Characterizations of commutative rings by their simple, cyclic, uniform and uniserial modules

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History and Preliminaries

The subject of determining structure of rings and algebras over which all modules are direct sums of certain modules (such as **simple, cyclic, unuserial, uniform** or **indecomposible** modules) has a long history.

One of the first important contributions in this direction is due to Wedderburn and Artin.

Wedderburn [2] showed that every module over a finite-dimensional K-algebra A is a direct sum of simple modules if and only if $A \cong \prod_{i=1}^{m} M_{n_i}(D_i)$ where $m, n_1, \ldots, n_m \in \mathbb{N}$ and each D_i is finite-dimensional division algebra over K. In 1927, E. Artin [1] generalizes the Wedderburn's theorem for semisimple algebras.

[1] Emil Artin, *Zur Theorie der hyperkomplexen Zahlen*, Abh. Math. Sem. U. Hamburg (1927) 251-260.

[2] J. H. MacLagan Wedderburn, *On hypercomplex numbers*, Proc. London Math. Soc. Ser. 2, **6** (1907) 77-118.

Rings, over which all modules are direct simple modules (semisimple rings)

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Wedderburn-Artin's result is a landmark in the theory of non-commutative rings. We recall this theorem as follows:

Wedderburn-Artin Theorem. For a ring R, the following conditions are equivalent:

(1) The module ${}_RR$ is a direct sum of simple modules.
(1′) The module R_R is a direct sum of simple modules.
(2) Every f.g left R-module is a direct sum of simple modules.
(2′) Every f.g right R-module is a direct sum of simple modules.
(3) Every left R-module is a direct sum of simple modules.
(3′) Every right R-module is a direct sum of simple modules.
(4) $R \cong \prod_{i=1}^{k} M_{n_i}(D_i)$ where $k, n_i \in \mathbb{N}$ and each D_i is a division ring.

Rings, over which all modules are direct sum of cyclics (Köthe rings)

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Another one is due to G. Köthe. He considered rings over which all modules are direct sums of cyclic modules.

Theorem (Köthe [3]). *Ovr er an Artinian principal ideal ring, each module is a direct sum of cyclic modules. Furthermore, if a commutative Artinian ring has the property that all its modules are direct sums of cyclic modules, then it is necessarily a principal ideal ring.*

[3] G. Köthe, Verallgemeinerte abelsche gruppen mit hyperkomplexen operatorenring, Math. Z. 39 (1935) 31-44.

Later I. S. Cohen and I. Kaplansky obtained the following.

Theorem (Cohen and Kaplansky [4]). *If R is a commutative ring such that each R-module is a direct sum of cyclic modules, then R must be an Artinian principal ideal ring.*

Result (Köthe-Cohen-Kaplansky). *A commutative ring R is a Köthe ring if and only if R is an Artinian principal ideal ring.*

The corresponding problem in the non-commutative case is still open

[4] I. S. Cohen, I. Kaplansky, *Rings for which every module is a direct sum of cyclic modules*, Math. Z. 54 (1951) 97-101.

Rings, over which all finitely generated modules are direct sum of cyclics (FGC-rings)

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The question of which commutative rings have the property that every finitely generated module is a direct sum of cyclic modules has been around for many years. We will call these rings FGC-rings. The problem originated in I. Kaplanskys papers [5] and [6].

Theorem. (See [7, Theorem 9.1]) *A commutative ring R is an FGC-ring exactly if it is a finite direct sum of commutative rings of the following kinds:*

(a) *maximal valuation rings;*

(b) *almost maximal Bézout domains;*

(c) *torch rings.*

[5] I. Kaplansky, *Elementary divisors and modules*, Trans. Amer. Math. Soc. **66**, (1949) 464-491.

[6] I. Kaplansky, *Modules over Dedekind rings and valuation rings*, Trans. Amer. Math. Soc. **72** (1952), 327-340.

[7] W. Brandal, *Commutative Rings Whose Finitely Generated Modules*

Rings, over which all modules are serial (direct sum of uniserial modules)

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Rings, over which all modules are serial, were first systematically considered by T. Nakayama. A module M is called uniserial if its submodules are linearly ordered by inclusion. Also M is called serial if it is a direct sum of uniserial modules.

Theorem. (Nakayama [8]). *If R is an Artinian serial ring and n is the nilpotency index of $J(R)$, then every left R-module is a direct sum of uniserial modules of $length \leq n$.*

The converse of the above result was also proved by Skornyakov in [9]. We record that below.

