INTRODUCTION TO HOMOLOGICAL THEORY OF DG MODULES

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1. DG Algebras

Some of the references on dg algebras and dg modules are [1, 2, 5, 6]. Throughout the notes, R is a commutative ring, often assumed to be Noetherian.

Definition 1.1. A, or more precisely (A, d^A), is called a dg R-algebra if it satisfies the following conditions:

(a) $A = \bigoplus_{n\in\mathbb{Z}} A_n$ is a graded R-algebra[1], that is, each A_n is an R-module and the product $A_n \times A_m \to A_{n+m}$ is R-bilinear. We say that elements of A_n are homogeneous of degree n, and write $n = |a|$ for $a \in A_n$. Notice that R always centrally acts on A, i.e., $ra = ar$ for all $r \in R$ and $a \in A$.

(b) A is an R-complex with a differential d^A, that is, $d^A : A \to A(-1)$ is a graded R-linear map[2] (a degree-preserving R-module homomorphism) with $(d^A)^2 = 0$.

(c) d^A satisfies the *Leibniz rule*: for all homogeneous elements $a, b \in A$ the equality $d^A(ab) = d^A(a)b + (-1)^{|a|}ad^A(b)$ holds.

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[1]Some authors use the cohomological notation for dg algebras. In such a case, A is described as $A = \bigoplus_{n\in\mathbb{Z}} A^n$, where $A^n = A_{-n}$.

[2]In general, $A(m)$ is the shifted graded algebra of A by $m \in \mathbb{Z}$, whose grading is defined by $A(m)_n = A_{m+n}$ for $n \in \mathbb{Z}$.

In this note we only consider *non-negatively* graded dg R-algebras, that is, $A = \bigoplus_{n\geqq 0} A_n$. [3]

Exercise 1.2. Show that $d^A(1) = 0$. More generally, $d^A(r) = 0$ for $r \in R$. Show that A_0 is an R-subalgebra of A.

1.3. Let A, B be dg R-algebras. A *homomorphism* $f\colon A \to B$ of dg R-algebras (or a *dg R-algebra homomorphism*) is a graded R-algebra homomorphism of degree 0 which is also a chain map, that is, $d^B f = f d^A$.

Exercise 1.4. You can now define a dg R-algebra isomorphism, a dg R-subalgebra and a residue dg algebra, etc. Give their precise definitions.

Exercise 1.5. A subset of A which is the kernel of a dg R-algebra homomorphism from A to some other dg algebra is called a dg ideal of A. Show that $I \subseteq A$ is a dg ideal if and only if I is a graded ideal of A and satisfies $d^A(I) \subseteq I$.

1.6. Let A be a dg R-algebra. Set

$$Z(A) = \mathrm{Ker}(d^A) = \bigoplus_n Z_n(A), \quad B(A) = \mathrm{Im}(d^A) = \bigoplus_n B_n(A).$$

Call $Z(A)$ (resp. $B(A)$) the cycle (resp. the boundary) of A. Note that $Z(A) \subseteq A$ is a dg R-subalgebra with the zero differential, and that $B(A) \subseteq Z(A)$ is a dg ideal, i.e. $B(A)$ is a graded ideal of $Z(A)$. Hence the quotient

$$H(A) = Z(A)/B(A) = \bigoplus_n H_n(A)$$

is a dg R-algebra with zero differential, and it is called the homology of A.

Exercise 1.7. Give a proof for each statement in 1.6.

Definition 1.8. A dg R-algebra A is called *commutative* if $ab = (-1)^{|a||b|}ba$ for all homogeneous $a, b \in A$. A is called *strongly commutative* if it is commutative and satisfies $a^2 = 0$ if $a \in A$ and $|a|$ is odd.

1.9. *In the rest of this section, unless otherwise stated, all the dg R-algebras are assumed to be strongly commutative and non-negatively graded.*

Example 1.10. The first trivial example of dg R-algebras is R itself that is concentrated in degree zero with zero differential. A little more generally, an associative R-algebra T can be regarded as a dg R-algebra concentrated in degree zero with $d^T = 0$.

Example 1.11. Let $C_\bullet$ be a chain complex of R-modules;

$$\cdots \longrightarrow C_1 \xrightarrow{d_1^C} C_0 \xrightarrow{d_0^C} C_{-1} \xrightarrow{d_{-1}^C} \cdots .$$

For each $n \in \mathbb{Z}$ we set

$$\mathrm{Hom}_R(C_\bullet, C_\bullet)_n := \prod_{i\in\mathbb{Z}} \mathrm{Hom}_R(C_i, C_{i+n})$$

and

$$\mathrm{Hom}_R(C_\bullet, C_\bullet) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_R(C_\bullet, C_\bullet)_n$$

[3] In the cohomological notation, it is equivalent to $A = \bigoplus_{n\leqq 0} A^n$, and it is said to be non-positively graded.

that is a (non-commutative) graded R-algebra. Define the differential by

$$d((f_i)_{i\in\mathbb{Z}}) = \left(d^C_{i+n} \circ f_i - (-1)^n f_{i-1} \circ d^C_i\right)_{i\in\mathbb{Z}}$$

for $(f_i)_{i\in\mathbb{Z}} \in \mathrm{Hom}_R(C_\bullet, C_\bullet)_n$. Then $\mathrm{Hom}_R(C_\bullet, C_\bullet)$ is a dg R-algebra, but in general, it is not necessarily commutative nor non-negatively graded.

Exercise 1.12. Verify that $\mathrm{Hom}_R(C_\bullet, C_\bullet)$ is a dg R-algebra.

Example 1.13 (Koszul complex)**.** Let $t_1, \dots, t_n$ be a sequence of elements of R, and set $V = \bigoplus_{i=1}^n Re_i$ that is a free R-module of rank n. Consider the exterior algebra over V;

$$K := \bigwedge V = \bigoplus_{j=0}^{n} \bigwedge^{j} V = \bigoplus_{j=0}^{n} \left(\bigoplus_{i_1<i_2<\cdots<i_j} R(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_j}) \right).$$

By setting $d^K(e_i) = t_i$ for $1 \leqslant i \leqslant n$, the unique differential d^K on K is defined by Leibniz rule and it yields the dg R-algebra structure on K. This is a strongly commutative dg R-algebra, called the Koszul complex on the sequence $t_1, \dots, t_n$.

Example 1.14 (polynomial algebra)**.** Let A be a strongly commutative dg R-algebra, and let $n \in \mathbb{Z}$ be even. For a cycle element $t \in Z_{n-1}(A)$ of degree $n-1$, consider the variable X whose degree is n. Now take the polynomial algebra $A[X]$ whose degree m part is

$$A[X]_m = \bigoplus_{i+nj=m} A_i X^j.$$

On this graded R-algebra we define the differential by $d^{A[x]}(X) = t$ so that $A[X]$ is again a strongly commutative dg R-algebra. Note that $d^{A[x]}(X^m) = mX^{m-1}t$ for $m \geqq 1$. Also note that $(d^{A[x]})^2(X^m) = m(m-1)X^{m-2}t^2 = 0$ for $m \geqq 2$, since $t^2 = 0$ by the strong commutativity property of A.

Let n be an odd integer and $t \in Z_{n-1}(A)$ be a cycle. In this case X is a variable of degree n, and in order to keep the strong commutative property, the polynomial ring $A[X]$ is defined as $A[X] = A + XA$ with $X^2 = 0$ and $dX = t$.

Definition 1.15. An R-algebra U is a *divided power algebra* if a sequence of elements $u^{(i)} \in U$ with $i \in \mathbb{N} \cup \{0\}$ is correspondent to every element $u \in U$ with $|u|$ positive and even such that the following conditions are satisfied:
(1) $u^{(0)} = 1$, $u^{(1)} = u$, and $|u^{(i)}| = i|u|$ for all i;
(2) $u^{(i)}u^{(j)} = \binom{i+j}{i} u^{(i+j)}$ for all i, j;
(3) $(u+v)^{(i)} = \sum_j u^{(j)} v^{(i-j)}$ for all i;
(4) for all $i \geqslant 2$ we have

$$(vw)^{(i)} = \begin{cases} 0 & |v| \text{ and } |w| \text{ are odd} \\ v^i w^{(i)} & |v| \text{ is even and } |w| \text{ is even and positive} \end{cases}$$

(5) For all $i \geqslant 1$ and $j \geqslant 0$ we have

$$\left(u^{(i)}\right)^{(j)} = \frac{(ij)!}{j!(i!)^j} u^{(ij)}.$$

A *divided power dg R-algebra* is a dg R-algebra whose underlying graded R-algebra is a divided power algebra.

1.16. If R contains the field of rational numbers and U is a graded R-algebra, then U has a structure of a divided power R-algebra by defining $u^{(m)} = (1/m!)u^m$ for all $u \in U$ and integers $m \geqslant 0$; see [6, Lemma 1.7.2]. Also, R considered as a graded R-algebra concentrated in degree 0 is a divided power R-algebra.

