Cotangent cohomology of quadratic monomial ideals

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Introduction

For any $\mathbb{k}$-algebra A, there is a cohomology theory providing modules $T^i(A)$ for $i = 0, 1, 2$.

- $T^0(A) = \mathrm{Der}_{\mathbb{k}}(A, A)$.
- $T^1(A)$ characterizes the first order deformations of A.
- $T^2(A)$ contains the obstructions for lifting these deformations to decent parameter spaces.

In this talk we study these modules when A is the quotient of a polynomial ring $\mathbb{k}[x_1, \ldots, x_n]$ by a quadratic monomial ideal I.

A quadratic monomial ideal is an ideal generated by monomials of degree 2.

A quadratic monomial ideal I in a polynomial ring $R = \Bbbk[x_1, \ldots, x_n]$ gives rise to a (not necessarily simple) graph $G = (V(G), E(G))$ where $V(G) = \{x_1, \ldots, x_n\}$ and $E(G) = \{\{x_i, x_j\} \mid x_i x_j \in I\}$.

We use the combinatorics of the corresponding graph to describe

1. a generating set for the first cotangent cohomology module of the ring R/I;
2. characterize rigid edge ideals of graphs;
3. vanishing results for the second cotangent cohomology module.

Polarization

Let I be a monomial ideal in a polynomial ring $R = \Bbbk[X]$ where $X = \{x_1, \ldots, x_n\}$ is a set of indeterminates.
Let $G(I) = \{g_1, \ldots, g_r\}$ be the set of minimal generators of I and for each $i = 1, \ldots, n$ let e_i be the highest power of x_i among the elements of $G(I)$.
Let $S = R[y_{i,j} | i = 1, \ldots, n; j = 2, \ldots, e_i]$ be a new polynomial ring containing R.
Now for $g \in G(I)$ if $g = x_1^{a_1} \cdots x_n^{a_n}$ then define

$$\tilde{g} = x_1^{\min\{a_1,1\}} y_{1,2} \cdots y_{1,a_1} \cdots x_n^{\min\{a_n,1\}} y_{n,2} \cdots y_{n,a_n}.$$

The ideal J defined by $\tilde{g}_1, \ldots, \tilde{g}_n$ in S is called the *polarization of I*.
Note that the variable differences $x_i - y_{i,j}$ are regular elements of S/J.

Separations

Let I be a monomial ideal in the polynomial ring $R = \Bbbk[X]$. Let $x \in X$ be an indeterminate of R and let y be an indeterminate over R.

A monomial ideal J in $S = R[y]$ is a called a *separation* of I at the variable x if

1. I is the image of J under the $\Bbbk$-algebra map $S \to R$ sending y to x and any other variable of S to itself,
2. x and y occur in some minimal generators of J and
3. $y - x$ is a regular element of the quotient ring S/J.

We shall call a succession of separations also a *separation*.

Polarization is a separation.

Let I be an ideal in a polynomial ring R. Let $J \subseteq R[y]$ be a separation of I at variable x.
We apply the coordinate change $y \rightsquigarrow x + t$ and we get an ideal $\tilde{I} \subseteq R[t]$ such that $R[y]/J \cong R[t]/\tilde{I}$.
Since $y - x$ is a nonzero divisor on $R[y]/J$, t is a nonzero divisor on $R[t]/\tilde{I}$. Hence $R[t]/\tilde{I}$ is flat over $\mathbb{k}[t]$.
Furthermore,

$$\frac{R[t]}{\tilde{I}} \otimes_{\mathbb{k}[t]} \frac{\mathbb{k}[t]}{(t)} = \frac{R[t]}{\tilde{I}} \otimes_{\mathrm{R}[t]} R[t] \otimes_{\mathbb{k}[t]} \frac{\mathbb{k}[t]}{(t)} = \frac{R[t]}{\tilde{I}} \otimes_{\mathrm{R}[t]} \frac{R[t]}{(t)} = \frac{R}{I}$$