Theorem. (Skornyakov [9]). *If R is a ring such that all left R-modules are serial, then R is an Artinian serial ring.*

[8] T. Nakayama, On Frobeniusean algebras. II, Ann. Math. (2) 42 (1941), 1-21.

[9] L. A. Skornyakov, When are all modules serial, Mat. Zametki, 5 (1969), 173-182.

Rings, over which all modules are are direct sums of indecomposable modules

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In the following, the implications $(iii) \Rightarrow (i), (ii)$ are due to I. S. Cohen and I. Kaplansky [same Z. 54 (1951), 97101; MR0043073].

Theorem. (Warfield [10]). *The following conditions on a commutative ring R are shown to be equivalent:*
(i) *There is a cardinal number k such that every R-module is a direct summand of a direct sum of k-generator modules;*
(ii) *Every R-module is a direct sum of indecomposable modules;*
(iii) *R is an Artinian principal ideal ring.*

[10] Warfield, Robert B., Jr. Rings whose modules have nice decompositions. Math. Z. 125 1972 187192.

Rings, over which all modules are are direct sums of uniform modules

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Theorem. (Köthe-Cohen-Kaplansky-Nakayama-Skornyakov-Warfield
). *The following conditions on a commutative ring R are shown to be equivalent:*

(i) *Every R-module is a direct sum of cyclic modules;*

(ii) *Every R-module is a direct sum of indecomposable modules;*

(iii) *Every R-module is a direct sum of serial modules;*

(iv) *Every R-module is a direct sum of uniform modules;*

(v) *Every R-module is a direct sum of compleatly cyclic modules;*

(vi) *R is an Artinian principal ideal ring.*

An R-module M is called *completely cyclic* if each submodule of M is cyclic. Completely cyclic modules are obvious generalizations of principal ideal rings.

The First Motivation for Our Study

Now, some natural problems arise from this situation. Instead of considering rings for which all modules are direct sums of simple, cyclic, uniserial, uniform or indecomposable modules, we weaken these conditions and study the structures of rings R for which it is assumed only that the ideals or proper ideals of R are direct sums of such modules. For instance, we will discuss the following natural questions in the commutative cases:

(1) Which commutative rings have the property that every ideal is a direct sum of cyclic modules?

(2) Which commutative rings have the property that every prime ideal is a direct sum of cyclic modules?

(3) Which commutative rings have the property that every maximal ideal is a direct sum of cyclic modules?

(4) Which commutative rings have the property that every (proper) ideal is a direct sum of completly cyclic modules?

(5) Which commutative rings have the property that every ideal is serial?

(6) Which commutative rings have the property that every proper ideal is serial?

(7) Which commutative rings have the property that prime ideal is serial?

(8) Which commutative rings have the property that every maximal ideal is serial?

(9) Which commutative rings have the property that every proper ideal a direct sum of uniform modules?

(10) Which commutative rings have the property that every proper ideal a direct sum of indecomposable modules?

Theorem . (BehboodiGhorbani-Moradzadeh) *For a Noetherian commutative local ring* $(R, \mathcal{M})$ *the following statements are equivalent:*

(1) *Every ideal of* R *is a direct sum of cyclic* R*-modules.*

(2) *Every ideal of* R *is a direct sum of cyclic* R*-modules, at most two of which are not simple.*

(3) $\mathcal{M} = Rx \oplus Ry \oplus (\bigoplus_{i=1}^{n} Rw_i)$ *with each* Rw_i *a simple* R*-module*

(4) *Every ideal of* R *is a direct summand of a direct sum of cyclic* R*-modules.*

[11] M. Behboodi, A. Ghorbani and A. Moradzadeh-Dehkordi, Commutative Noetherian local rings whose ideals are direct sums of cyclic modules J. Algebra 345 (2011), 257-265

Theorem . (Behboodi-Shojsee) *For a commutative local ring $(R, \mathcal{M})$ the following statements are equivalent:*

(1) *Every ideal of R is a direct sum of cyclic R-modules.*

(2) *Every ideal of R is a direct sum of cyclic R-modules, at most two of which are not simple.*

(3) *There is an index set Λ and a set of elements $\{x, y\} \cup \{w_\lambda\}_{\lambda \in \Lambda} \subseteq R$ such that $\mathcal{M} = Rx \oplus Ry \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda)$ with each Rw_λ a simple R-module, $R/\mathrm{Ann}(x)$ and $R/\mathrm{Ann}(y)$ principal ideal rings.*

(4) *Every ideal of R is a direct summand of a direct sum of cyclic R-modules.*

[12] M. Behboodi, A. Ghorbani, A. Moradzadeh-Dehkordi, and S. H. Shojaee, On left Kothe rings and an analogue of the Cohen-Kaplansky theorem Proc. Amer. Math. Soc. 142 (2014), 2625-2631