Definition 1.17 (free extension)**.** Let $t \in A$ be a cycle, and let $A\langle X\rangle$ with the differential d denote the *simple free extension of* A obtained by adjunction of a variable X of degree $|t|+1$ such that $dX = t$. The dg R-algebra $A\langle X\rangle$ can be described as $A\langle X\rangle = \bigoplus_{m\geqslant 0} X^{(m)}A$ with the conventions $X^{(0)} = 1$ and $X^{(1)} = X$, where $\{X^{(m)} \mid m \geqslant 0\}$ is a free basis of $A\langle X\rangle$ such that:

(a) If $|X|$ is odd, then $X^{(m)} = 0$ for all $m \geqslant 2$, and for all $a + Xb \in A\langle X\rangle$ we have
$$d(a + Xb) = d^A a + tb - X d^A b.$$

(b) If $|X|$ is even, then $A\langle X\rangle$ is a divided power dg R-algebra with the algebra structure given by $X^{(m)}X^{(\ell)} = \binom{m+\ell}{m}X^{(m+\ell)}$ and the differential structure defined by $dX^{(m)} = X^{(m-1)}t$ for all $m \geqslant 1$. Note that the action of d on a general element of $A\langle X\rangle$ is given by
$$d\left(\sum_{i=0}^{n} X^{(i)}a_i\right) = \sum_{i=0}^{n-1} X^{(i)}\left(d^A(a_i) + ta_{i+1}\right) + X^{(n)}d^A(a_n). \tag{1.17.1}$$

If R contains the field of rational numbers, then $A\langle X\rangle = A[X]$.

Lemma 1.18. *(Assume that A is a strongly commutative, non-negatively graded, dg R-algebra as in 1.9.)*

Let $A\langle X\rangle$ be a free extension of A with the variable X such that $dX = t \in A$ is a homogeneous cycle of degree n. Then the natural inclusion mapping $A \hookrightarrow B$ is a dg algebra homomorphism and it induces the following isomorphisms of homology;
$$H_i(B) = H_i(A) \quad \text{if} \quad i < n, \qquad H_n(B) = H_n(A)/[t]H_0(A),$$
where $[t]$ is the homology class of t in $H(A)$.

Proof. Just note that $B_i = A_i$ if $i \leqq n$, and $B_{n+1} = A_{n+1} + XA_0$. □

By this lemma we sometimes call $A\langle X\rangle$ a free extension of A by adjunction of variable X that kills the cycle t.

1.19. Let n be a positive integer, and let $A\langle X_1, \dots, X_n\rangle$ (which is also denoted by $A\langle X_i \mid 1 \leqslant i \leqslant n\rangle$) be a *finite free extension of the dg R-algebra A* obtained by adjunction of n variables. In fact, setting $A^{(0)} = A$ and $A^{(i)} = A^{(i-1)}\langle X_i\rangle$ for all $1 \leqslant i \leqslant n$ such that $d^{A^{(i)}}X_i$ is a cycle in $A^{(i-1)}$, we have $A\langle X_1, \dots, X_n\rangle = A^{(n)}$. We also assume that $0 < |X_1| \leqslant \cdots \leqslant |X_n|$. Note that there is a sequence of dg R-algebras $A = A^{(0)} \subset A^{(1)} \subset \cdots \subset A^{(n)} = A\langle X_1, \dots, X_n\rangle$.

In a similar way, one can define the *finite polynomial extension of the dg R-algebra A*, which is denoted by $A[X_1, \dots, X_n]$.

1.20. Our discussion in 1.19 can be extended to the case of adjunction of infinitely countably many variables to the dg R-algebra A. Let $\{X_i \mid i \in \mathbb{N}\}$ be a set of variables. Attaching a degree to each variable such that $0 < |X_1| \leqslant |X_2| \leqslant \cdots$, similar to 1.19, we construct a sequence $A = A^{(0)} \subset A^{(1)} \subset A^{(2)} \subset \cdots$ of dg R-algebras. We define an *infinite free extension of the dg R-algebra A* obtained by adjunction of the variables $X_1, X_2, \dots$ to be $A\langle X_i \mid i \in \mathbb{N}\rangle = \bigcup_{n\in\mathbb{N}} A^{(n)}$. It is sometimes convenient for us to use the notation $A\langle X_1, \dots, X_n\rangle$ with $n = \infty$ instead of $A\langle X_i \mid i \in \mathbb{N}\rangle$.

Theorem 1.21 (Tate resolution)**.** *Let $f : R \twoheadrightarrow S$ be a surjective ring homomorphism of commutative Noetherian rings. Then there are countably infinite number of variables $\{X_i\}_{i\in\mathbb{N}}$ such that f extends to a quasi-isomorphism $R\langle X_i \mid i \in \mathbb{N}\rangle \to S$ of dg R-algeberas, that is,*

$$H_n(R\langle X_i \mid i \in \mathbb{N}\rangle) = \begin{cases} S & (n = 0), \\ 0 & (n \neq 0). \end{cases}$$

The dg R-algebra $R\langle X_i \mid i \in \mathbb{N}\rangle$ is called a Tate resolution *of S over R.*

Furthermore we can take a Tate resolution $R\langle X_i \mid i \in \mathbb{N}\rangle$ so that it satisfies the degree-wise finiteness condition, that is, for all $n \in \mathbb{N}$, we may assume that the set $\{i \mid |X_i| = n\}$ is finite.

1.22 (Outline of the proof)**.** By induction on n we construct a sequence of dg R-algebras $R = A^{(0)} \subset A^{(1)} \subset A^{(2)} \subset \cdots \subset A^{(n)}$ such that it satisfies the following conditions for all $n \geqslant 1$:

(i) $A^{(n)}$ is a finite free extension of $A^{(n-1)}$ with variables of degree n,
(ii) $H_0(A^{(n)}) = S$,
(iii) $H_i(A^{(n)}) = 0$ for $1 \leqslant i < n$,
(iv) $H_j(A^{(n)})$ is a finitely generated R-module for each $j \in \mathbb{N}$.

Once you have such a sequence, $\bigcup_{n\in\mathbb{N}} A^{(n)}$ is a required Tate resolution.

For $n = 1$, take a generating set $\{t_1, \dots, t_m\}$ of the ideal $\operatorname{Ker} f$, and set $A^{(1)} = R\langle X_1, \dots, X_m\rangle$ with $dX_i = t_i$ and $|X_i| = 1$ for all i. For $n \geqslant 2$ assume $A^{(n-1)}$ have been constructed. Then, take a finite set $\{z_1, \dots, z_\ell\}$ of elements of $Z_{n-1}(A^{(n-1)})$ whose homology class generates $H_{n-1}(A^{(n-1)})$ and we set $A^{(n)} = R\langle Z_1, \dots, Z_\ell\rangle$ with $dZ_i = z_i$ and $|Z_i| = n$ for all i. Then, since all the cycles of degree $n - 1$ are killed by Lemma 1.18, we see that $H_0(A^{(n)}) = S$ and $H_i(A^{(n)}) = 0$ for $1 \leqslant i < n$. Since R is Noetherian, we can easily see that all the homology modules of $A^{(n)}$ are finitely generated as well as those of $A^{(n-1)}$. □

1.23. In a similar way to 1.20, one can define the *infinite polynomial extension of the dg R-algebra A*, which is denoted by $A[X_i \mid i \in \mathbb{N}]$ or $A[X_1, \dots, X_n]$ with $n = \infty$. The same proof as Theorem 1.21 applies to polynomial extensions instead of free extensions, and we get Avramov resolutions.

Theorem 1.24 (Avramov resolution)**.** *Let $f : R \twoheadrightarrow S$ be a surjective ring homomorphism of commutative Noetherian rings. Then there are countably infinite number of variables $\{X_i\}_{i\in\mathbb{N}}$ such that f extends to a quasi-isomorphism $R[X_i \mid i \in \mathbb{N}] \to S$ of dg R-algeberas, that is,*

$$H_n(R[X_i \mid i \in \mathbb{N}]) = \begin{cases} S & (n = 0), \\ 0 & (n \neq 0). \end{cases}$$

The dg R-algebra $R[X_i \mid i \in \mathbb{N}]$ is called an Avramov resolution *of S over R.*

Furthermore we can take an Avramov resolution $R[X_i \mid i \in \mathbb{N}]$ so that it satisfies the degree-wise finiteness condition, that is, for all $n \in \mathbb{N}$, we may assume that the set $\{i \mid |X_i| = n\}$ is finite.

1.25 (Gulliksen-Levine)**.** Let $(R, \mathfrak{m}, k)$ be a commutative Noetherian ring. Gulliksen proved that the minimally constructed Tate resolution $R\langle X_i \mid i \in \mathbb{N}\rangle$ of $R \twoheadrightarrow k$ gives a minimal free resolution of k over R. In such a situation we set ϵ_i to be a number of degree i variables for each $i \in \mathbb{N}$. This number ϵ_i is called the ith deviation of the local ring R. Once we know the deviations we can easily get the Betti numbers of k over R. It is then easy to see the following description for the Poincaré series;

$$P_R(T)\left(\overset{\text{def}}{=}\sum_{i=0}^{\infty}\left(\dim_k \operatorname{Tor}_i^R(k,k)\right)T^i\right) = \frac{\prod_{i>0;\text{odd}}(1+T^i)^{\epsilon_i}}{\prod_{j>0;\text{even}}(1-T^j)^{\epsilon_j}}$$

Exercise 1.26. Prove the above equality concerning the Poincaré series.

Exercise 1.27. Let k be a field, and let $R := k[t]/(t^2) \twoheadrightarrow S := R/(t) = k$. In this case, consider the dg R-algebra $A = R\langle X, Y\rangle$, with $dX = t$ and $dY = Xt$ ($|X| = 1$, $|Y| = 2$). Show that $H(A) \cong k$, and so A is a Tate resolution of k over R. What is the Poincaré series?