Deformations

Let I be an ideal in a $\Bbbk$-algebra R. Let B be another $\Bbbk$-algebra with a distinguished $\Bbbk$-point $b \in \operatorname{Spec} B$ corresponding to a morphism $B \to \Bbbk$. A *deformation* of I over B is an ideal J in $R \otimes_{\Bbbk} B$ satisfying the following

1. $(R \otimes_{\Bbbk} B)/J$ is flat over B,
2. the natural map $R \otimes_{\Bbbk} B \to R$ induces an isomorphism $(R \otimes_{\Bbbk} B)/J \otimes_B \Bbbk \to R/I$.

If B is a local Artinian $\Bbbk$-algebra such that $B/m_B \cong \Bbbk$ then a deformation over B is called an *infinitesimal deformation*. A deformation over the local Artinian ring $\Bbbk[\epsilon] = \Bbbk[t]/(t^2)$ is called a *first order deformation* of R/I.

A separation J of an ideal I at a variable x is a flat deformation of I over the polynomial ring $\Bbbk[t]$.

One can iterate the separation of an ideal I until getting an *inseparable* ideal J in a larger polynomial ring.

Question

Does the ideal J have any further deformations?

The parameter ring $\mathbb{k}[t]$ is quite big to study the deformations of an ideal.

Instead it is better to start with the smallest possible parameter ring, i.e. the ring of *dual numbers* $\mathbb{k}[\epsilon] = \mathbb{k}[t]/(t^2)$.

Suppose $J \subseteq R[\epsilon]$ is an ideal such that

$$R[\epsilon]/J \otimes_{\Bbbk[\epsilon]} \Bbbk[t]/(t) \cong R/I.$$

If $I = (f_1, \ldots, f_r)$ then $J = (f_1 + g_1\epsilon, \ldots, f_r + g_r\epsilon)$ and $R[\epsilon]/J$ is flat over $\Bbbk[\epsilon]$ if and only if the map sending $f_i \mapsto g_i + I$ defines a well-defined R-module homomorphism $I \to R/I$.

Therefore the set of first order deformations of R/I are in one-to-one correspondence with elements of $\mathrm{Hom}_R(I, R/I)$.

First cotangent cohomology module

Let $I = (f_1, \ldots, f_r)$ be an ideal in a polynomial ring R and let $A = R/I$. Let $\mathrm{Der}_{\Bbbk}(R)$ be the module of derivations of R.
There is a map

$$\delta^* : \mathrm{Der}_{\Bbbk}(R) \longrightarrow \mathrm{Hom}_R(I, R/I)$$

which sends ∂ to the homomorphism sending $f_i \mapsto \partial f_i + I$ for $i = 1 \ldots, r$.
The cokernel of the map δ^* is called the *first cotangent cohomology module* of A and it is denoted by $T^1(A)$.
A homomorphisms in $\mathrm{Hom}_R(I, R/I)$ is called a *trivial first order* deformation if it lies in the image of δ^*.
Therefore $T^1(A)$ characterizes all the nontrivial first order deformations of A.
A ring $A = R/I$ as well as the ideal I is called *rigid* if $T^1(A)$ vanishes.

Deformation theory of square-free monomial ideals have been studied by Klaus Altmann and Jan Arthur Christophersen, [2, 3].
If I is a square-free monomial ideal in a polynomial ring $R = \Bbbk[x_1, \ldots, x_n]$ then $T^1(R/I)$ is $\mathbb{Z}^n$-graded.