[13] M. Behboodi and S.H. Shojaee, Commutative local rings whose ideals are direct sum of cyclic modules Algebr. Represent Theor. 17 (2014), 971-982

Theorem . (Behboodi-Heidari-Roointan) *For a commutative ring R the following statements are equivalent:*

(1) *Every proper ideal of R is a direct sum of completely cyclic R-modules.*

(2) *Either R is a principal ideal ring or $(R, \mathcal{M})$ is a local ring such that there is an index set Λ and a set of elements $\{x\} \cup \{w_\lambda\}_{\lambda \in \Lambda} \subseteq R$ such that $\mathcal{M} = Rx \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda)$ with each Rw_λ a simple R-module and $R/\mathrm{Ann}(x)$ a principal ideal ring.*

Theorems (Behboodi-Heidari) *The following statements are equivalent for a commutative ring R:*

(1) *Every proper ideal of R is serial.*

(2) *Either R is serial or R is a local ring with maximal ideal $\mathcal{M}$ such that there is an index set Λ, a set of elements $\{w_\lambda\}_{\lambda\in\Lambda} \subseteq R$ and a uniserial ideal U of R such that $\mathcal{M} = U \oplus (\bigoplus_{\lambda\in\Lambda} Rw_\lambda)$ with: each Rw_λ a simple R-module.*

Theorems (Behboodi-Heidari) *Let R be a commutative Noetherian ring. Then the following statements are equivalent:*

(1) *Every proper ideal of R is serial.*

(2) *Either R is a finite direct products of discrete valuation domains and special principal rings or $(R, \mathcal{M})$ is a local ring such that $\mathcal{M} = Rx \oplus (\bigoplus_{i=1}^{n} Rw_i)$ with each Rw_i a simple R-module and $R/\mathrm{Ann}(x)$ is a principal ideal ring.*

(3) *Either R is a serial ring or $(R, \mathcal{M})$ is a local ring such that $\mathcal{M} = U \oplus (\bigoplus_{i=1}^{n} Rw_i)$ with each Rw_i a simple R-module and U is a uniserial R-module.*

(4) *There is an integer $n \geq 1$ such that every proper ideal of R is a direct sum of at most n uniserial R-module.*

Theorem (Behboodi-Daneshvar-Vedadi). The following statements are equivalent for a commutative ring R:

(1) Every proper ideal of R is a virtually semisimple R-module.

(2) Every proper ideal of R is a direct sum of virtually simple R-modules.

(3) R is one of the following forms:

(i) R is virtually semisimple.
(ii) $(R, \mathcal{M})$ is a local ring such that there exist an index set Λ and a set of ele- ments $\{x\} \cup \{y_\lambda\}_{\lambda \in \Lambda}$ of R such that $\mathcal{M} = Rx \oplus (\bigoplus_{\lambda \in \Lambda} Ry_\lambda)$ with: every Ry_λ a simple R-module and Rx a virtually simple R-module.

(4) R is one of the following forms:

(i) R is a finite direct product of principal ideal domains.
(ii) $(R, \mathcal{M})$ is a local ring with $\mathcal{M} = \mathrm{Soc}(R)$.
(iii) $(R, \mathcal{M})$ is a local ring with $\mathcal{M} = Rx \oplus \mathrm{Soc}(R)$ for some $0 \neq x \in R$ and every proper ideal of R is semisimple or is isomorphic to $Rx \oplus W'$ where $W' \leq \mathrm{Soc}(R)$.

Rings, Whose Proper Ideals Are Direct Sums of Uniform Modules

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Theorem (Behboodi-Asgari). The following statements are equivalent for a commutative local ring R.

(1) Every proper ideal of R is a direct sum of uniform modules.

(2) Either R is a (finite) direct sum of uniform modules or $(R, \mathcal{M})$ is a local ring and $\mathcal{M} = U \oplus T$, where U is uniform and T is semisimple.

Moreover, in the local case every proper ideal I of R has a decomposition $I = V \oplus W$, where V is uniform and W is semisimple.

The Second Motivation for Our Study

Using $\cong$ instant = for generalizing the notions of *simple, semisimple, uniserial, uniform, indecomposable and etl.*

For instance, we introduced and study *almost serial modules* as a generalizations of serial modules.

Note: A module M is **semisimple** if and only if **every submodule of M is a direct summand.** In fact; a semisimple module is a type of module that **can be understood easily from its parts.**

Motivation: This property motivates us to study rings and modules for which **every submodule is isomorphic to a direct summand.**

In fact, we give the following generalization of semisimple modules.