2. DG Modules

Definition 2.1. Let A be a dg R-algebra. A right *dg A-module* M, or more precisely (M, ∂^M), is a graded right A-module $M = \bigoplus_{i\in\mathbb{Z}} M_i$ with graded R-linear mapping $\partial^M : M \to M[-1]$ satisfying $(\partial^M)^2 = 0$ and the Leibniz rule;

$$\partial^M(ma) = \partial^M(m)\ a + (-1)^{|m|}m\ d^A(a),$$

for all homogeneous elements $a \in A$ and $m \in M$.

2.2. *All dg modules considered in these notes are right dg modules, unless otherwise stated.*

Definition 2.3. Let A be a dg R-algebra, and let M and N be dg A-modules. We say that a mapping $f : M \to N$ is a dg A-module homomorphism if f is a graded A-module homomorphism (that is, f is A-linear and $f(M_n) \subset N_n$ for all $n \in \mathbb{Z}$) and satisfies the equality $f \circ \partial^M = \partial^N \circ f$.

Exercise 2.4. Considering R as a dg R-algebra, prove that a dg R-module is nothing but a chain complex of R-modules and a dg R-module homomorphism is a chain map between chain complexes.

Exercise 2.5. If A is graded commutative, then a right dg A-module M is also a left dg A-module with the left A-action defined by $am = (-1)^{|m||a|}ma$ for $a \in A$ and $m \in M$. Verify this left action is well-defined.

Definition 2.6. Let A be a dg R-algebra. We denote by $\mathcal{C}(A)$ the category of dg A-modules and dg A-module homomorphisms.

Exercise 2.7. Prove that the morphism set $\operatorname{Hom}_{\mathcal{C}(A)}(M, N)$ is naturally an R-module for $M, N \in \mathcal{C}(A)$.

Exercise 2.8. Let $f : M \to N$ be a morphism in $\mathcal{C}(A)$, i.e., a dg A-module homomorphism.

(1) Show that the kernel, the cokernel, the image and the coimage of f as underlined graded A-modules, have the natural structure of dg A-modules. Therefore the category $\mathcal{C}(A)$ is closed under such operations.

(2) Prove the natural isomorphism $M/\operatorname{Ker}(f) \cong \operatorname{Im}(f)$ in $\mathcal{C}(A)$.

Theorem 2.9. *The category $\mathcal{C}(A)$ is an abelian category which admits the product $\prod$, the sum $\coprod$, the inverse limit $\varprojlim$ and the inductive limit $\varinjlim$.*

2.10 (Outline of proof)**.** It follows from Exercise 2.8 above that $\mathcal{C}(A)$ is an abelian category. For a family of dg A-modules M_λ ($\lambda \in \Lambda$), we define the graded A-modules;

$$\prod_\lambda M_\lambda = \bigoplus_{n\in\mathbb{Z}} \left(\prod_\lambda (M_\lambda)_n \right), \quad \coprod_\lambda M_\lambda = \bigoplus_{n\in\mathbb{Z}} \left(\coprod_\lambda (M_\lambda)_n \right),$$

where the product and the coproduct in the right hand sides are the ones as R-modules. Furthermore we define the differentials diagonally as;

$$\partial^{\prod M_\lambda}((m_\lambda)_\lambda) = \left((\partial^{M_\lambda}(m_\lambda) \right)_\lambda, \quad \partial^{\coprod M_\lambda}((m_\lambda)_\lambda) = \left((\partial^{M_\lambda}(m_\lambda) \right)_\lambda,$$

for $(m_\lambda)_\lambda \in \prod_\lambda (M_\lambda)_n$ (resp. $\in \coprod_\lambda (M_\lambda)_n$).

Then $(\prod_\lambda M_\lambda, \partial^{\prod M_\lambda})$ and $(\coprod_\lambda M_\lambda, \partial^{\coprod M_\lambda})$ are dg A-modules. It is easy to see the following functorial isomorphism;

$$\operatorname{Hom}_{\mathcal{C}(A)}(-, \prod_\lambda M_\lambda) \cong \prod_\lambda \operatorname{Hom}_{\mathcal{C}(A)}(-, M_\lambda), \ \operatorname{Hom}_{\mathcal{C}(A)}(\coprod_\lambda M_\lambda, -) \cong \prod_\lambda \operatorname{Hom}_{\mathcal{C}(A)}(M_\lambda, -),$$

where the products in the RHS are taken as R-modules. This completes the proof to the existence of products and coproducts in $\mathcal{C}(A)$. We leave the proof for $\varprojlim$ and $\varinjlim$ to the reader. □

Definition 2.11 (degree shift)**.** Let $n \in \mathbb{Z}$. We define the n-shift functor $[n] : \mathcal{C}(A) \to \mathcal{C}(A)$ as follows:

For a dg A-module $(M, \partial^M) \in \mathcal{C}(A)$, define the graded A-module $M[n]$ as $M[n]_i = M_{n+i}$ for $i \in \mathbb{Z}$, and the differential $\partial^{M[n]} = (-1)^n \partial^M$.

For a dg A-module homomorphism $f : M \to N$, define $f[n] : M[n] \to N[n]$ as the same mapping as f, that is, $f[n](x) = f(x)$ for $x \in M[n]_i = M_{n+i}$.

Exercise 2.12. Show the equality $M[n][m] = M[n+m]$ for $n, m \in \mathbb{Z}$. In particular the shift functor $[n] : \mathcal{C}(A) \to \mathcal{C}(A)$ is an isomorphism of categories.

Definition 2.13 (cone)**.** Let $f : M \to N$ be a morphism in $\mathcal{C}(A)$. We define the dg A-module $C(f)$, which is called the cone (or the mapping cone) of f, as follows;

$$C(f) = \bigoplus_{n\in\mathbb{Z}} C(f)_n \text{ where } C(f)_n = N_n \oplus M_{n-1}$$

$$\partial^{C(f)} = \begin{pmatrix} \partial^N & f[-1] \\ 0 & \partial^{M[-1]} \end{pmatrix}$$

More precisely, the differential on $C(f)$ is defined by

$$\partial^{C(f)} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \partial^N(x) + f(y) \\ -\partial^M(y) \end{pmatrix} \text{ for } \begin{pmatrix} x \\ y \end{pmatrix} \in C(f)_n = N_n \oplus M_{n-1}.$$

Exercise 2.14. Verify the equality $\left(\partial^{C(f)}\right)^2 = 0$ and the Leibniz rule for $\partial^{C(f)}$.

2.15. The graded A-module homomorphism $\iota = \begin{pmatrix} 1 \\ 0 \end{pmatrix} : N \to C(f) = N \oplus M[-1]$ is a dg A-module homomorphism, since

$$\begin{pmatrix} \partial^N & f[-1] \\ 0 & \partial^{M[-1]} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \partial^N.$$

Similarly the graded A-module homomorphism $\pi = \begin{pmatrix} 0 & 1 \end{pmatrix} : C(f) = N \oplus M[-1] \to M[-1]$ is also a dg A-module homomorphism, since

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \partial^N & f[-1] \\ 0 & \partial^{M[-1]} \end{pmatrix} = \partial^{M[-1]} \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

As a consequence of this observation we have the following lemma.

Lemma 2.16. *For a dg A-module homomorphism $f : M \to N$, there is a short exact sequence in $\mathcal{C}(A)$;*

$$0 \longrightarrow N \xrightarrow{\iota} C(f) \xrightarrow{\pi} M[-1] \longrightarrow 0.$$

Definition 2.17. Let

$$0 \longrightarrow L \xrightarrow{p} M \xrightarrow{q} N \longrightarrow 0$$

be a short exact sequence in $\mathcal{C}(A)$. It is called an *admissible* short exact sequence if it splits as a short exact sequence of underlying graded A-modules. In such a case, p is called an *admissible monomorphism* (or admissible mono), and q an *admissible epimorphism* (or admissible epi).

The short exact sequence in Lemma 2.16 is an admissible short exact sequence, $\iota : N \to C(f)$ is an admissible mono, and $\pi : C(f) \to M[-1]$ is an admissible epi.

2.18. Let $f, g \in \mathrm{Hom}_{\mathcal{C}(A)}(M, N)$ be morphisms. We say that f and g are *homotopic*, denoted by $f \sim g$, if there is a graded A-module homomorphism $h \colon M \to N[1]$ of underlying graded A-modules such that $f - g = \partial^N h + h\partial^M$.

The following theorem shows that every admissible short exact sequence comes from mapping cone construction.

Theorem 2.19. *Let*

$$0 \longrightarrow L \xrightarrow{p} M \xrightarrow{q} N \longrightarrow 0$$

be an admissible short exact sequence in $\mathcal{C}(A)$.

(1) There is a morphism $f : N[1] \to L$ such that M is isomorphic to $C(f)$ in $\mathcal{C}(A)$ and this isomorphism makes the following diagram commutative;

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{\iota} & C(f) & \xrightarrow{\pi} & N & \longrightarrow & 0 \\ & & \| & & \downarrow{\cong} & & \| & & \\ 0 & \longrightarrow & L & \xrightarrow{p} & M & \xrightarrow{q} & N & \longrightarrow & 0 \end{array}$$

(2) The morphism f in (1) is unique up to homotopy. More precisely saying, if $g : N[1] \to L$ is a morphism that makes the following diagram commutative;

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & C(f) & \longrightarrow & N & \longrightarrow & 0 \\ & & \| & & \downarrow{\cong} & & \| & & \\ 0 & \longrightarrow & L & \longrightarrow & C(g) & \longrightarrow & N & \longrightarrow & 0, \end{array}$$

then it holds that $f \sim g$ *in* $\mathrm{Hom}_{\mathcal{C}(A)}(N[1], L)$.

