Thank organizers & participants

refs – especially Avr. IFR & my notes w/ Kristen Beck?

← typo in codim 4 Gor?

Examples of DG Algebra Resolutions. I

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Workshop on Differential Graded Algebra Techniques in Commutative Algebra

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Fundamental references:

- L.L. Avramov, Infinite Free Resolutions
- K. Beck & K. Sather-Wagstaff, A Somewhat Gentle Introduction to Differential Graded Commutative Algebra

Goals: To exhibit examples of DG algebras including some DG algebra resolutions.

Take-Home Points

- Some resolutions admit DGA structures
- Many do not
- Canonical / minimal $\not\Rightarrow$ DGA

Assumptions. R is a commutative noetherian ring w/ $1 \neq 0$ and $S = R/I$ where $I = \langle \underline{f} \rangle$ and $\underline{f} = f_1, \dots, f_n \in R$.

Definition. A strongly commutative, non-negatively graded differential graded R-algebra (DGA or DG R-algebra) is an R-complex

$$A = \quad \cdots \xrightarrow{\partial_3^A} A_2 \xrightarrow{\partial_2^A} A_1 \xrightarrow{\partial_1^A} A_0 \to 0$$

equipped with an R-bilinear binary operation $(a,b) \mapsto ab$ that is unital, associative, and strongly graded commutative: $\forall\ a, b \in A$

- $ba = (-1)^{|a||b|} ab$ $\quad$ · $a^2 = 0$ if $|a|$ is odd

satisfying the Leibniz Rule

- $\partial(ab) = \partial(a)b + (-1)^{|a|} a\,\partial(b)$.

the structures interact well

count # hops

$\therefore$ R-linear chain map $\mu: A \otimes_R A \to A$ s.t. the underlying R-algebra $A^{\natural} = \bigoplus_{i \geq 0} A_i$ is unital, associative and strongly graded commutative.

A <u>DGA resolution</u> of S over R is an R-free resolution of S that is a DGA.

<u>Theorem</u> (Avramov, Tate 1957) DGA resolutions exist. Moreover, $\exists$ DGA resolution A of $S \cong R/I$ over R that is degreewise finite and as short as possible $\mathrm{rank}_R(A_i) < \infty \ \forall i$ & $A_i = 0 \ \forall i > \mathrm{pd}_R(S)$.

<u>Note</u>. A is as short as possible but may not be minimal. (More on this later.)

<u>Questions</u> : What does A look like?

: when is A minimal?

<u>Example</u> : R is a DGA resolution of itself.

: S is a DG R-algebra

<u>Example</u> : (Koszul complex with exterior algebra)

$\underline{f} = f_1, \dots, f_n \in R$. $K = K^R(\underline{f})$:

$$K = \left(0 \to R \to R^n \to \cdots \to R^{\binom{n}{i}} \to \cdots \to R^n \to R \to 0 \right)$$

h. deg.	n	$n-1$	i	1	0
basis	$e_{1\cdots n}$		$e_{j_1 \cdots j_i}$	$e_1, \dots, e_n$	1

$1 \leq j_1 < \cdots < j_i \leq n$

or e_Λ where $\Lambda = \{j_1, \dots, j_i\}$.

$$\partial(e_{j_1 \cdots j_i}) = \sum_{p=1}^{i} (-1)^{p-1} f_{j_p} e_{j_1 \cdots \hat{j_p} \cdots j_i}$$

sign counts # of hops

$$\partial(e_\Lambda) = \sum_{j\in\Lambda} \sigma(j,\Lambda)\, f_j\, e_{\Lambda\setminus\{j\}} \quad \checkmark$$

e.g. $\partial(e_{23}) = f_2 e_3 - f_3 e_2$

commutativity $\Rightarrow \partial(\partial(e_{pq})) = f_p f_q - f_q f_p = 0$

$$e_\Lambda e_\Gamma = \begin{cases} \sigma(\Lambda,\Gamma)\, e_{\Lambda\cup\Gamma} & \text{if } \Lambda\cap\Gamma = \phi \\ 0 & \text{if } \Lambda\cap\Gamma \neq 0 \end{cases}$$