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Definition. We say that an R-module M is **virtually semisimple** if each submodule of M is isomorphic to a direct summand of M.

Note: **A module M is semisimple if and only if it is a direct sum (finite or not) of simple modules.**

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This property motivates us to give the following definition.

Definition. We say that an R-module M is **virtually simple** or (**isosimple**) if $M \neq (0)$ and $N \cong M$ for each $(0) \neq N \leq M$.

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Note 1: A direct sum of virtually simple modules need not be virtually semisimple.

Note 2: A virtually semisimple module need not be a direct sum of virtually simple modules.

Note 3: A submodule of a virtually semisimple modules need not be virtually semisimple.

Note 4: The class of virtually semisimple modules is not closed under homomorphic image.

Counterexamples

Example 1. Let F be a field and we set $R = F[[x, y]]/ < xy >$. The maximal ideal of R is a direct sum of two virtually simple R-module, but it is not virtually semisimple.

Example 2. The $\mathbb{Z}$-module $\mathbb{Z}$ is clearly virtually semisimple, but the $\mathbb{Z}$-module $\mathbb{Z}/4\mathbb{Z}$ is not virtually semisimple.

Example 3. Let $R = A_1(F)$, the first Weyl algebra over a field F with characteristic zero, and let $S = M_2(R)$. By using several proposition, we can see that S is a virtually semisimple left S-module, but there exists a left ideal I of R such I is not virtually semisimple.

Example 4. By the notations of Example 3, $S = M_2(R)$ is a virtually semisimple left S-module, but it is not a direct sum of virtually simple modules.

The above notes motivates us to give the following definitions

Definitions. *We say that an R-module M is:*

- ***completely virtually semisimple*** *if each submodule of M is a virtually semisimple module.*

- ***fully virtually semisimple*** *if each factor module of M is a virtually semisimple module.*

- ***isosemisimple*** *if M is a direct sum of virtually simple (isosimple) modules.*

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Definitions. *We say that a ring R is a:*

- ***left virtually semisimple ring*** *if ${}_RR$ is a virtually semisimple module.*

- ***left completely virtually semisimple ring*** *if ${}_RR$ is a completely virtually semisimple module.*

- ***left fully virtually semisimple ring*** *if ${}_RR$ is a fully virtually semisimple module.*

- ***left isosemisimple ring*** *if R_R is a isoemisimple module.*

The Wedderburn-Artin Structure Theorem motivated us to study the following interesting natural questions:

Question 1. Characterize a left virtually semisimple ring.

Question 2. Characterize a left fully virtually semisimple ring.

Question 3. Characterize a left completely virtually semisimple ring.

Question 4. Characterize a left isosemisimple ring.

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Question 5. Is every left virtually semisimple ring also a right virtually semisimple?

Question 6. Is every left fully virtually semisimple ring also a right fully virtually semisimple?

Question 7. Is every left completely virtually semisimple ring also a right completely virtually semisimple?

Question 8. Is every left isosemisimple ring also a right isosemisimple?

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Question 9. Characterize rings over which all left (right) modules are virtually semisimple.

Question 10. Characterize rings over which all finitely generated left (right) modules are virtually semisimple.

Question 11. Characterize rings over which all left (right) modules are direct sums of virtually isimple modules.

Question 12. Characterize rings over which all injective modules are virtually semisimple.

Question 13. Characterize rings over which all projective modules are virtually semisimple.

Question 14. Whether the Krull-Schmidt Theorem holds for direct sums of virtually simple modules?

and others................................

Some of our main results are as follows:

When every R-module is virtually semisimple.

Proposition. *Every quasi-injective virtually semisimple module M is semisimple.*

Corollary. *The following conditions are equivalent for a ring R.*

(1) *Every left (right) R-module is a direct sum of virtually simple module.*

(2) *Every left (right) R-module is virtually semisimple.*

(3) *R is a semisimple ring.*

When R is a left isosemisimple ring.

When is a left completely virtually semisimple ring.

Theorem. *The following statements are equivalent for a ring R.*

(1) *The left R-module R is a direct sum of virtually simple modules.*

(2) *R is a left completely virtually semisimple ring.*

(3) *$R \cong \prod_{i=1}^{k} M_{n_i}(D_i)$ where $k, n_1, ..., n_k \in \mathbb{N}$ and each D_i is a principal left ideal domain.*

Moreover, in the statement (3), the integers k, $n_1, ..., n_k$ and the principal left ideal domains $D_1, ..., D_k$ are uniquely determined (up to isomorphism) by R.

