$.

(b) $({}^{\perp}\mathbb{C}(\mathcal{Q},\mathcal{B}),\mathbb{C}(\mathcal{Q},\mathcal{B}))$ and $({}^{\perp}\mathbb{C}(\mathcal{Q},\mathcal{C})),\mathbb{C}(\mathcal{Q},\mathcal{C}))$ is a Hovey pair in $\mathbb{C}(\mathrm{Rep}(\mathcal{Q},\mathcal{G}))$.

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Consider the following subclasses of $\mathbb{C}(\mathcal{G})$:

$$\mathbb{C}(\mathcal{F}) = \{X^\bullet \in \mathbb{C}(\mathcal{G}) \mid X^i \in \mathcal{F}, \forall i \in \mathbb{Z}\}$$

$$\mathrm{ex}(\mathcal{F}) = \mathbb{C}(\mathcal{F}) \cap \mathcal{E}.$$

$$\widetilde{\mathcal{F}} = \{X^\bullet \in \mathcal{E} \mid \mathrm{Z}^i(X^\bullet) \in \mathcal{F}, \forall i \in \mathbb{Z}\}$$

$$\widetilde{\mathcal{C}} = \{X^\bullet \in \mathcal{E} \mid \mathrm{Z}^i(X^\bullet) \in \mathcal{C}, \forall i \in \mathbb{Z}\}$$

$$\text{dg-}\widetilde{\mathcal{F}} = \{X^\bullet \in \mathbb{C}(\mathcal{F}) \mid \mathrm{Hom}(X^\bullet, C^\bullet) \text{ is exact}, \ \forall C^\bullet \in \mathcal{C}\}$$

$$\text{dg-}\widetilde{\mathcal{C}} = \{X^\bullet \in \mathbb{C}(\mathcal{C}) \mid \mathrm{Hom}(F^\bullet, X^\bullet) \text{ is exact}, \ \forall F^\bullet \in \mathcal{F}\}$$

Hovey pairs in $\mathbb{C}(\mathcal{G})$:

[Gillespie]

If $\mathcal{F}$ is closed under taking kernels of epimorphisms, then

$$(\text{dg-}\widetilde{\mathcal{F}},\widetilde{\mathcal{C}}) \quad \text{and} \quad (\widetilde{\mathcal{F}},\text{dg-}\widetilde{\mathcal{C}})$$

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are a Hovey pair.

- By putting $\mathcal{C} = \text{Inj-}R$ we have injective model structure on $\mathbb{C}(R)$ that is constructed by Joyal.
- By putting $\mathcal{F} = \text{Prj-}R$ we have projective model structure on $\mathbb{C}(R)$ that is constructed by Hovey.
- By putting $\mathcal{F} = \text{Flat-}R$ we have flat model structure on $\mathbb{C}(R)$ that is constructed by Gillespie.

Hovey pairs in $\mathbb{C}(\mathcal{G})$:

[Enochs et al.]

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$$({}^{\perp}\mathbb{C}(\mathcal{C}), \mathbb{C}(\mathcal{C})) \text{ and } ({}^{\perp}\mathrm{ex}(\mathcal{F}), \mathrm{ex}(\mathcal{F}))$$

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Corollary:

Let $(\mathcal{F},\mathcal{C})$ be a complete cotorsion pair in Grothendieck category $\mathcal{G}$ and such that the class $\mathcal{F}$ contains a generator of $\mathcal{G}$ and $\mathcal{F}$ is closed under kernels of epimorphisms. Then there is a model structure on $\mathbb{C}(\mathrm{Rep}(\mathcal{Q},\mathcal{G}))$ which we call componentwise $\widetilde{\mathcal{F}}$-model structure, where the weak equivalences are the homology isomorphisms, the cofibrations (resp. trivial cofibrations) are the monomorphisms with cokernels in $(\mathcal{Q},\text{dg-}\widetilde{\mathcal{F}})$ (resp, $(\mathcal{Q},\widetilde{\mathcal{F}})$), and the fibrations (resp. trivial fibrations) are the epimorphisms whose kernels are in $(\mathcal{Q},\widetilde{\mathcal{F}})^{\perp}$ (resp. $(\mathcal{Q},\text{dg-}\widetilde{\mathcal{F}})^{\perp}$).

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- $\mathcal{E}$: the class of exact complexes of R-modules.

Definition:

A complex $X^\bullet$ is DG-projective (resp. DG-injective) if each X^n is projective (resp. injective) and if $\mathcal{H}\mathrm{om}(X^\bullet, E^\bullet)$ (resp. $\mathcal{H}\mathrm{om}(E^\bullet, X^\bullet)$) is an exact complex for all $E^\bullet \in \mathcal{E}$. We denote by DGPrj-$R$ (resp. DGInj-R) the class of all DG-projective (resp. DG-injective) complexes of R-modules.