Proof. (1) From the definition of admissible sequence, M is a direct sum of L and N as an underlying graded A-module; $M = L \oplus N$, and according to this direct decomposition, the morphisms p and q are described as $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \end{pmatrix}$ respectively. We also write the differential as

$$\partial^M = \begin{pmatrix} a & f \\ b & c \end{pmatrix} : L \oplus N \to L[-1] \oplus N[-1]$$

where a, b, c, f are R-linear mappings. That p is a dg A-module homomorphism forces that $\partial^M p = p\partial^L$, that is,

$$\begin{pmatrix} a & f \\ b & c \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \partial^L,$$

hence we have $b = 0$ and $a = \partial^L$. Similarly, since $\begin{pmatrix} 0 & 1 \end{pmatrix} \partial^M = \partial^N \begin{pmatrix} 0 & 1 \end{pmatrix}$, we have $c = \partial^N = -\partial^{N[1]}$. Therefore ∂^M is described as;

$$\partial^M = \begin{pmatrix} \partial^L & f \\ 0 & -\partial^{N[1]} \end{pmatrix} : L \oplus N \to L[-1] \oplus N[-1]$$

where $f : N \to L[-1]$ is just a graded R-linear map. Now the Leibniz rule for ∂^M implies

$$\partial^M \left(\begin{pmatrix} 0 \\ x \end{pmatrix} a \right) = \left(\partial^M \begin{pmatrix} 0 \\ x \end{pmatrix} \right) a + \begin{pmatrix} 0 \\ x \end{pmatrix} da,$$

for all $x \in N$ and $a \in A$. By computation it implies that $f(xa) = f(x)a$ for $x \in N$ and $a \in A$. Thus f is a graded A-linear map. Finally $(\partial^M)^2 = 0$ implies that $f\partial^{N[1]} = \partial^L f$, and so f is a dg A-module homomorphism. By the above construction we can see that $M \cong C(f)$ in $\mathcal{C}(A)$.

(2) Let $\varphi : C(f) \to C(g)$ be a dg A-module isomorphism that fits in the diagram. Similar to the proof of (1), we have $C(f) = L \oplus N = C(g)$ as underlying graded A-modules. According to this decomposition, φ is described as

$$\varphi = \begin{pmatrix} a & h \\ b & c \end{pmatrix} : L \oplus N \to L \oplus N,$$

where a, b, h and c are graded A-linear map. The commutativity of the left square in the diagram means $\varphi \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, hence $a = 1_L$ and $b = 0$. By the commutativity of the right square we also have $c = 1_N$. Thus

$$\varphi = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} : L \oplus N \to L \oplus N.$$

Since φ is a dg A-module homomorphism, it commutes with the differential on M;

$$\begin{pmatrix} \partial^L & g \\ 0 & -\partial^{N[1]} \end{pmatrix} \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \partial^L & f \\ 0 & -\partial^{N[1]} \end{pmatrix}.$$

This shows that $\partial^L h + g = f - h\partial^{N[1]}$, thus h is the homotopy that connects f and g, therefore $f \sim g$ as desired. □

3. The Homotopy Category $\mathcal{K}(A)$

In this section A is a dg R-algebra that is not necessarily commutative nor positively graded.

Definition 3.1. We define the category $\mathcal{K}(A)$ as follows:

- Objects of $\mathcal{K}(A)$ are all of (right) dg A-modules, i.e. $Ob(\mathcal{K}(A)) = Ob(\mathcal{C}(A))$.
- Morphisms in $\mathcal{K}(A)$ are the homotopy classes of the morphisms in $\mathcal{C}(A)$;

$$\mathrm{Hom}_{\mathcal{K}(A)}(X,Y) = \mathrm{Hom}_{\mathcal{C}(A)}(X,Y)/\sim \quad \text{for objects } X, Y \in \mathcal{K}(A).$$

We call $\mathcal{K}(A)$ the homotopy category of a dg algebra A.

Remark 3.2. Let $X, Y \in \mathcal{C}(A)$. For $n \in \mathbb{Z}$, we denote by $\mathrm{Hom}_{\mathrm{gr}A\text{-mod}}(X,Y)_n$ the set of all graded A-module homomorphism from X to Y of degree n, that is, $f \in \mathrm{Hom}_{\mathrm{gr}A\text{-mod}}(X,Y)_n$ means that $f : X \to Y$ is a A-module homomorphism with $f(X_i) \subset Y_{i+n}$ for all $i \in \mathbb{Z}$. Now set

$$\mathrm{Hom}_A(X,Y) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{gr}A\text{-mod}}(X,Y)_n$$

and define

$\partial^{\mathrm{Hom}}(f) := \partial^Y f - (-1)^n f \partial^X \in \mathrm{Hom}_{\mathrm{gr}A\text{-mod}}(X,Y)_{n-1}$ for $f \in \mathrm{Hom}_{\mathrm{gr}A\text{-mod}}(X,Y)_n$.

Then $(\mathrm{Hom}_A(X,Y), \partial^{\mathrm{Hom}})$ is a chain complex of R-modules. By definition, we have equalities;

$$\begin{cases} \mathrm{Hom}_{\mathcal{C}(A)}(X,Y) = Z_0(\mathrm{Hom}_A(X,Y)) \text{ (cycles)}, \\ \mathrm{Hom}_{\mathcal{K}(A)}(X,Y) = H_0(\mathrm{Hom}_A(X,Y)) \text{ (homology)}. \end{cases}$$

Exercise 3.3. Prove the above equalities.

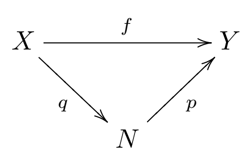
Exercise 3.4. Show that an object X is isomorphic to 0 in $\mathcal{K}(A)$ if and only if the identity mapping 1_X on X is null-homotopic, i.e. $1_X \sim 0$. We call such an object X *null*.

Exercise 3.5. Let $X \in \mathcal{K}(A)$. Then prove that $C(1_X)$ is null in $\mathcal{K}(A)$.

Hint: $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \partial^X & 1_X \\ 0 & -\partial^X \end{pmatrix} - \begin{pmatrix} \partial^X & 1_X \\ 0 & -\partial^X \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1_X & 0 \\ 0 & 1_X \end{pmatrix}$, which shows $1_{C(1_X)} \sim 0$.

Lemma 3.6. *The following conditions are equivalent for* $f \in \mathrm{Hom}_{\mathcal{C}(A)}(X,Y)$.

(1) $f = 0$ *in* $\mathcal{K}(A)$.
(2) $f \sim 0$.
(3) f *factors through a null object, i.e. there is a commutative diagram in* $\mathcal{C}(A)$;



where N *is a null object.*

Proof. The equivalence (1) $\Leftrightarrow$ (2) follows from the definition. For (3) $\Rightarrow$ (2), it is enough to apply the following well-known and easily proved fact:

- Let $U_1 \xrightarrow{a} U_2 \overset{f}{\underset{g}{\rightrightarrows}} U_3 \xrightarrow{b} U_4$ be morphisms in $\mathcal{C}(A)$. If $f \sim g$, then we have $bfa \sim bga$.

Actually, since $1_N \sim 0$, we have $f = p1_N q \sim 0$.

Now we prove (2) $\Rightarrow$ (3). Assume $f \sim 0$, and h is its homotopy, that is, $f = \partial^Y h - h\partial^X$. Then set $N = C(1_{Y[1]})$ that is null, and note that $N = Y[1] \oplus Y$ as underlying graded A-module and the differential is $\partial^N = \begin{pmatrix} -\partial^Y & 1 \\ 0 & \partial^Y \end{pmatrix}$. The natural projection $p = \begin{pmatrix} 0 & 1 \end{pmatrix} : Y[1] \oplus Y \to Y$ is easily seen to be a dg A-module homomorphism. Consider $q = \begin{pmatrix} h \\ f \end{pmatrix} : X \to Y[1] \oplus Y$. Since

$$\begin{pmatrix} -\partial^Y & 1 \\ 0 & \partial^Y \end{pmatrix} \begin{pmatrix} h \\ f \end{pmatrix} = \begin{pmatrix} f - \partial^Y h \\ \partial^Y f \end{pmatrix} = \begin{pmatrix} h \\ f \end{pmatrix} \partial^X,$$

q is a morphism in $\mathcal{C}(A)$. Finally, since we have $f = pq$, condition (3) holds. □

Lemma 3.7. *Let*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X' & \xrightarrow[f']{} & Y' \end{array}$$

be a commutative diagram in $\mathcal{K}(A)$. Then replacing Y with $Y \oplus N$ where N is null, there is a commutative square of the same form in $\mathcal{C}(A)$.

