REFERENCES AND COMMENTS ON WORKSHOP

RUSS WOODROOFE

Overview and examples. The connection between simplicial complexes and square-free monomial ideals can be found in any number of sources, for example [18, 21, 39, 37]. A useful general reference from the topological viewpoint is [4]. A nice book written from the rather specific point of view (applications of the Borsuk-Ulam theorem of topology) is [23].

A thorough discussion of matroids from the standpoint of topological and algebraic combinatorics may be found in [41, Chapter 7]. There is a smaller amount of material in [37, Chapter III.3]; matroids are also mentioned in [39] and more lightly in [18]. The standard reference and textbook on the topic is [26].

Independence complexes appear in the graph theory literature, mostly through questions about f-vector. Some selected papers are [13, 22].

The topology of the order complex of a poset is particularly important in topological and algebraic combinatorics. Useful tools are surveyed in [4]. A general overview of results in the field is in [40]. The result connecting the probability of generating a vector space (group) with the topology of the subgroup lattice was first observed by Hall [17]; it is explained clearly and in modern notation in [10]. Möbius inversion in general is discussed in many sources: an early one is [30], but you can also find this topic in [8, 38, 40].

The result that a finite group is solvable if and only if its subgroup lattice is sequentially Cohen-Macaulay was essentially proved in [34]. In this paper, Shareshian discusses only shellability. That the subgroup lattice of a minimal simple group is not sCM follows since the possible h-triangles of shellable and sCM complexes are the same [15], and since CM complexes are strongly connected [4, Proposition 11.7]. The latter is used only for $SL_3(\mathbb{F}_3)$, and an alternative approach for this group can be found in [28]. A direct connection with commutative algebra would be especially interesting in this direction, particularly if the reliance on the classification could be removed. In the other direction ("solvable $\implies$ sCM"), there are at least two later proofs that solvable groups have shellable subgroup lattices [33, 42].

Hochster's formula. Hochster's formula comes out of work of Hochster, Reisner, and Stanley in the 70s, including [29, 35, 36]. The proof sketch that we looked at in some detail is sketched lightly in [37], which is also an excellent general reference. See also [19]. A standard reference for simplicial cohomology is [24].

That depth and the Cohen-Macaulay property are topological invariants was proved in [25]. This paper connects (co)homology of links in a simplicial complex with local homology of topological spaces.

Collapses. A general overview on the power and limitations of collapses to compute homotopy type is [14]. The method described to check if a simplicial complex is a sphere is from one point of view a search for a shelling. From another point of view, it is a search for a Brown-Chari (discrete Morse) matching [9, 12] – see [2] for more on the method.

Shellable, Cohen-Macaulay, and sequentially Cohen-Macaulay complexes. The term *shelling* was first introduced, so far as I can determine, in [32] (although similar conditions had been considered earlier). The term regained currency after the paper [11] showed every simplicial polytope to be shellable; unfortunately, this paper did not reference [32], causing much later confusion. Björner and Wachs wrote a series of papers developing shellability as a general tool, including the extension to non-pure complexes [3, 5, 6, 7]. A textbook treatment of shellability from the point of view of polytope complexes may be found in [44, Chapter 8].

The first non-shellable ball (at least under that name) was constructed in [31], using specialized techniques. An excellent and very readable description of another non-shellable ball is on the blog "College Math Teaching" at [1].

Vertex-decomposability and k-decomposability were first defined in [27]. Vertex-decomposability was extended to the nonpure case in [7], while k-decomposability was so extended in [20]. See also [43]. Complexes that are shellable and not vertex-decomposable were already known to Provan and Billera [27]. A flag complex (the independence complex of a graph) with the same combination of properties was found in [16].

I did not have time to talk about the powerful techniques of EL-labelings and CL-labelings, used to construct shellings of order complexes of posets. These can be found in [3, 5, 6, 7].

