Thank organizers & participants
refs — especially Avr. IFR & my notes w/ Kristen Beck?
← typo in codim 4 Gor?

Examples of DG Algebra Resolutions. II

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Workshop on Differential Graded Algebra Techniques in Commutative Algebra
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Fundamental references:
- L.L. Avramov, Infinite Free Resolutions
- K. Beck & K. Sather-Wagstaff, A Somewhat Gentle Introduction to Differential Graded Commutative Algebra
https://ssather.people.clemson.edu/research.html

Goals: To exhibit examples of DG algebras including some DG algebra resolutions.

Take-Home Points

- Some resolutions admit DGA structures
- Many do not
- Canonical / minimal $\not\Rightarrow$ DGA

Assumptions. R is a commutative noetherian ring w/ $1 \neq 0$ and $S = R/I$ where $I = \langle \underline{f} \rangle$ and $\underline{f} = f_1, \dots, f_n \in R$.

Last time.

Theorem (Avramov, Tate 1957) DGA resolutions exist.

Example: Koszul complex with exterior algebra structure is a DGA $\therefore$ if $\underline{f}$ is R-regular then minimal R-free resolution of S is DGA. (Exercise)

Theorem (Buchsbaum-Eisenbud). If $\operatorname{pd}_R S \leq 3$, then the minimal R-free resolution of S is a DGA.

Theorem (Kustin-Miller 1980, Kustin 1987) If $\operatorname{pd}_R S = 4$ and I is Gorenstein, then the minimal R-free resolution of S is a DGA.

Theorem (Buchsbaum-Eisenbud) Associativity is the challenge

Questions: What if $\operatorname{pd}_R S = 4$ but I is not Gorenstein?

: What if $\operatorname{pd}_R S = 5$ and I is Gorenstein?

The Taylor Resolution. Assume $R = k[x_1, \dots, x_d]$ and each f_i is a monomial in R, so I is a monomial ideal. The Taylor resolution fixes an inherent problem with the Koszul complex. Koszul cycles: $f_i f_j - f_j f_i = 0$

but there are usually other relations/syzygies.

e.g. $f_1 = xy$ & $f_2 = xz$ & $f_3 = yz$

$$0 = -xz \cdot xy + xy \cdot xz = (xy \quad xz \quad yz)\begin{pmatrix} -xz \\ xy \\ 0 \end{pmatrix}$$

but there is a smaller relation obtained by factoring out the x here

$$(xy \quad xz \quad yz)\begin{pmatrix} -z \\ y \\ 0 \end{pmatrix} = -z \cdot xy + y \cdot xz = 0 \qquad (\star) \text{ for later}$$

You may think of this as a "reduced commutativity relation." Diana Taylor addressed this in her dissertation (University of Chicago, 1966). Set $f_\emptyset = 1$, and for each subset $\Lambda \subseteq [n] = \{1, \dots, n\}$ set

$$f_\Lambda = \operatorname{lcm}(f_\lambda \mid \lambda \in \Lambda)$$

e.g. $f_{123} = \operatorname{lcm}(f_1, f_2, f_3)$. Then $T = T(\underline{f}; k)$ is

$$T = (\, 0 \to R \to R^n \to \cdots \to R^{(?)} \to \cdots \to R^n \to R \to 0)$$

homological degree	n	$n-1$	i	1	0
basis	$e_{1\dots n}$		$e_{j_1 \cdots j_i}$ or e_Λ	e_1 ⋮ e_n	1

$$\partial(e_\Lambda) = \sum_{j\in\Lambda} \sigma(j,\Lambda)\frac{f_\Lambda}{f_{\Lambda\setminus\{j\}}} e_{\Lambda\setminus\{j\}}$$

conservation of sub-script

e.g. $$\partial(e_{12}) = +\frac{f_{12}}{f_2}e_2 - \frac{f_{12}}{f_1}e_1$$

e.g. if $\underline{f} = xy, xz, yz$

$$\partial(e_{12}) = +\frac{xyz}{xz}e_2 - \frac{xyz}{xy}e_1 = ye_2 - ze_1 = \begin{pmatrix} -z \\ y \\ 0 \end{pmatrix}$$

from earlier

$$\partial(\partial(e_{pq})) = \partial\left(\frac{f_{pq}}{f_q}e_q - \frac{f_{pq}}{f_p}e_p\right)$$

$$= \frac{f_{pq}}{\cancel{f_q}}\cancel{f_q} - \frac{f_{pq}}{\cancel{f_p}}\cancel{f_p} = 0$$

<u>Theorem</u> (Taylor) T is a resolution.

<u>Note</u>. T is not usually minimal, e.g., because it is usually too long to satisfy the Auslander-Buchsbaum formula.

<u>Example</u> $\underline{f} = xy, xz, yz$

$$T = \quad 0 \to R \xrightarrow{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} -z & -z & 0 \\ y & 0 & -y \\ 0 & x & x \end{pmatrix}} R^3 \xrightarrow{(xy \;\; xz \;\; yz)} R \to 0$$

units $\Rightarrow$ not minimal.

