DGA Structures on Free Res.

R conn. noeth. ring

M f.g. R-module

$$\cdots \to F_1 \to F_0 \to M \to 0$$

- linear alg.

  ⋮

Koszul cx.

$$T^p(M) = \begin{cases} 0, & p<0 \\ R, & p=0 \\ M^{\otimes p}, & p \geq 1 \end{cases}$$

A graded algebra with

$$(x_1 \otimes \cdots \otimes x_p)(y_1 \otimes \cdots \otimes y_q)$$

$$= x_1 \otimes \cdots \otimes x_p \otimes y_1 \otimes \cdots \otimes y_p$$

$$J = \langle x \otimes x \mid x \in M \rangle$$

$$\Lambda(M) = T(M)/J$$

$$[x_1 \otimes \cdots \otimes x_p]_J = x_1 \wedge \cdots \wedge x_p$$

$$x \otimes y + y \otimes x = (x+y) \otimes (x+y) - x \otimes x - y \otimes y \in J$$

$$x_1 \wedge \cdots \wedge x_p = - x_1 \wedge \cdots \wedge x_{i+1} \wedge x_i \cdots \wedge x_p$$

If $M \cong R^n$ then

$\Lambda^p(M)$ is free of rank $\binom{n}{p}$

with basis

$$x_{i_1} \wedge \cdots \wedge x_{i_p} \quad i_1 < \cdots < i_p$$

Given a homomorphism:

$$M \xrightarrow{\varepsilon} R$$

Define

$$\delta : T(M) \longrightarrow \Lambda(M)$$

$$\delta(x_1 \otimes \cdots \otimes x_p) =$$

$$\sum (-1)^{i-1} \varepsilon(x_i)\, x_1 \wedge \cdots \wedge x_{i-1} \wedge x_{i+1} \wedge \cdots \wedge x_p$$

$$y_1 \otimes \cdots \otimes y_n \otimes x \otimes x \otimes z_1 \otimes \cdots \otimes z_m$$

$\in \operatorname{Ker} \delta$

$$\begin{tikzcd}
T(M) \arrow[rr, "\delta"] \arrow[dr] & & \Lambda(M) \\
& \Lambda(M) \arrow[ur, "\partial"'] &
\end{tikzcd}$$

$\partial^2 = 0$

$$\partial(x \wedge y) = \varepsilon(x) y - \varepsilon(y) x$$
$$= \partial(x) y - x \partial(y)$$

$$\partial(x_1 \wedge \dots \wedge x_p \wedge y_1 \wedge \dots \wedge y_q) =$$
$$\partial(x_1 \wedge \dots \wedge x_p) \wedge y_1 \wedge \dots \wedge y_q +$$
$$(-1)^p x_1 \wedge \dots \wedge x_p \, \partial(y_1 \wedge \dots \wedge y_q)$$

DGA: $\partial(ab) = \partial(a) b + (-1)^{|a|} a \partial(b)$

Let $I = (u_1, \dots, u_n) \subseteq R$

$\varepsilon : R^n \xrightarrow{[u_1 \dots u_n]} R$

By depth sensitivity of the Koszul cx the Koszul cx is a free resolution of R/I if and only if $x_1, \dots, x_n$ is a regular sequence.

$$n = 1 : 0 \to F_1 \xrightarrow{u_1} R \to 0$$

<u>Example</u> $R = k[x, y, z]$

$I = (x^2, y^2, z^2)$

$$R \xrightarrow{\begin{bmatrix} z^2 \\ -y^2 \\ -x^2 \end{bmatrix}} R^3 \xrightarrow{\begin{bmatrix} -y^2 & -z^2 & 0 \\ x^2 & 0 & -z^2 \\ 0 & x^2 & y^2 \end{bmatrix}} R^3 \longrightarrow R$$