References

1. Anonymous, *Mary Ellen Rudin tribute (1924-2013): Shelling a triangulated disk or ball.*, https://collegemathteaching.wordpress.com/2013/03/25/mary-ellen-rudin-tribute-1924-2013-shelling-a-triangulated-disk-or-ball/, Mar 2013.
2. Bruno Benedetti and Frank H. Lutz, *Random discrete Morse theory and a new library of triangulations*, Exp. Math. **23** (2014), no. 1, 66–94. MR 3177457

3. Anders Björner, *Shellable and Cohen-Macaulay partially ordered sets*, Trans. Amer. Math. Soc. **260** (1980), no. 1, 159–183. MR 81i:06001
4. ———, *Topological methods*, Handbook of combinatorics, Vol. 1, 2, Elsevier, Amsterdam, 1995, pp. 1819–1872. MR 1373690 (96m:52012)
5. Anders Björner and Michelle L. Wachs, *On lexicographically shellable posets*, Trans. Amer. Math. Soc. **277** (1983), no. 1, 323–341. MR 84f:06004
6. ———, *Shellable nonpure complexes and posets. I*, Trans. Amer. Math. Soc. **348** (1996), no. 4, 1299–1327. MR 96i:06008
7. ———, *Shellable nonpure complexes and posets. II*, Trans. Amer. Math. Soc. **349** (1997), no. 10, 3945–3975. MR 98b:06008
8. Miklós Bóna, *A walk through combinatorics*, third ed., World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011, An introduction to enumeration and graph theory, With a foreword by Richard Stanley. MR 2849213
9. Kenneth S. Brown, *The geometry of rewriting systems: a proof of the Anick-Groves-Squier theorem*, Algorithms and classification in combinatorial group theory (Berkeley, CA, 1989), Math. Sci. Res. Inst. Publ., vol. 23, Springer, New York, 1992, pp. 137–163. MR MR1230632 (94g:20041)
10. ———, *The coset poset and probabilistic zeta function of a finite group*, J. Algebra **225** (2000), no. 2, 989–1012. MR 2000k:20082
11. H. Bruggesser and P. Mani, *Shellable decompositions of cells and spheres*, Math. Scand. **29** (1971), 197–205 (1972). MR 48 #7286
12. Manoj K. Chari, *On discrete Morse functions and combinatorial decompositions*, Discrete Math. **217** (2000), no. 1-3, 101–113, Formal power series and algebraic combinatorics (Vienna, 1997). MR 1766262 (2001g:52016)
13. Maria Chudnovsky and Paul Seymour, *The roots of the independence polynomial of a clawfree graph*, J. Combin. Theory Ser. B **97** (2007), no. 3, 350–357. MR 2305888
14. Marshall M. Cohen, *A course in simple-homotopy theory*, Springer-Verlag, New York, 1973, Graduate Texts in Mathematics, Vol. 10. MR 0362320 (50 #14762)
15. Art M. Duval, *Algebraic shifting and sequentially Cohen-Macaulay simplicial complexes*, Electron. J. Combin. **3** (1996), no. 1, Research Paper 21, approx. 14 pp. (electronic). MR MR1399398 (98b:06009)
16. Jonathan Earl, Kevin N. Vander Meulen, and Adam Van Tuyl, *Independence complexes of well-covered circulant graphs*, Exp. Math. **25** (2016), no. 4, 441–451. MR 3499708
17. Philip Hall, *The Eulerian functions of a group*, Quart. J. Math. **7** (1936), 134–151.
18. Jürgen Herzog and Takayuki Hibi, *Monomial ideals*, Graduate Texts in Mathematics, vol. 260, Springer-Verlag London Ltd., London, 2011. MR 2724673 (2011k:13019)
19. Srikanth B. Iyengar, Graham J. Leuschke, Anton Leykin, Claudia Miller, Ezra Miller, Anurag K. Singh, and Uli Walther, *Twenty-four hours of local cohomology*, Graduate Studies in Mathematics, vol. 87, American Mathematical Society, Providence, RI, 2007. MR 2355715 (2009a:13025)
20. Jakob Jonsson, *Optimal decision trees on simplicial complexes*, Electron. J. Combin. **12** (2005), Research Paper 3, 31 pp. (electronic). MR 2134166 (2006d:05186)
21. ———, *Simplicial complexes of graphs*, Lecture Notes in Mathematics, vol. 1928, Springer-Verlag, Berlin, 2008. MR MR2368284