Proof. The commutativity in $\mathcal{K}(A)$ shows that $f'a \sim bf$. Then, by Lemma 3.6, $f'a - bf : X \to Y'$ factors through a null N;

$$\begin{array}{ccc} X & \xrightarrow{f'a-bf} & Y' \\ & {\scriptstyle q}\searrow \quad \nearrow{\scriptstyle p} & \\ & N & \end{array}$$

Since $f'a - bf = pq$, the following is commutative in $\mathcal{C}(A)$;

$$\begin{array}{ccc} X & \xrightarrow{\begin{pmatrix} f \\ q \end{pmatrix}} & Y \oplus N \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle \begin{pmatrix} b & p \end{pmatrix}} \\ X' & \xrightarrow[f']{} & Y' \end{array}$$

□

Exercise 3.8. Prove that every morphism in $\mathcal{K}(A)$ is represented by an admissible monomorphism (resp. an admissible epimorphism).

Hint: If $f : X \to Y$ is a morphism in $\mathcal{C}(A)$, then $\begin{pmatrix} i \\ f \end{pmatrix} : X \to C(1_X) \oplus Y$ is an admissible monomorphism, wehre $i : X \to C(1_X)$ is a natural injection.

Definition 3.9 (shift functor)**.** Recall that $[-1] : \mathcal{C}(A) \to \mathcal{C}(A)$ is the (-1)-shift functor that sends $X \xrightarrow{f} Y$ in $\mathcal{C}(A)$ to $X[-1] \xrightarrow{f[-1]} Y[-1]$ in $\mathcal{C}(A)$. It is easy to see that $f \sim g$ in $\mathcal{C}(A)$ if and only if $f[-1] \sim g[-1]$. Therefore it naturally induces the functor $\Sigma : \mathcal{K}(A) \to \mathcal{K}(A)$, which we call the shift functor in $\mathcal{K}(A)$.

Definition 3.10 (triangle)**.** Let

$$0 \longrightarrow L \xrightarrow{p} M \xrightarrow{q} N \longrightarrow 0$$

be an admissible short exact sequence in $\mathcal{C}(A)$. Then, by Theorem, 2.19 there is a unique morphism $r : N \to \Sigma L$ in $\mathcal{K}(A)$, and thus the following diagram in $\mathcal{K}(A)$ is uniquely determined by the admissible short exact sequence;

$$L \xrightarrow{p} M \xrightarrow{q} N \xrightarrow{r} \Sigma L,$$

which we call a *distinguished triangle* in $\mathcal{K}(A)$.

Every diagram which is isomorphic to a distinguished triangle is called an *exact triangle* or simply a *triangle.* [4]

Theorem 3.11 (triangulated category)**.** *The homotopy category $\mathcal{K}(A)$ (with the notion of shift and triangle) is a triangulated category, i.e. $\mathcal{K}(A)$ satisfies the following axioms TR1- TR4:*

TR1 (*i*) $X \xrightarrow{1_X} X \to 0 \to \Sigma X$ *is a triangle for* $X \in \mathcal{K}(A)$.
(*ii*) *For any morphism* $f : X \to Y$ *in* $\mathcal{K}(A)$, *there is a triangle of the form* $X \xrightarrow{f} Y \to Z \to \Sigma X$.

TR2 $X \xrightarrow{a} Y \xrightarrow{b} Z \xrightarrow{c} \Sigma X$ *is a triangle if and only if* $Y \xrightarrow{b} Z \xrightarrow{c} \Sigma X \xrightarrow{-\Sigma a} \Sigma Y$ *is a triangle.*

TR3 For a diagram

$$\begin{array}{ccccccc} X & \xrightarrow{a} & Y & \xrightarrow{b} & Z & \xrightarrow{c} & \Sigma X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & & & \downarrow{\scriptstyle \Sigma f} \\ X' & \xrightarrow{a'} & Y' & \xrightarrow{b'} & Z' & \xrightarrow{c'} & \Sigma X', \end{array}$$

where the left square is commutative, there is a morphism $h : Z \to Z'$ that makes the following diagram commutative;

$$\begin{array}{ccccccc} X & \xrightarrow{a} & Y & \xrightarrow{b} & Z & \xrightarrow{c} & \Sigma X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Sigma f} \\ X' & \xrightarrow{a'} & Y' & \xrightarrow{b'} & Z' & \xrightarrow{c'} & \Sigma X' \end{array}$$

[4]As one can easily imagine, a "morphism between diagrams" is the sets of morphisms on each object in the diagrams that commute with morphisms in the diagrams, and an "isomorphisms between diagrams" is defined to be an invertible morphism between diagrams.

TR4 (Octahedron axiom) Let

$$\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\
\| & & \downarrow & & & & \| \\
X & \longrightarrow & U & \longrightarrow & V & \longrightarrow & \Sigma X \\
& & \downarrow & & & & \\
& & W & & & & \\
& & \downarrow & & & & \\
& & \Sigma Y & & & &
\end{array}$$

be a diagram in $\mathcal{K}(A)$ whose rows and columns are triangles and whose left square is commutative. Then there is a triangle $Z \to V \to W \to \Sigma Z$ that makes the following diagram commutative;

$$\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\
\| & & \downarrow & & \downarrow & & \| \\
X & \longrightarrow & U & \longrightarrow & V & \longrightarrow & \Sigma X \\
& & \downarrow & & \downarrow & & \\
& & W & = & W & & \\
& & \downarrow & & \downarrow & & \\
& & \Sigma Y & \longrightarrow & \Sigma Z & &
\end{array}$$

3.12 (Outline of proof)**.** TR1(i) : $0 \to X \xrightarrow{1_X} X \to 0 \to 0$ is an admissible short exact sequence. □

TR1(ii) : Let $i : X \to C(1_X)$ be a natural injection. Then it is easily seen that $\begin{pmatrix} i \\ f \end{pmatrix} : X \to C(1_X) \oplus Y$ is an admissible monomorphism. Therefore there is an admissible short exact sequence of the form;

$$0 \to X \xrightarrow{\begin{pmatrix} i \\ f \end{pmatrix}} C(1_X) \oplus Y \to Z \to 0.$$

Since $C(1_X) \cong 0$ in $\mathcal{K}(A)$, there is a triangle $X \xrightarrow{f} Y \to Z \to \Sigma X$. □

TR2 : Assume $X \xrightarrow{a} Y \xrightarrow{b} Z \xrightarrow{c} \Sigma X$ is a triangle. Without loss of generality we may assume there is an admissible short exact sequence $0 \to X \xrightarrow{a} Y \xrightarrow{b} Z \to 0$. By Theorem 2.19 we may also assume that $Y = C(c)$ for $c : Z \to X[-1]$ and $a : X \to Y = C(c)$ is a natural injection as an underlying graded A-module. Then, by computation, one can show that $C(a) \cong Z$. (Prove this as an exercise.) Thus there is an admissible short exact sequence $0 \to Y \xrightarrow{b} Z \xrightarrow{c} \Sigma X \to 0$, and a triangle $Y \xrightarrow{b} Z \xrightarrow{c} \Sigma X \xrightarrow{-\Sigma a} \Sigma Y$. □

TR3 : Using Lemma 3.7, we may assume that there is a commutative diagram in $\mathcal{C}(A)$;

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{a} & Y & \xrightarrow{b} & Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & & & \\ 0 & \longrightarrow & X' & \xrightarrow{a'} & Y' & \xrightarrow{b'} & Z' & \longrightarrow & 0, \end{array}$$

where both rows are admissible short exact sequences. It is obvious that the morphism $h : Z \to Z'$ is induced.

TR4 : Using Lemma 3.7 and Excercise 3.8 we may assume that there is a commutative diagram in $\mathcal{C}(A)$;

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \| & & \downarrow \\ X & \longrightarrow & U, \end{array}$$

where all the morphism are admissible monomorphisms. Then we have the commutative diagram;

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X & \longrightarrow & U & \longrightarrow & V & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & W & = & W & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0, & & \end{array}$$

where all the rows and columns are admissible short exact sequences. □

Exercise 3.13. This is an exercise for a general triangulated category. Prove the following questions only using the axioms TR1 - TR4.

Assume that $X \xrightarrow{a} Y \xrightarrow{b} Z \xrightarrow{c} \Sigma X$ is a triangle.

(1) Prove that $ba = 0$.
(2) For any morphism $f : W \to Y$ in $\mathcal{K}(A)$, if $bf = 0$, then there is $g : W \to X$ with $f = ag$.
(3) For any object $W \in \mathcal{K}(A)$, the following induced sequence is exact (as abelian groups);

$$\mathrm{Hom}_{\mathcal{K}(A)}(W, X) \longrightarrow \mathrm{Hom}_{\mathcal{K}(A)}(W, Y) \longrightarrow \mathrm{Hom}_{\mathcal{K}(A)}(W, Z) \longrightarrow \mathrm{Hom}_{\mathcal{K}(A)}(W, \Sigma X)$$

(Using TR2 successively, we have a log exact sequence of infinite length.)
(4) There is a homology long exact sequence;

$$\cdots \longrightarrow H_n(X) \longrightarrow H_n(Y) \longrightarrow H_n(Z) \longrightarrow H_{n-1}(X) \longrightarrow \cdots$$

(Just notice that $H_n(X) = \mathrm{Hom}_{\mathcal{K}(A)}(A, X[n])$ for $n \in \mathbb{N}$.)
(5) If $c = 0$ then $Y \cong X \oplus Z$ in $\mathcal{K}(A)$.

4. Semi-Free DG Modules

Let A be a dg R-algebra.