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- $\mathcal{E}$: the class of exact complexes of R-modules.

- Prj^{op}-$\mathcal{Q}$ = all representations $\mathcal{X}\in\mathrm{Rep}(\mathcal{Q},R)$ such that for every vertex v, $\mathcal{X}_v$ is a projective module and the map $\eta_{\mathcal{X},v}:\mathcal{X}_v\to\oplus_{s(a)=v}\mathcal{X}_{t(a)}$ is split epimorphism.

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- DGPrj^{op}-$\mathcal{Q}$ = all representation $\mathcal{X}^\bullet \in \mathrm{Rep}(\mathcal{Q},\mathbb{C}(R))$ such that for every vertex v, $\mathcal{X}_v^\bullet$ is DG-projective complexes of R-modules and the map $\eta_{\mathcal{X}^\bullet,v}$ is split epimorphism.

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We have the componentwise projective model structure on $\mathbb{C}(\mathcal{Q})$ such that

$$((\mathcal{Q}, \text{DGPrj-}R), (\mathcal{Q}, \text{DGPrj-}R)^{\perp}) \ , \ ((\mathcal{Q}, \text{Prj-}\mathbb{C}(R)), (\mathcal{Q}, \text{Prj-}\mathbb{C}(R))^{\perp})$$

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- $\mathcal{C} = (\mathcal{Q}, \text{DGPrj-}R)$
- $\mathcal{F} = (\mathcal{Q}, \text{Prj-}\mathbb{C}(R))^{\perp}$
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Clearly the homotopy category of this model structure is equal to $\mathbb{D}(\mathcal{Q})$

Homotopy relation:

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$\mathcal{C}\cap\mathcal{W}\cap\mathcal{F}$ is exactly equal to all objects $\mathcal{X}^\bullet\in\mathbb{C}(\mathcal{Q})$ such that satisfy in the following conditions:

$$(\divideontimes)\quad \begin{array}{l} (1)\ \mathcal{X}_v^\bullet\in \text{Prj-}\mathbb{C}(R) \text{ for each vertex } v\in V \\ (2)\ \text{For each vertex } v\in V, \eta_{\mathcal{X}^\bullet,v}:\mathcal{X}_v^\bullet\to\bigoplus_{s(a)=v}\mathcal{X}_{t(a)}^\bullet \\ \quad\ \text{is epimorphism.}\end{array}.$$

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(⋇) (1) $\mathcal{X}^\bullet_v \in \text{Prj-}\mathbb{C}(R)$ for each vertex $v \in V$
(2) For each vertex $v \in V, \eta_{\mathcal{X}^\bullet,v} : \mathcal{X}^\bullet_v \to \bigoplus_{s(a)=v} \mathcal{X}^\bullet_{t(a)}$.
is epimorphism.

If $\mathcal{X}^\bullet \in \mathcal{C}$, $\mathcal{Y}^\bullet \in \mathcal{F}$ and $f, g : \mathcal{X}^\bullet \to \mathcal{Y}^\bullet$ then we say that f and g are homotopic, written $f \sim_{\text{cw}} g$, if and only if $f - g$ factor through an object $\mathcal{P}^\bullet$ such that satisfying two conditions in (⋇) as above.

Lemma:

Let $\mathcal{Q}$ be an acyclic finite quiver. Consider componentwise projective model structure on $\mathbb{C}(\mathcal{Q})$. If $f, g : \mathcal{X}^\bullet \longrightarrow \mathcal{Y}^\bullet$ are two morphisms of fibrant and cofibrant objects, then $f \sim_{\mathrm{cw}} g$ if and only if $f \sim g$.

Theorem:

Let $\mathcal{Q}$ be an acyclic finite quiver. Then we have the following equivalence

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Remark:

Note that in theorem above we introduce a subcategory, differ from subcategory of DG-projective complexes of $\mathbb{K}(\mathcal{Q})$ such that equivalent to $\mathbb{D}(\mathcal{Q})$ under the canonical functor $\mathbb{K}(\mathcal{Q}) \longrightarrow \mathbb{D}(\mathcal{Q})$.

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Theorem:

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$$\mathbb{D}(\mathcal{Q}) \cong \mathbb{K}(\mathcal{Q}, \text{Prj-}R)/\mathbb{K}_{\text{ac}}(\mathcal{Q}, \text{Prj-}R)$$

where $\mathbb{K}(\mathcal{Q}, \text{Prj-}R)$ (resp. $\mathbb{K}_{\text{ac}}(\mathcal{Q}, \text{Prj-}R)$) is the homotopy category of all (resp. acyclic) complexes $\mathcal{X}^\bullet \in \mathbb{C}(\mathcal{Q}, \text{Prj-}R)$.