22. Vadim E. Levit and Eugen Mandrescu, *The independence polynomial of a graph—a survey*, Proceedings of the 1st International Conference on Algebraic Informatics, Aristotle Univ. Thessaloniki, Thessaloniki, 2005, pp. 233–254. MR 2186466 (2006k:05163)
23. Jiří Matoušek, *Using the Borsuk-Ulam theorem*, Universitext, Springer-Verlag, Berlin, 2003, Lectures on topological methods in combinatorics and geometry, Written in cooperation with Anders Björner and Günter M. Ziegler. MR MR1988723 (2004i:55001)
24. James R. Munkres, *Elements of algebraic topology*, Addison-Wesley Publishing Company, Menlo Park, CA, 1984. MR 755006 (85m:55001)
25. ______, *Topological results in combinatorics*, Michigan Math. J. **31** (1984), no. 1, 113–128. MR 736476 (85k:13022)
26. James Oxley, *Matroid theory*, second ed., Oxford Graduate Texts in Mathematics, vol. 21, Oxford University Press, Oxford, 2011. MR 2849819
27. J. Scott Provan and Louis J. Billera, *Decompositions of simplicial complexes related to diameters of convex polyhedra*, Math. Oper. Res. **5** (1980), no. 4, 576–594. MR MR593648 (82c:52010)
28. Victor Reiner and Dumitru Ioan Stamate, *Koszul incidence algebras, affine semigroups, and Stanley-Reisner ideals*, Adv. Math. **224** (2010), no. 6, 2312–2345. MR 2652208
29. Gerald Allen Reisner, *Cohen-Macaulay quotients of polynomial rings*, Advances in Math. **21** (1976), no. 1, 30–49. MR 0407036
30. Gian-Carlo Rota, *On the foundations of combinatorial theory. I. Theory of Möbius functions*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **2** (1964), 340–368 (1964). MR 0174487 (30 #4688)
31. Mary Ellen Rudin, *An unshellable triangulation of a tetrahedron*, Bull. Amer. Math. Soc. **64** (1958), 90–91. MR 0097055 (20 #3535)
32. D. E. Sanderson, *Isotopy in 3-manifolds. I. Isotopic deformations of 2-cells and 3-cells*, Proc. Amer. Math. Soc. **8** (1957), 912–922. MR 0090052 (19,760d)
33. Jay Schweig and Russ Woodroofe, *A broad class of shellable lattices*, Adv. Math. **313** (2017), 537–563. MR 3649231
34. John Shareshian, *On the shellability of the order complex of the subgroup lattice of a finite group*, Trans. Amer. Math. Soc. **353** (2001), no. 7, 2689–2703. MR 1828468 (2002k:06006)
35. Richard P. Stanley, *The upper bound conjecture and Cohen-Macaulay rings*, Studies in Appl. Math. **54** (1975), no. 2, 135–142. MR 0458437
36. ______, *Cohen-Macaulay complexes*, Higher combinatorics (Proc. NATO Advanced Study Inst., Berlin, 1976), Reidel, Dordrecht, 1977, pp. 51–62. NATO Adv. Study Inst. Ser., Ser. C: Math. and Phys. Sci., 31. MR 0572989 (58 #28010)
37. ______, *Combinatorics and commutative algebra*, second ed., Progress in Mathematics, vol. 41, Birkhäuser Boston Inc., Boston, MA, 1996. MR 1453579 (98h:05001)
38. ______, *Enumerative combinatorics. Volume 1*, second ed., Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 2012. MR 2868112
39. Rafael H. Villarreal, *Monomial algebras*, second ed., Monographs and Research Notes in Mathematics, CRC Press, Boca Raton, FL, 2015. MR 3362802

40. Michelle L. Wachs, *Poset topology: Tools and applications*, Geometric combinatorics, IAS/Park City Math. Ser., vol. 13, Amer. Math. Soc., Providence, RI, 2007, pp. 497–615. MR MR2383132
41. Neil White (ed.), *Matroid applications*, Encyclopedia of Mathematics and its Applications, vol. 40, Cambridge University Press, Cambridge, 1992. MR 1165537
42. Russ Woodroofe, *Vertex decomposable graphs and obstructions to shellability*, Proc. Amer. Math. Soc. **137** (2009), no. 10, 3235–3246. MR MR2515394
43. ———, *Chordal and sequentially Cohen-Macaulay clutters*, Electron. J. Combin. **18** (2011), no. 1, Paper 208, 20 pages.
44. Günter M. Ziegler, *Lectures on polytopes*, Graduate Texts in Mathematics, vol. 152, Springer-Verlag, New York, 1995. MR 96a:52011

University of Primorska, FAMNIT, Glagoljaška 8, 6000 Koper, Slovenia
E-mail address: russ.woodroofe@famnit.upr.si