$e_x \wedge e_y \wedge e_z$ $\qquad$ $e_x \wedge e_y$, $e_x \wedge e_z$, $e_y \wedge e_z$ $\qquad$ $e_x \longrightarrow x^2$, $e_y \longrightarrow y^2$, $e_z \longrightarrow z^2$

$$\partial(e_x \wedge e_y) = x^2 e_y - y^2 e_x$$

$$R \xrightarrow{\begin{bmatrix} z^2 \\ -y^2 \\ x^2 \end{bmatrix}} R^3 \xrightarrow{\begin{bmatrix} -y^2 & -z^2 & 0 \\ x^2 & 0 & -z^2 \\ 0 & x^2 & y^2 \end{bmatrix}} R^3 \xrightarrow{(x^2\ y^2\ z^2)} R$$

$$g \qquad\qquad f_1, f_2, f_3 \qquad\qquad e_1 e_2 e_3$$

Can I put a product on this resolution?

$$\partial(e_1 e_2) = \partial(e_1) e_2 - \partial(e_2) e_1$$
$$= x^2 e_2 - y^2 e_1$$
$$= \partial(f_1)^{*}$$

Can choose $e_1 e_2 = f_1$

$$\partial(e_1 e_3) = x^2 e_3 - z^2 e_1 = \partial(f_2)$$

$\therefore\ e_1 e_3 = f_2$

$$\partial(e_2 e_3) = y^2 e_3 - z^2 e_2 = \partial(f_3)$$

$\therefore\ e_2 e_3 = f_3$

$$\begin{aligned} \partial(e_1 f_1) &= x^2 f_1 - e_1(x^2 e_2 - y^2 e_1) \\ &= x^2 f_1 - x^2 f_1 + 0 \\ &= 0 \end{aligned}$$

$\therefore\ e_1 f_1 = 0$

$$\begin{aligned} \partial(e_1 f_2) &= x^2 f_2 - e_1(x^2 e_3 - z^2 e_1) \\ &= x^2 f_2 - x^2 f_2 + 0 \\ &= 0 \end{aligned}$$

$\therefore\ e_1 f_2 = 0$

$$\begin{aligned} \partial(e_1 f_3) &= x^2 f_3 - e_1(y^2 e_3 - z^2 e_2) \\ &= x^2 f_3 - y^2 f_2 + z^2 f_1 \\ &= \partial(g) \end{aligned}$$

$\therefore\ e_1 f_3 = g$

*) $e_1 e_2 = (1+z^2) f_1 - y^2 f_2 + x^2 f_3$

Res. of length 1:

$$0 \to F_1 \to R \to 0$$

$$(F_1)^2 = 0$$

Res of length 2

$$0 \to R^{n-1} \xrightarrow{\partial_2} R^m \xrightarrow{\partial_1} R \twoheadrightarrow R/I$$

$I = a(I_{n-1}(\partial_2))$, a nzd

$$e_i \cdot e_j = -a \sum (\pm 1) \det(\partial_2)^k_{ij} f_k$$

$n = 2$ a complete intersection

$$0 \to R \xrightarrow{[u\,v]} R^2 \xrightarrow{[-v\,u]} R \to 0$$

Thm [5,6] If $I \subseteq R$ with $\mathrm{pd}_R R/I \leq 3$ then every free resolution of R/I over R has a DGA structure.

__Exa__ [1] Associativity may fail.

__Prop__ [16,15] If ~~pd_R~~ R local and $\mathrm{pd}_R R/I = 4$ and I is Gorenstein (i.e. $\mathrm{Ext}_R^i(R/I, R) = 0$ for $i \neq 4$ and $\mathrm{Ext}_R^4(R/I, R) \cong R/I$) then the min'l free res of R/I over R has DGA structure.

<u>Prop</u> [17] If R local, I is a.c.i ($\mathrm{Ext}^i_R(R/I, R) = 0$ for $i \neq 4$ and $\mu(I) = 5$), and char $R \neq 2$ then the min'l free resolution of R/I over R has a DGA structure.

Thm [15] If R is local then the min'l free res of the residue field has a DGA structure.

Thm.[2] DGA resolutions always exist of R-algebras.

Thm[2] If S is a module-finite R-algebra and M a f.g. S-module, then S has a DGA resolution over R and M has a degwise f.g. free res over R which is a DG module over A.