(also Auslander-Buchsbaum not satisfied)

Theorem (Geneda 1976) T is a DGA (Exercise):

$$e_\Lambda e_\Gamma = \begin{cases} \sigma(\Lambda,\Gamma)\dfrac{f_\Lambda f_\Gamma}{f_{\Lambda\cup\Gamma}} e_{\Lambda\cup\Gamma} & \text{if } \Lambda\cap\Gamma=\emptyset \\ 0 & \text{if } \Lambda\cap\Gamma\neq 0 \end{cases}$$

compare w/ mult. on Koszul

e.g. $e_{23}e_3 = 0$

$$e_{23}e_1 = +\frac{f_{23}\cdot f_1}{f_{123}} e_{123}$$

e.g. $\underline{f} = XY, XZ, YZ$

$$e_{23}e_1 = +\frac{\cancel{XYZ}\cdot XY}{\cancel{XYZ}} e_{123} = XY e_{123}$$

check Leibniz Rule

$$\partial(e_{23}e_1) \overset{?}{=} \partial(e_{23})e_1 + e_{23}\partial(e_1)$$

$$\partial\left(\frac{f_{23}\cdot f_1}{f_{123}} e_{123}\right) \qquad \left[\frac{f_{23}}{f_3}e_3 - \frac{f_{23}}{f_2}e_2\right]e_1 + f_1 e_{23}$$

$$\frac{f_{23}\cdot f_1}{\cancel{f_{123}}}\left(\frac{\cancel{f_{123}}}{f_{23}}e_{23} - \frac{\cancel{f_{123}}}{f_{13}}e_{13} + \frac{\cancel{f_{123}}}{f_{12}}e_{12}\right) \qquad -\frac{f_{23}}{f_3}\frac{f_3 f_1}{f_{13}}e_{13} + \frac{f_{23}}{f_2}\frac{f_2 f_1}{f_{12}}e_{12} + f_1 e_{23}$$

Question Can we make this smaller while still keeping DGA? not usually

Simplicial Resolutions. One advantage of Koszul & Taylor is the easy description of the bases: all subsets of $[n] = \{1, \dots, n\}$. Geometrically, this is the $(n-1)$-simplex Δ_{n-1}. Bayer, Peeva, & Sturmfels recognized that similar resolutions can be constructed from other simplicial complexes (and, more generally, regular cell complexes).

Definition. Let $V = \{v_1, \dots, v_n\}$ with $|V| = n$, e.g. $V = [n]$.
A simplicial complex on V is a non-empty subset $\Delta \subseteq \Delta_{n-1} = \mathcal{P}(V)$ that is closed under subsets:

If $F, G \subseteq V$ s.t. $G \subseteq F$ & $F \in \Delta$, then $G \in \Delta$.

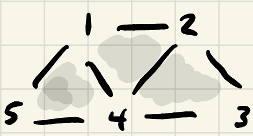
Examples: Δ_{n-1} is a simplicial complex on V called the $(n-1)$-simplex

: $\{\phi\}$ is a simplicial complex on V called the empty complex

: ϕ is not a simplicial complex.

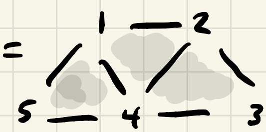
: every finite simple graph with vertex set V is a simplicial complex on V

:



Definition. The elements of a simplicial complex are its faces. The maximal faces are its facets

Construction (Bayer, Peeva, Sturmfels 1998) Let Δ be a simplicial complex on $V = \{v_1, \dots, v_n\}$ & consider a list of monomials $\underline{f} = f_1, \dots, f_n \in R = k[x_1, \dots, x_d]$. The *simplicial chain complex* $C = C(\Delta; \underline{f}; k)$ is the subcomplex $C \subseteq T(\underline{f}; k)$ with basis given by all e_F s.t. $F \in \Delta$.

Example. $\Delta =$ 

$$C = (0 \to R^3 \longrightarrow R^7 \longrightarrow R^5 \longrightarrow R \longrightarrow 0)$$

h.deg	3	2	1	0
basis	e_{124} e_{145} e_{234}	e_{12} e_{14} e_{15} e_{23} e_{24} e_{34} e_{45}	e_1 $\vdots$ e_5	1

$$e_{124} \mapsto \frac{f_{124}}{f_{24}} e_{24} - \frac{f_{124}}{f_{14}} e_{14} + \frac{f_{124}}{f_{12}} e_{12}$$

Example. $\Delta = \Delta_{n-1}$: $C(\Delta_{n-1}; \underline{f}; k) = T(\underline{f}; k)$

Facts : Δ simplicial complex $\Rightarrow$ C is an R-complex

: ~~Δ~~ C need not be minimal, e.g., if $C = T$

: ~~Δ~~ C need not be a resolution.

: Bayer, Peeva, & Sturmfels give a combinatorial (homological) criterion for Δ to be a resolution in terms of acyclicity of certain subcomplexes $\Delta_{\leq b}$ over k.

E.g., if Δ is not acyclic then C is not a resolution.

: C is minimal iff $\forall F \in \Delta \ \forall p \in F$:

$$f_F \neq f_{F'} \quad \text{where} \quad F' = F \setminus \{p\}.$$

~~Theorem~~ (Bayer, Peeva, Sturmfels) $C = C(\Delta; \underline{f}; k)$ is a DGA.