4.1. For an object $X \in \mathcal{K}(A)$, note from Remark 3.2 that $\mathrm{Hom}_{\mathcal{K}(A)}(A, X) = H_0(X)$. More generally, it holds for any $n \in \mathbb{Z}$ that

$$\mathrm{Hom}_{\mathcal{K}(A)}(A[-n], X) = H_n(X).$$

As a dg A-module, $A[-n]$ is free of rank one as a graded A-module with generator e of degree n and $\partial^{A[-n]}(e) = 0$;

$$A[-n] = eA \quad \text{with } |e| = n,\ \partial(e) = 0.$$

In this case, we have an equality $\partial(ea) = ed^A(a)$ for any $a \in A$.

Definition 4.2. A dg A-module $F \in \mathcal{C}(A)$ is called a *free* dg A-module if $F = \coprod_\lambda e_\lambda A$ in $\mathcal{C}(A)$ with homogeneous free basis $\{e_\lambda\}_{\lambda \in \Lambda}$, that is, $F \cong \bigoplus_\lambda e_\lambda A$ is free as an underlying graded A-module and $\partial^F(e_\lambda) = 0$ for $\lambda \in \Lambda$. Note in this case that $F \cong \coprod_\lambda A(-|e_\lambda|)$.

Definition 4.3. A dg A-module $F \in \mathcal{C}(A)$ is called *semi-free* (abbreviation; sf) if there exists an increasing filtration

$$0 = F_{-1} \subseteq F_0 \subseteq F_1 \subseteq \cdots \subseteq F_i \subseteq F_{i+1} \cdots \subseteq F$$

of dg A-submodules such that $F = \bigcup_{i \in \mathbb{N}} F_i$ and each dg A-module F_i/F_{i-1} is a free dg A-module. See [3] or [5]. The filtration $\{F_i\}_{i \in \mathbb{N}}$ is said to be a semi-free filtration of the semi-free dg module F.

Note in this case that each natural sequence $0 \longrightarrow F_{i-1} \longrightarrow F_i \longrightarrow F_i/F_{i-1} \longrightarrow 0$ is an admissible short exact sequence for $i \in \mathbb{N}$. Thus, for each $i \in \mathbb{N}$, we can take a subset $\mathbb{B}_i = \{e_{i\lambda_i} \mid \lambda_i \in \Lambda_i\}$ of F_i which gives a free basis of F_i/F_{i-1}. Then the disjoint union $\mathbb{B} = \bigcup_{i \in \mathbb{N}} \mathbb{B}_i$ is a free basis of the underlying graded A-module F and satisfies

$$\partial^F(\mathbb{B}_n) \in \sum_{i<n} \mathbb{B}_i A. \tag{4.3.1}$$

The set $\mathbb{B}$ is called a semi-free basis of the semi-free dg A-module F. Note that a dg A-module is semi-free if and only if it has a semi-free basis, more precisely saying, if a dg A-module F is free as an underlying graded A-module and if there is a basis $\mathbb{B} = \bigcup_{i \in \mathbb{N}} \mathbb{B}_i$ that satisfies (4.3.1), then F is semi-free.

Lemma 4.4. *The following are equivalent for $F \in \mathcal{C}(A)$.*

(1) F is semi-free.
(2) F is free as an underlying graded A-module with basis $\{e_\lambda | \lambda \in \Lambda\}$ that is indexed by a well-ordered set $(\Lambda, <)$ and satisfies;

$$\partial^F(e_\lambda) \in \sum_{\mu<\lambda} e_\mu A, \tag{4.4.1}$$

for all $\lambda \in \Lambda$.

Proof. (1) $\Rightarrow$ (2): Assume F has a semi-free basis $\mathbb{B} = \bigcup_{i \in \mathbb{N}} \mathbb{B}_i$ that satisfies (4.3.1). By the axiom of choice we can make each $\mathbb{B}_i$ a well-ordered set, then $\mathbb{B}$ is given the well-order as the sum of ordinals $\mathbb{B}_0 \sqcup \mathbb{B}_1 \sqcup \mathbb{B}_2 \sqcup \cdots$. Then the well-ordered set $\mathbb{B}$ satisfies (4.4.1).

(2) $\Rightarrow$ (1): Assume we have a well-ordered free basis $\mathbb{B} = \{e_\lambda | \lambda \in \Lambda\}$ satisfying (4.4.1). We construct a subset $\mathcal{E}_n$ of $\mathbb{B}$ by induction on n: Starting from $\mathcal{E}_{-1} = \emptyset$, we set $\mathcal{E}_n = \{e \in \mathbb{B} | \ \partial^F(e) \in \sum_{e' \in \mathcal{E}_{n-1}} e'A\}$. Then it is an increasing sequence of subsets;

$$\mathcal{E}_{-1} = \emptyset \subset \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n \subset \cdots \subset \mathbb{B}$$

and it satisfies that $\partial(\mathcal{E}_n) \subset \sum_{e \in \mathcal{E}_{n-1}} eA$ for each $n \in \mathbb{N}$. Set $\mathcal{E} = \bigcup_{n \in \mathbb{N}} \mathcal{E}_n$. If $\mathcal{E} = \mathbb{B}$ then the subsets $\mathbb{B}_n = \mathcal{E}_n \backslash \mathcal{E}_{n-1}$ satisfy (4.3.1), and the proof is completed. To show $\mathcal{E} = \mathbb{B}$, we prove $e_\lambda \in \mathcal{E}$ by transfinite induction on $\lambda \in \Lambda$. If e_0 is the initial element, then $\partial^E(e_0) = 0$ by (4.4.1), and thus $e_0 \in \mathcal{E}_0 \subset \mathcal{E}$. Assume $e_\mu \in \mathcal{E}$ for all $\mu < \lambda$. By (4.4.1) we have $\partial^E(e_\lambda) = e_{\mu_1}a_1 + \cdots + e_{\mu_r}a_r$ where $\mu_i < \lambda$ and $a_i \in A$ for $1 \leqslant i \leqslant r$. Since $e_{\mu_i} \in \mathcal{E}$ for $1 \leqslant i \leqslant r$ by induction hypothesis, taking $m \in \mathbb{N}$ enough large we have $e_{\mu_i} \in \mathcal{E}_m$ for $1 \leqslant i \leqslant r$. Then $e_\lambda \in \mathcal{E}_{m+1}$ by the definition, hence $e_\lambda \in \mathcal{E}$ as desired. □

Lemma 4.5. *Let $\{E_\lambda \mid \lambda \in \Lambda\} \subset \mathcal{C}(A)$ be any set of semi-free dg A-modules. Then the coproduct (direct sum) $\coprod_{\lambda \in \Lambda} E_\lambda$ is semi-free.*

The lemma follows only from the definition. The proof is left to the reader as an exercise.

Lemma 4.6. *Let $f : E \to F$ be a morphism in $\mathcal{C}(A)$. Assume E and F are semi-free. Then the mapping cone $C(f)$ is also semi-free.*

Proof. By Lemma 4.4, E (resp. F) has a free basis $\{e_\lambda | \ \lambda \in \Lambda\}$ (resp. $\{e'_{\lambda'} | \ \lambda' \in \Lambda'\}$) indexed by a well-ordered set that satisfies (4.4.1). Then the cone $C(f)$ has a free basis $\{e_\lambda | \ \lambda \in \Lambda\} \coprod \{e'_{\lambda'} | \ \lambda' \in \Lambda'\}$ as an underlying graded A-module. Note that this basis is indexed by the well-ordered set $\Lambda' \sqcup \Lambda$ (the sum of ordinals), and from the definition of the mapping cone the indexed basis elements satisfy (4.4.1). □

Exercise 4.7. Let $X_1, X_2, \ldots$ be (maybe an infinite number of) variables over a dg R-algebra A and let B be either a free extension $A\langle X_1, X_2, \ldots \rangle$ or a polynomial extension $A[X_1, X_2, \ldots]$, which we can regard as a (right) dg A-module. Prove the dg A-module B is semi-free.

Lemma 4.8. *Let A be a non-negatively graded dg R-algebra and let M be a dg A-module that is bounded below, i.e. there is an integer n_0 such that $M_n = 0$ for $n < n_0$. Then, M is semi-free if and only if M is free as an underlying graded A-module.*

Proof. Assume that M is free as an underlying graded A-module with homogeneous basis $\mathbb{B}$. Let n_0 be the least degree $|e|$ for $e \in \mathbb{B}$. Then set $\mathbb{B}_i = \{e | \ e \in \mathbb{B}, \ |e| = n_0 + i\}$ for $i \in \mathbb{N}$. Since ∂^M is of degree -1, it is easy to see that (4.3.1) holds. □

Exercise 4.9. Let $R = k[t]/(t^2)$ where k is a field. We consider R as a dg algebra concentrated in degree zero. Let $M = \bigoplus_{i \in \mathbb{Z}} e_i R$ be a free R-module with basis $\{e_i | \ i \in \mathbb{Z}\}$ with $|e_i| = i$. Defining the differential ∂ on M by $\partial(e_i) = e_{i-1}t$ for $i \in \mathbb{Z}$, we have a dg R-module (M, ∂). Note that, as a complex over R, it can be described as $\cdots \longrightarrow R \xrightarrow{t} R \xrightarrow{t} R \xrightarrow{t} \cdots$. Then show that M is not semi-free, even though it is free as underlying graded module. (Hint: Show $H(M) = 0$. Later in 4.16 we shall show if F is semi-free and if $H(F) = 0$ then $F \cong 0$ in $\mathcal{K}(A)$.)