Let $\mathbf{c} \in \mathbb{Z}^n$ be a multidegree and suppose $\mathbf{c} = \mathbf{a} - \mathbf{b}$ with $\mathbf{a}, \mathbf{b} \in \mathbb{N}^n$ and $\operatorname{Supp} \mathbf{a} \cap \operatorname{Supp} \mathbf{b} = \emptyset$. Recall that for a multidegree $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, $\operatorname{Supp} \mathbf{a} = \{i \in [n] | a_i \neq 0\}$. We have

Theorem (Altmann and Christophersen - 2004, [2])

1. *if* $\mathbf{b} \notin \{0, 1\}^n$ *then* $T^1(R/I)_{\mathbf{a}-\mathbf{b}} = 0$;
2. *if* $\mathbf{b} \in \{0, 1\}^n$ *then* $T^1(R/I)_{\mathbf{a}-\mathbf{b}} = T^1(\operatorname{lk}_\Delta \operatorname{Supp} \mathbf{a})_{-\mathbf{b}}$.

First order deformations of graphs - Type I

Let ab be an edge of G. The vertices a and b of edge ab are not necessarily distinct.

For a vertex v let $N(v)$ denote the neighborhood of v.

Let $\Lambda = (N(a)\backslash\{b\}) \cup (N(b)\backslash\{a\})$.

For any $g \in \Lambda$ let Λ_g be the set all vertices adjacent to g other than a and b, i.e. $\Lambda_g = N(g)\backslash\{a, b\}$.

Let $|\Lambda| = d$. Any ordered d-tuple $(x_1, \ldots, x_d)$ in $\prod_{g\in\Lambda} \Lambda_g$ gives a monomial $x_1 \cdots x_d$.

Now define Λ_{ab} as

$$\Lambda_{ab} = \{\sqrt{m} \mid m \in \prod_{g\in\Lambda} \Lambda_g\}.$$

If $\Lambda = \emptyset$ that is when ab is an isolated edge or an isolated loop then $\Lambda_{ab} = \{1\}$.

Now for $\lambda \in \Lambda_{ab}$, we define a linear map

$$\phi_{ab}^{\lambda} : I_2 \to R/I$$

which sends ab to λ and any other minimal generator of I to zero.

Lemma

The map ϕ_{ab}^{λ} algebraically extends to a well-defined homomorphism in $\mathrm{Hom}_R(I, R/I)$. Furthermore, if ϕ_{ab}^{λ} is nonzero then it corresponds to a nontrivial deformation.

For any $\lambda \in \Lambda_{ab}$, we call ϕ_{ab}^{λ} a *type I deformation associated with the edge ab*.

When there is no confusion we denote ϕ_{ab}^{λ} simply by $ab \mapsto \lambda$.

First order deformations of graphs - Type II

Let $a \in V(G)$ be a vertex. Let $N(a)$ be the neighborhood of a. Let $G_{N(a)}$ be the induced subgraph of G on the vertex set $N(a)$. We denote the complementary graph of the underlying simple graph of $G_{N(a)}$ by $\overline{N}(a)$. Let L be a nonempty subset of the vertex set of $\overline{N}(a)$.

Let $\Gamma(L)$ be the set of all vertices in $\overline{N}(a)$ which are adjacent to some vertex of L but does not belong to L.

For any $g \in \Gamma(L)$, let Γ_g be the set of vertices adjacent to g other than a.

Let

$$\Gamma_{a,L} = \{\sqrt{m} \mid m \in \prod_{g \in \Gamma} \Gamma_g\}.$$

Now define a linear map

$$\phi^{\lambda}_{a,L} : I_2 \to R/I$$

by

$$\phi^{\lambda}_{a,L}(e) = \begin{cases} \lambda x & e = ax \text{ and } x \in L \\ 0 & \text{otherwise.} \end{cases}$$

Lemma

The map $\phi^{\lambda}_{a,L}$ algebraically extends to a well-defined homomorphism in $\mathrm{Hom}_R(I, R/I)$.

For any $\lambda \in \Gamma_{a,L}$ we call $\phi^{\lambda}_{a,L}$ a *type II deformation associated with the vertex a*.