e.g. $e_{23}\, e_{13} = 0$

$e_{23}\, e_{14} = +e_{1234}$ $\quad (-1)^2$

check Leibniz Rule

$$\partial(\underbrace{e_{23} e_{14}}_{e_{1234}}) \stackrel{?}{=} \underline{\partial(e_{23})} e_{14} + \underline{e_{23}}\, \underline{\partial(e_{14})}$$

$\partial(e_{1234}) = f_1 e_{234} - f_2 e_{134} + f_3 e_{124} - f_4 e_{123}$

$$(f_2 e_3 - f_3 e_2) e_{14} + e_{23}(f_1 e_4 - f_4 e_1)$$

$$= f_2 e_3 e_{14} - f_3 e_2 e_{14} + f_1 e_{23} e_4 - f_4 e_{23} e_1$$

$$= -f_2 e_{134} + f_3 e_{124} + f_1 e_{234} - f_4 e_{123}$$

This is the starting point for Avramov-Tate construction.

Notes: Because multiplication must be R-linear, it suffices to define it on basis vectors & to check axioms for products of basis vectors.

: K not always resolution

Theorem (Buchsbaum-Eisenbud 1977). If A is a degreewise finite R-free resolution of $S \cong R/I$ s.t. $A_0 = R$, then A is almost a DGA: it has a multiplication satisfying all the axioms except possibly associativity.

Pf. (sketch). For two elements $a, b \in A$ need to find $c \in A$ s.t. $\partial(c) = \partial(a)b + (-1)^{|a|} a\partial(b)$, then set $ab = c$. I.e., need to show $\partial(a)b + (-1)^{|a|} a \partial(b)$ is a boundary, i.e., it is a cycle (since A is a resolution).

induct on $|a| + |b|$

$$\partial(\partial(a)b + (-1)^{|a|} a\partial(b))$$
$$= \cancel{\partial(\partial(a))b} + (-1)^{|a|-1}\partial(a)\partial(b)$$
$$+ (-1)^{|a|}\partial(a)\partial(b) + \cancel{a\,\partial(\partial(b))}) = 0. \quad \square$$

Conclusion. Associativity is the hard part.

Corollary. If A is a resolution s.t. $A_i = 0$ $\forall i \geq 3$, then A is a DGA. $(ab)c \in A_{1+1+1} = A_3 = 0$

Pf. Associativity is automatic for degree reasons. $\square$

Corollary. If $\mathrm{pd}_R S \leq 2$, then the minimal R-free resolution of S is a DGA.

Exercise. Explicitly describe the DGA structure in this case using the Hilbert Burch Theorem. ★

Theorem (Buchsbaum-Eisenbud). If $\mathrm{pd}_R S \leq 3$, then the minimal R-free resolution of S is a DGA.

Pf. Suffices to check $(ab)c = a(bc)$ $\forall\, a,b,c \in A_1$. ∂_3^A is injective $\therefore$ suffices to check

$$\partial((ab)c) = \partial(a(bc))$$

$$\partial(ab)c + (ab)\partial(c) \qquad\qquad \partial(a)(bc) - a\,\partial(bc)$$

$$= \big(\partial(a)b\big)c - \big(a\partial(b)\big)c + (ab)\partial(c) \qquad\qquad = \partial(a)(bc) - a(\partial(b)c) + a(b\,\partial(c))$$

(notice sign counts # hops) ∎

Theorem (Kustin-Miller 1980, Kustin 1987) If $\mathrm{pd}_R S = 4$ and I is Gorenstein, then the minimal R-free resolution of S is a DGA.

Exercise. Prove this using the Buchsbaum-Eisenbud resolution in terms of Pfaffians. ★