Lemma 4.10. *Let $F \in \mathcal{C}(A)$ be a semi-free dg A-module with semi-free filtration $\{F_i | i \in \mathbb{N}\}$. Then there is an admissible short exact sequence of the following form;*

$$0 \longrightarrow \coprod_{i\in\mathbb{N}} F_i \xrightarrow{\lambda} \coprod_{i\in\mathbb{N}} F_i \xrightarrow{\rho} F \longrightarrow 0,$$

where λ maps $\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \end{pmatrix}$ *to* $\begin{pmatrix} a_0 \\ a_1 - a_0 \\ a_2 - a_1 \\ a_3 - a_2 \\ \vdots \end{pmatrix}$, *while ρ maps* $\begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ \vdots \end{pmatrix}$ *to* $\sum_{i\in\mathbb{N}} b_i$.

Proof. One can verify that λ and ρ are commutative with the differentials, hence they are dg A-module homomorphisms. It is also easy to verify that $\rho\lambda = 0$, λ is injective, and ρ is surjective, To show that it is exact at the middle term, let $\beta = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \end{pmatrix}$ be an element of $\coprod_{i\in\mathbb{N}} F_i$ and assume $\rho(\beta) = \sum_{i\in\mathbb{N}} b_i = 0$. Then setting

$$\alpha = \begin{pmatrix} b_0 \\ b_0 + b_1 \\ b_0 + b_1 + b_2 \\ \vdots \end{pmatrix},$$

we see that α is a well-defined element of $\coprod_{i\in\mathbb{N}} F_i$ and $\lambda(\alpha) = \beta$. Hence the sequence is exact. (Note that $\beta = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \end{pmatrix}$ is an element of $\coprod_{i\in\mathbb{N}} F_i$ if and only if $a_i \in F_i$ for all $i \in \mathbb{N}$ and $a_j \neq 0$ only for a finite number of $j \in \mathbb{N}$.)

Since F is free as an underlying graded A-module, the sequence is split as a sequence of underlying graded A-modules, and hence it is an admissible short exact sequence. □

Definition 4.11. Denote by $\mathcal{K}^{sf}(A)$ the full subcategory of $\mathcal{K}(A)$ consisting of all the dg A-modules that are isomorphic to semi-free dg A-modules.

By definition an additive full subcategory $\mathcal{U}$ of a triangulated category $\mathcal{T}$ is called a triangulated subcategory if $\mathcal{U}$ satisfies the following conditions;

(1) Let $U \in \mathcal{T}$. Then $U \in \mathcal{U}$ if and only if $\Sigma U \in \mathcal{U}$.
(2) Let $X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$ be a triangle in $\mathcal{T}$. If two of X, Y, Z belong to $\mathcal{U}$ then all belong to $\mathcal{U}$.

It is clear that a triangulated subcategory itself is a triangulated category. A triangulated subcategory is said to be a localizing subcategory if it is closed under taking coproducts. (See Definition 8.1 below.)

Theorem 4.12. *$\mathcal{K}^{sf}(A)$ is a triangulated subcategory of $\mathcal{K}(A)$. More precisely $\mathcal{K}^{sf}(A)$ is a localizing subcategory of $\mathcal{K}(A)$.*

Proof. It is obvious that $\mathcal{K}^{sf}(A)$ satisfies the condition (1) above. Condition (2) follows from Lemma 4.6. In fact if there is a triangle $X \xrightarrow{f} Y \longrightarrow Z \longrightarrow \Sigma X$ with

$X, Y \in \mathcal{K}^{sf}(A)$, then $Z \cong C(f)$ in $\mathcal{K}(A)$, and hence $Z \in \mathcal{K}^{sf}(A)$ by Lemma 4.6. Lemma 4.5 asserts that $\mathcal{K}^{sf}(A) \subset \mathcal{K}(A)$ is closed under taking coproducts. □

Remark 4.13. The subcategory $\mathcal{K}^{sf}(A) \subset \mathcal{K}(A)$ is a thick subcategory. (Recall that 'thick' means 'closed under direct summands'.) Actually $\mathcal{K}^{sf}(A)$ is a localizing subcategory of $\mathcal{K}(A)$, and it is known that any localizing subcategory is thick. See Remark 8.2.

Theorem 4.14. *$\mathcal{K}^{sf}(A)$ is the smallest localizing subcategory of $\mathcal{K}(A)$ that contains A.* (Under the notation in Definition 8.1, this statement is simply denoted as $\mathcal{K}^{sf}(A) = \mathrm{Loc}(A)$.)

Precisely saying the following holds: Let $\mathcal{U} \subset \mathcal{K}(A)$ be an additive full subcategory. Assume the following conditions are satisfied:

(1) $A \in \mathcal{U}$

(2) $\mathcal{U}$ is closed under shift functor; $Y \in \mathcal{U}$ if and only if $\Sigma Y \in \mathcal{U}$.

(3) $\mathcal{U}$ is closed under forming triangles; if there is a triangle $Y \to Z \to W \to \Sigma Y$ in $\mathcal{K}(A)$ and if two of $Y, Z, W \in \mathcal{U}$ then all belong to $\mathcal{U}$.

(4) $\mathcal{U}$ is closed under coproducts; if $Y_j \in \mathcal{U}$ for $j \in J$ then $\coprod_{j \in J} Y_j \in \mathcal{U}$.

Then all the semi-free A-modules belong to $\mathcal{U}$, i.e. $\mathcal{K}^{sf}(A) \subset \mathcal{U}$.

Proof. By (1)(2) and (4) all free dg A-modules are contained in $\mathcal{U}$. Let F be a semi-free dg A-module with semi-free filtration $\{F_i\}_{i \in \mathbb{N}}$. Starting from F_0 that is free, and using the condition (3) above, we can show that $F_i \in \mathcal{U}$ for each $i \in \mathbb{N}$ by induction on i. Thus we have $\coprod_{i \in \mathbb{N}} F_i \in \mathcal{U}$ by (4), hence using Lemma 4.10 and the condition (3) above, we finally get $F \in \mathcal{U}$. □

Theorem 4.15. *Assume $F \in \mathcal{K}(A)$ is a semi-free dg A-module. If $X \in \mathcal{K}(A)$ is acyclic* (i.e. $H(X) = 0$), then $\mathrm{Hom}_{\mathcal{K}(A)}(F, X) = 0$.

This property of F is often called *$\mathcal{K}$-projective* property. Thus the theorem says that semi-free dg modules are $\mathcal{K}$-projective.

Proof. For $X \in \mathcal{K}(A)$ and assume that $H(X) = 0$. We consider the full subcategory $\mathcal{U} = {}^{\perp}\langle X \rangle$ of $\mathcal{K}(A)$ which is defined by

$$\mathcal{U} = \{Y \in \mathcal{K}(A) | \ \mathrm{Hom}_{\mathcal{K}(A)}(Y, \Sigma^i X) = 0 \text{ for all } \ i \in \mathbb{Z}\}.$$

It is almost trivial that the subcategory $\mathcal{U} \subset \mathcal{K}^{sf}(A)$ satisfies the conditions (1) - (4) in Lemma 4.14. Thus we have $\mathcal{K}^{sf}(A) \subset \mathcal{U}$. □

Corollary 4.16. *Let $F \in \mathcal{K}(A)$ be semi-free. If $H(F) = 0$ then $F \cong 0$ in $\mathcal{K}(A)$.*

Proof. By the theorem $\mathrm{Hom}_{\mathcal{K}(A)}(F, F) = 0$, hence $1_F \sim 0$. □

Corollary 4.17. *Let $F, X \in \mathcal{K}(A)$ and assume that F is semi-free and $H(X) = 0$* (i.e. X is acyclic). Then the complex $\mathrm{Hom}_A(F, X)$ is also acyclic. (See Remark 3.2 for the definition of $\mathrm{Hom}_A(F, X)$.)

Proof. $H_n(\mathrm{Hom}_A(F, X)) = \mathrm{Hom}_{\mathcal{K}(A)}(F, \Sigma^{-n} X) = 0$. □

Recall that a morphism $f : X \to Y$ in $\mathcal{K}(A)$ is said to be a quasi-isomorphism (abbreviation; qis) if the homology mapping $H(f) : H(X) \to H(Y)$ is an isomorphism of graded $H(A)$-modules.