Theorem

As ab varies in the set of edges of G and a varies in the set of vertices of G, the homomorphisms ϕ^{λ}_{ab} for $\lambda \in \Lambda_{ab}$ alongside with the homomorphisms $\phi^{\lambda}_{a,L}$ for nonempty $L \subseteq V(\overline{N}(a))$ and $\lambda \in \Gamma_{a,L}$ define a generating set for $\mathrm{Hom}_R(I, R/I)$.

Recall that a *leaf vertex* is a vertex of degree 1. We call an edge a *leaf* if it contains a leaf vertex.

Lemma

Let a be a vertex of graph G with no loop on it. Suppose either

1. *vertex a does not lie on any 3-cycle, or*
2. *vertex a belongs to a leaf,*

then the derivation $\frac{\partial}{\partial a}$ is the only deformation of type II associated with a.

We call an edge having a common vertex with a leaf a *branch*.

Lemma

If an edge ab is a branch then there is no nonzero type I deformation associated with edge ab.

Rigidity of edge ideals of graphs

Suppose G is not a simple graph and G has a loop on some vertex x.
If $N(x) \neq \{x\}$, then the separation at x is a nontrivial deformation and G is not algebraically rigid.
Now suppose $N(x) = \{x\}$, i.e. the loop on x is an isolated loop. In this case the separation at x is a trivial deformation but the type I deformation $\phi^1_{a^2}$ is a nontrivial deformation.
It follows that non square-free quadratic monomial ideals are never rigid.

Let G be a simple graph on vertex set $x_1, \ldots, x_n$ and let $R = \mathbb{k}[x_1, \ldots, x_n]$ be a polynomial ring on variables x_i.
The *neighborhood* of a set X of vertices of G is defined to be $N(X) = \cup_{x \in X} N(x)$, and the *closed neighborhood* of X is defined to be $N[X] = X \cup N(X)$.
We also denote the induced subgraph of G on the vertex set $V(G) \backslash X$ by $G \backslash X$.

Theorem (Altmann, Bigdeli, Herzog and Lu - 2016, [1])

$R/I(G)$ is rigid if and only if any independent subset X of G satisfies both of the following conditions.

1. *$\overline{N}(x)$ is connected for all vertex x of graph $G \backslash N[X]$;*
2. *$G \backslash N[X]$ contains no isolated edge.*

Characterization of rigid graphs

Theorem

Let I be the edge ideal of a simple graph G. I is rigid if and only if

1. *for each edge ab of G,*
$$\prod_{x \in \Lambda^{ab}} \Lambda_x \subseteq I, \text{ and}$$
2. *for each vertex a of G and subset $L \subseteq V(\overline{N}(a))$,*
$$\Big(\prod_{x \in \Gamma(L)} \Gamma_x\Big) \times \big(V(\overline{N}(a)) \backslash (L \cup \Gamma(L)\big) \subseteq I.$$

Theorem (Altmann, Bigdeli, Herzog and Lu - 2016, [1])

Let G be a simple graph such that G does not contain any induced cycle of length 4,5 or 6. Then G is rigid if and only if each edge of G is a branch and each vertex of a 3-cycle of G belongs to a leaf.

Second cotangent cohomology module

Let

$$0 \longrightarrow K \longrightarrow R^m \xrightarrow{j} R \longrightarrow A \longrightarrow 0$$

be an exact sequence presenting A as an R-module. Let $\epsilon_1, \ldots, \epsilon_m$ be a basis for R^m and let K_0 be the submodule of K generated by relations $j(\epsilon_i)\epsilon_j - j(\epsilon_j)\epsilon_i$ for all $i \neq j$, $1 \leq i, j \leq m$. These relations are called the Koszul relations. The cokernel of the map

$$\Phi : \text{Hom}_R(R^m, A) \longrightarrow \text{Hom}_A(K/K_0, A)$$

is called the *second cotangent cohomology module* of A and is denoted by $T^2(A)$.

