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On the Ratliff-Rush Closure of Ideals

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Let R be a Noetherian ring with identity and I be an ideal in R. The Ratliff-Rush closure of I is defined by $\widetilde{I} = \bigcup_{n=1}^{\infty}(I^n :_R I^{n-1})$. A regular ideal I for which $\widetilde{I} = I$ is called Ratliff-Rush closed. The Ratliff-Rush closure of ideals is a good operation with respect to many properties, it carries information about associated primes of powers of ideals, about zerodivisors in the associated graded ring, preserves the Hilbert function of zero-dimensional ideals, etc. In this talk, we review some of the known properties, and compares properties of Ratliff-Rush closure of an ideal with its integral closure. For a proper regular ideal I, we denote by $G(I)$ the graded ring (or form ring) $R = I \bigoplus I/I^2 \bigoplus I^2/I^3 \bigoplus \ldots$. All powers of I are Ratliff-Rush ideals if and only if its positively graded ideal $G(I)_+ = I/I^2 \bigoplus I^2/I^3 \bigoplus \ldots$ contains a nonzerodivisor. An ideal $J \subseteq I$ is called a reduction of I if $I^{n+1} = JI^n$ for some $n \in \mathbb{N}$. A reduction J is called a minimal reduction of I if it does not properly contain a reduction of I. The least such n is called the reduction number of I with respect to J, and denoted by $r_J(I)$. Northcott and Rees proved that minimal reductions of I always exist if the residue field of R is infinite. Rossi and Swanson ([2]) denoted by $\widetilde{r_J(I)} := \min\{n : \widetilde{I^{m+1}} = J\widetilde{I^m} form \geq n\}$ and they called it the Ratliff-Rush reduction number of I with respect to J.

An element x of the ideal I is said to be superficial element for I if there exists a nonnegative integer k such that $(I^{m+1} : x) \bigcap I^k = I^m$ for all $m \geq k$ and so, with our assumption, there exists a non-negative integer k_0 such that $(I^{m+1} : x) = I^m$ for all $m \geq k_0$. A set of elements $x_1, \ldots, x_s \in I$ is a superficial sequence of I if x_i is a superficial element of $I/(x_1, \ldots, x_{i-1})$ for $i = 1, \ldots, s$. Swanson ([3]) proved that if $x_1, \ldots, x_d$ is a superficial sequence of I, then $J = (x_1, \ldots, x_d)$ is a minimal reduction of I. Elias ([1]) defined that a superficial sequence $x_1, \ldots, x_s$ of I is tame if x_i is a superficial element of I, for all $i = 1, \ldots, s$. Also, he proved that a tame superficial sequence always exists. In this talk, we will show that if $(R, \mathfrak{m})$ is a Cohen-Macaulay local ring of dimension $d \geq 3$, I an $\mathfrak{m}$-primary ideal $x_1, \ldots, x_d$ a tame superficial sequence of I and $J = (x_1, \ldots, x_d)$, then $\widetilde{r_J(I)} \leq r_J(I)$. This answer to a question that made by Rossi and Swanson in ([2], Question 4.6). The Hilbert-Samuel function of I is the numerical function that measures the growth of the length of R/I^n for all $n \in \mathbb{N}$. For all n large this function $\lambda(R/I^n)$ is a polynomial in n . Finally, in the last section, we review some facts on Hilbert function of the Ratliff-Rush closure of an ideal.

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References

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