5) Morphism categories

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If $f : A \to B$ is an object of $\mathbf{H}(R)$ we will write either

$$(A \xrightarrow{f} B) \quad \text{or} \quad \begin{array}{c} A \\ \downarrow^{f} \\ B \end{array}$$

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- $\mathbf{S}(R)$ =the full subcategory of $\mathbf{H}(R)$ consisting of all monomorphisms in Mod-R
- $\mathbf{F}(R)$ =the full subcategory of $\mathbf{H}(R)$ consisting of all epimorphisms in Mod-R

These three categories are related by the kernel and cokernel functors:

$$\mathrm{Cok} : \mathbf{H}(R) \to \mathbf{F}(R), \qquad (A \xrightarrow{f} B) \mapsto (B \xrightarrow{\mathrm{can}} \mathrm{Coker}(f))$$

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The restrictions of the kernel and cokernel functors

$$\mathrm{Ker}:\mathbf{F}(R)\to\mathbf{S}(R),\qquad \mathrm{Cok}:\mathbf{S}(R)\to\mathbf{F}(R)$$

induce a pair of inverse equivalences.

- $\mathrm{Cok}^{\bullet} : \mathbb{C}(\mathbf{S}(R)) \longrightarrow \mathbb{C}(\mathbf{F}(R))$

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There is an equivalence

$$\overline{\mathrm{Cok}^{\bullet}} : \mathbb{K}(\text{DGPrj-}A_2) \xrightarrow{\cong} \mathbb{K}(\text{DGPrj}^{op}\text{-}A_2)$$

So we define an auto-equivalence $\psi : \mathbb{D}(\mathbf{H}(R)) \longrightarrow \mathbb{D}(\mathbf{H}(R))$ as composition of the following equivalence functors

$$\begin{array}{ccc} \mathbb{D}(\mathbf{H}(R)) \cong \mathbb{D}(A_2) & \overset{\psi}{\dashrightarrow} & \mathbb{D}(A_2) \cong \mathbb{D}(\mathbf{H}(R)) \\ \cong \uparrow & & \uparrow \cong \\ \mathbb{K}(\mathrm{DGPrj}\text{-}A_2) & \xrightarrow{\overline{\mathrm{Cok}^{\bullet}}} & \mathbb{K}(\mathrm{DGPrj}^{op}\text{-}A_2) \end{array}$$

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By using this equivalence we can define an equivalence

$$\psi_0 : \mathbf{H}(R) \longrightarrow \mathbf{H}(R)$$

such that it is an extension of equivalence between $\mathbf{S}(R)$ and $\mathbf{F}(R)$.

$$\begin{array}{ccc} \mathbb{D}(\mathbf{H}(R)) & \underset{\psi^{-1}}{\overset{\psi}{\rightleftarrows}} & \mathbb{D}(\mathbf{H}(R)) \\ \uparrow & & \uparrow \\ \mathbf{H}(R) & \underset{\psi_0^{-1}}{\overset{\psi_0}{\rightleftarrows}} & \mathbf{H}(R) \\ \uparrow & & \uparrow \\ \mathbf{S}(R) & \underset{\mathrm{Ker}}{\overset{\mathrm{Cok}}{\rightleftarrows}} & \mathbf{F}(R) \end{array}$$

- $\mathcal{H}(R)$ the category of all maps f in mod-R
- $\mathcal{S}(R)$ (resp. $\mathcal{F}(R)$) the full subcategory of $\mathcal{H}(R)$ consisting of all monomorphism (resp. epimorphism) maps.

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Lemma:

Let R be a noetherian ring. Then we have the following equivalence

$$\mathbb{K}^{-,\mathrm{b}}(\mathrm{prj}^{\mathrm{op}}\text{-}A_2) \cong \mathbb{D}^{\mathrm{b}}(\mathrm{rep}(A_2,R)) \cong \mathbb{D}^{\mathrm{b}}(\mathcal{H}(R))$$

where $\mathbb{K}^{-,\mathrm{b}}(\mathrm{prj}^{\mathrm{op}}\text{-}A_2)$ is the homotopy category of all bounded above complexes with bounded homologies and all entries in $\mathrm{prj}^{\mathrm{op}}\text{-}A_2$.

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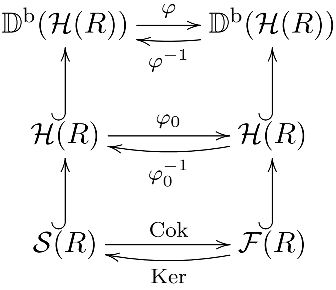

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Thank you all for your attention