We fix a total order $\prec$ on $E(G)$ the edge set of G. For $ab \in E(G)$, let ϵ_{ab} be the standard basis of R^m. As a submodule of R^m, K is generated by relations $r_{ab,bc}$ and $r_{ab,cd}$ defined below,

1. for $ab, bc \in I$ with $ab \prec bc$, $r_{ab,bc} = r_{bc,ab} = -c\epsilon_{ab} + a\epsilon_{bc}$ and,
2. for $ab, cd \in I$ with $ab \prec cd$, $r_{ab,cd} = r_{cd,ab} = -cd\epsilon_{ab} + ab\epsilon_{cd}$.

The relations of second form are Koszul relations and they vanish in the sub-quotient K/K_0.

Therefore any minimal generator of K/K_0 can be denoted by two adjacent edges ab and bc of G.

For a subset F of edges of G and for an edge $ab \in F$, $\sigma(F, ab)$ is defined to be the number of elements less than ab in the totally ordered set $(F, \prec)$.

Let I be the edge ideal of a graph G and let ab be an edge of G. Let L_a (resp. L_b) be a subset of $N(a)\backslash\{b\}$ (resp. $N(b)\backslash\{a\}$) and $\overline{L_a}$ (resp. $\overline{L_b}$) be its complement. We shall choose L_a and L_b such that for any vertex $z \in N(a) \cap N(b)$ we have $z \in L_a$ if and only if $z \in L_b$.
We define

$$\Delta^a = \{x \in \overline{L_a} \mid \exists\ y \in L_b \text{ s.t. } xy \notin I \text{ or } \exists y \in L_a \text{ s.t. } xy \notin I\}$$

and similarly

$$\Delta^b = \{x \in \overline{L_b} \mid \exists\ y \in L_a \text{ s.t. } xy \notin I \text{ or } \exists y \in L_b \text{ s.t. } xy \notin I\}.$$

Let $\Delta = \Delta^a \cup \Delta^b$. We define homomorphisms in $\mathrm{Hom}_R(K/K_0, R/I)$ without making any further choices.

For any $x \in \Delta$ let Δ_x to be the set $N(x) \backslash \{a, b\}$. Now define

$$\Delta_{L_a, L_b} = \{\sqrt{m} \mid m \in \prod_{x \in \Delta} \Delta_x\}.$$

The generators of K/K_0 are in degree 3. Now for any $\lambda \in \Delta_{L_a, L_b}$ define a $\Bbbk$-linear map

$$\phi^{\lambda}_{L_a, L_b} : (K/K_0)_3 \longrightarrow R/I$$

by

$$\phi^{\lambda}_{L_a, L_b}(r_{e,e'}) = \begin{cases} (-1)^{\sigma(\{ab, ax\}, ax)} \lambda x & e = ab, e' = ax \text{ and } x \in L_a \\ (-1)^{\sigma(\{ab, bx\}, bx)} \lambda x & e = ab, e' = bx \text{ and } x \in L_b \\ 0 & \text{otherwise.} \end{cases}$$

Lemma

For L_a and L_b as above and for any $\lambda \in \Delta_{L_a,L_b}$, $\phi^{\lambda}_{L_a,L_b}$ algebraically extends to a well-defined homomorphism in $\mathrm{Hom}_R(K/K_0, R/I)$.

Let G be a graph with no 3-cycles. As ab varies in the edge set $E(G)$, the homomorphisms $\phi^{\lambda}_{L_a,L_b}$ form a generating set for $T^2(R/I(G))$.

Vanishing of the second cotangent cohomology

Theorem

Let G be a graph with no 3-cycles and let I be its edge ideal. The second cotangent cohomology module $T^2(R/I)$ vanishes if and only if for any edge ab of G and any L_a and L_b as above we have

$$\prod_{x \in \Delta} \Delta_x \times ((N(a) \cup N(b)) \setminus (\{a, b\} \cup L_a \cup L_b \cup \Delta) \subseteq I.$$

Corollary

If G is a graph with no induced 3 or 4 cycles then $T^2(R/I(G))$ vanishes.

Thank you for your attention.

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