We recall that a ring R is called a *left* (resp., *right*) V-*ring* if each simple left (resp., right) R-module is injective. We say that R is V-*ring* if it is both left and right V-ring.

Remark. Although there exists an example of a non-domain which is a left V-ring but not a right V-ring, the question whether a left (right) V-domain is a right (left) V-domain remains open in general. See [4], where the authors proved that the answer is positive for principal ideal domains.

[3] S. K. Jain, K. S. Ashish, and A. T. Askar, *Cyclic modules and the structure of rings*, Oxford University Press, 2012.

When all f.g left R-modules are virtually semisimple.

When all f.g left R-modules are completely virtually semisimple.

Theorem. *The following statements are equivalent for a ring R.*

(1) *All finitely generated left R-modules are virtually semisimple.*

(1′) *All finitely generated right R-modules are virtually semisimple.*

(2) *All finitely generated left R-modules are completely virtually semisimple.*

(2′) *All finitely generated right R-modules are completely virtually semisimple.*

(3) $R \cong \prod_{i=1}^{k} M_{n_i}(D_i)$ *where each D_i is a principal ideal V-domain.*

Cozzens's Example

The following example, originally from Cozzens [2], shows that there are principal ideal V-domains which are not division rings.

Example. Let K be a universal differential field with derivation d and let $D = K[y; d]$ denote the ring of differential polynomials in the indeterminate y with coefficients in K, i.e., the additive group of $K[y; d]$ is the additive group of the ring of polynomials in the indeterminate y with coefficients in field K, and multiplication in D is defined by: $ya = ay + d(a)$ for all a in K. It is shown that D is both left and right principal ideal domain, the simple left D-modules are precisely of the form $V_a = D/D(y - a)$ where a in K and each simple left D-module is injective . Hence D is a left V-ring. Similarly, D is a right V-ring.

[4] J. Cozzens, *Ph.D. thesis*, Rutgers, The State University, New Brunswick, New Jersey, 1969.

Krull-Schmidt Theorem holds for direct sums of virtually simple modules

Theorem. *Let* $V_1 \oplus \cdots \oplus V_n = M = U_1 \oplus \cdots \oplus U_m$ *where all* V_i*'s and* U_j*'s are virtually simple modules. Then* $n = m$ *and there is a permutation* σ *on* $\{1, ..., n\}$ *such that* $U_i \cong V_{\sigma(i)}$.

${}_RM$ Completely virtually semisimple iff ${}_RM$ is a direct sum of virtually simple modules.

As first important application of this theory, we give the following.

Proposition. *Every finitely generated completely virtually semisimple module is a direct sum of virtually simple modules.*
Up to a permutation, the virtually simple components in such a direct sum are uniquely determined up to isomorphism.

Second important application of this theory:

A fundamental structure theorem.

Theorem. *The following statements are equivalent for a ring R.*

(1) *Every finitely generated left R-modules is a direct sum of virtually simple modules.*

(1′) *Every finitely generated right R-modules is a direct sum of virtually simple modules.*

(2) $R \cong \prod_{i=1}^{k} M_{n_i}(D_i)$ *where* $k, n_1, ..., n_k \in \mathbb{N}$ *and each* D_i *is a principal ideal V-domain.*

(3) *Every finitely generated left R-modules is uniquely (up to isomorphism) a direct sum of cyclic left R-modules that are either simple or virtually simple direct summand of ${}_RR$.*

(3′)*Every finitely generated right R-modules is uniquely (up to isomorphism) a direct sum of cyclic left R-modules that are either simple or virtually simple direct summand of ${}_RR$.*

Finitely generated virtually semisimple modules over commutative rings

Proposition. *Let M be a finitely generated module over commutative ring R. Then M is a direct sum of virtually simple modules if and only if $M \cong \bigoplus_{i=1}^{k} R/P_i$ where $k \in \mathbb{N}$ and each R/P_i is principal ideal domain.*

The following is a structure theorem for finitely generated virtually semisimple modules over commutative rings.

Theorem. *Let M be a finitely generated module over a commutative ring R. Then the following conditions are equivalent:*

(1) *M is virtually semisimple.*

(2) *M is completely virtually semisimple.*

(3) *$M \cong \bigoplus_{i=1}^{n} R/P_i$ where R/P_i is a principal ideal domain for all i and for each pair i, j either P_i, P_j are comparable or $P_i + P_j = R$.*

Proposition. *Let M be a module over a commutative ring R. Then M is a fully virtually semisimple if and only if M is semisimple.*

Thank you!