Corollary 4.18. *Assume that F is a semi-free dg A-module. Then the functor $\operatorname{Hom}_A(F,-) : \mathcal{C}(A) \to \mathcal{C}(R)$ preserves quasi-isomorphisms, that is, if $f : X \to Y$ is a quasi-isomorphism in $\mathcal{C}(A)$, then $\operatorname{Hom}_A(F,f) : \operatorname{Hom}_A(F,X) \to \operatorname{Hom}_A(F,Y)$ is a quasi-isomorphism in $\mathcal{C}(R)$.*

Proof. Take a mapping cone $C(f)$ and we see $H(C(f)) = 0$ since f is a qis. Since there is an admissible short exact sequence; $0 \to Y \to C(f) \to X[-1] \to 0$, we have an exact sequence in $\mathcal{C}(R)$;

$$0 \to \operatorname{Hom}_A(F,Y) \to \operatorname{Hom}_A(F,C(f)) \to \operatorname{Hom}_A(F,X[-1]) \to 0.$$

(Note, since F is free as an underlying graded A-module, $\operatorname{Hom}_A(F,-)$ keeps exactness.) Since $H(\operatorname{Hom}_A(F,C(f))) = 0$ by the previous corollary, it follows from the homology long exact sequence that
$H(\operatorname{Hom}_A(F,f)) : H(\operatorname{Hom}_A(F,X)) \to H(\operatorname{Hom}_A(F,Y))$ is an isomorphism. □

Corollary 4.19. *Let $f : F \to F'$ be a morphism in $\mathcal{K}(A)$ between semi-free dg A-modules. If f is a quasi-isomorphism, then f is an isomorphism in $\mathcal{K}(A)$.*

Proof. Take the mapping cone $C(f)$, then it is semi-free and acyclic, hence $C(f) \cong 0$ in $\mathcal{K}(A)$ by Corollary 4.16. Thus f is an isomorphism in $\mathcal{K}(A)$. □

Theorem 4.20 (semi-free resolution)**.** *For any $M \in \mathcal{K}(A)$, there exists a morphism $f : F \to M$ such that F is semi-free and f is a quasi-isomorphism. Such an F (or f) is called a semi-free resolution of M.*

To prove this theorem we need a lemma:

Lemma 4.21. *Let $M \in \mathcal{C}(A)$.*

(1) There exists a morphism $f_0 : F_0 \to M$ in $\mathcal{C}(A)$ where F_0 is a free dg A-module and $H(f_0) : H(F_0) \to H(N)$ is surjective. (Call f_0 homologically surjective.)
(2) Let $f_0 : F_0 \to M$ be a homologically surjective morphism in $\mathcal{C}(A)$. Then there exists a sequence of morphisms $G_0 \xrightarrow{g_0} F_0 \xrightarrow{f_0} M$ in $\mathcal{C}(A)$ where G_0 is free and $H(G_0) \xrightarrow{H(g_0)} H(F_0) \xrightarrow{H(f_0)} H(M) \longrightarrow 0$ is an exact sequence of graded $H(A)$-modules.

Proof. (1) Note $\operatorname{Hom}_{\mathcal{K}(A)}(A[-n], M) \cong H_n(M)$ for $n \in \mathbb{Z}$, and this isomorphism is realized by sending $g \in \operatorname{Hom}_{\mathcal{K}(A)}(A[-n], M)$ to $[g(1)] \in H_n(M)$. Take a subset $\{g_{nk_n}\}_{k_n}$ in $\operatorname{Hom}_{\mathcal{C}(A)}(A[-n], M)$ whose image in $\operatorname{Hom}_{\mathcal{K}(A)}(A[-n], M)$ generates $H_n(M)$ as an $H_0(A)$-module. Then consider the mapping $g_n : \coprod_{k_n} A[-n] \to M$ defined by $\{g_{nk_n}\}_{k_n}$. By construction $H_n(g_n)$ is surjective for each n. Now take all the sum of g_n for $n \in \mathbb{Z}$, and $f_0 := \sum_n g_n : \coprod_n(\coprod_{k_n} A[-n]) \to M$ satisfies the required condition.

(2) follows from (1) by taking $G_0 \to C(f_0)[1]$ to be homologically surjective. □

4.22 (Proof of Theorem 4.20)**.** By induction on n, we construct the following diagram (4.22.1) in $\mathcal{C}(A)$ where the right squares are commutative as morphisms in $\mathcal{K}(A)$ (not in $\mathcal{C}(A)$), all i_n are admissible monos, and all the sequence $H(G_n) \to$

$H(F_n) \to H(M) \to 0$ are exact.

$$
\begin{array}{ccccc}
G_0 & \xrightarrow{g_0} & F_0 & \xrightarrow{f_0} & M \\
 & & \downarrow{\scriptstyle i_0} & & \| \\
G_1 & \xrightarrow{g_1} & F_1 & \xrightarrow{f_1} & M \\
 & & \downarrow{\scriptstyle i_1} & & \| \\
 & & \vdots & & \vdots \\
 & & \downarrow & & \| \\
G_{n-1} & \xrightarrow{g_{n-1}} & F_{n-1} & \xrightarrow{f_{n-1}} & M \\
 & & \downarrow{\scriptstyle i_{n-1}} & & \| \\
G_n & \xrightarrow{g_n} & F_n & \xrightarrow{f_n} & M \\
 & & \downarrow & & \| \\
 & & \vdots & & \vdots
\end{array}
\tag{4.22.1}
$$

First of all, applying Lemma 4.21, we get $G_0 \xrightarrow{g_0} F_0 \xrightarrow{f_0} M$ in $\mathcal{C}(A)$ so that $H(G_0) \xrightarrow{H(g_0)} H(F_0) \xrightarrow{H(f_0)} H(M) \longrightarrow 0$ is exact. Take the mapping cone of g_0 and set $F_1 = C(g_0)$. Then there is an admissible short exact sequence in $\mathcal{C}(A)$; $0 \longrightarrow F_0 \xrightarrow{i_0} F_1 \longrightarrow G_0[-1] \longrightarrow 0$. Note in particular that i_0 is an admissible mono. From the definition there is a triangle in $\mathcal{K}(A)$; $G_0 \xrightarrow{g_0} F_0 \xrightarrow{i_0} F_1 \longrightarrow G_0[-1]$. Since $H(f_0 g_0) = H(f_0)H(g_0) = 0$, and since G_0 is free, it follows $f_0 g_0 = 0$ as a morphism $F_0 \to M$ in $\mathcal{K}(A)$. Thus from the property of triangles there is a morphism $F_1 \xrightarrow{f_1} M$ with $f_1 i_0 = f_0$ in $\mathcal{K}(A)$. Note that $H(f_1)$ is surjective, since so is $H(f_0) = H(f_1)H(i_0)$. Then by Lemma 4.21 there is a morphism $G_1 \xrightarrow{g_1} F_1$ in $\mathcal{C}(A)$ such that $H(G_1) \xrightarrow{H(g_1)} H(F_1) \xrightarrow{H(f_1)} M \longrightarrow 0$ is exact.

Now assume we have constructed the diagram (4.22.1) up to the $(n-1)$th row; $G_{n-1} \xrightarrow{g_{n-1}} F_{n-1} \xrightarrow{f_{n-1}} M$. Then, take the mapping cone and set $F_n = C(g_{n-1})$, and we have an admissible short exact sequence; $0 \longrightarrow F_{n-1} \xrightarrow{i_{n-1}} F_n \longrightarrow G_{n-1}[-1] \longrightarrow 0$ and a triangle in $\mathcal{K}(A)$; $G_{n-1} \xrightarrow{g_{n-1}} F_{n-1} \xrightarrow{i_{n-1}} F_n \longrightarrow G_{n-1}[-1]$. Since $f_{n-1} g_{n-1} = 0$ in $\mathcal{K}(A)$, there is a morphism $F_n \xrightarrow{f_n} M$ satisfying $f_n i_{n-1} = f_{n-1}$ in $\mathcal{K}(A)$. Then $H(f_n)$ is surjective as well as $H(f_{n-1})$. Using Lemma 4.21 we have a sequence $G_n \xrightarrow{g_n} F_n \xrightarrow{f_n} M$ such that $H(G_n) \xrightarrow{H(g_n)} H(F_n) \xrightarrow{H(f_n)} H(M) \longrightarrow 0$ is exact. In such a way the nth row of the diagram (4.22.1) is constructed.

Now we obtain a sequence of admissible monomorphisms:

$$F_0 \xhookrightarrow{i_0} F_1 \xhookrightarrow{i_1} F_2 \hookrightarrow \cdots \hookrightarrow F_{n-1} \xhookrightarrow{i_n} F_n \hookrightarrow \cdots$$

Set $F = \bigcup_{n=0}^{\infty} F_n$. Since $F_n/F_{n-1} \cong G_{n-1}[-1]$ is a free dg A-module, the family of dg submodules $\{F_n\}_{n\in\mathbb{N}}$ of F gives a semi-free filtration, thus F is a semi-free dg A-module. The morphism $f : F \to M$ is now defined as its restriction onto F_n equals f_n for all $n \in \mathbb{N}$. Then, since each $H(f_n)$ is surjective, we see that $H(f) : H(F) \to H(M)$ is surjective.

Now it remains to prove $H(f)$ is injective as well. For this purpose let $x \in F$ be a cycle, and assume that the homology class $[x] \in H(F)$ is sent to 0 in $H(M)$. Since F is a union of F_n, there is an integer n with $x \in F_n$, and then the homology class $[f_n(x)] = 0$. Recalling that there is an exact sequence

$H(G_n) \xrightarrow{H(g_n)} H(F_n) \xrightarrow{H(f_n)} H(M) \longrightarrow 0$, we can find a cycle $y \in G_n$ that satisfies $[x] = [g_n(y)]$ in $H(F_n)$. Since $H(G_n) \xrightarrow{H(g_n)} H(F_n) \xrightarrow{H(i_n)} H(F_{n+1})$ is also exact, it follows that $[i_n(x)] = 0$. Thus $i_n(x)$ is a boundary in F, therefore $[x] = 0$ as an element of $H(F)$. □

Remark 4.23. Later in §9 we will prove Theorem 4.20 again. Actually the existence of semi-free resolution is a direct consequence of Brown representation theorem. See 9.7.

Theorem 4.24 (Uniqueness of sf resolution)**.** *Let $M \in \mathcal{K}(A)$ and let $p : F \to M$ and $p' : F' \to M$ be semi-free resolutions of M. Then there is an isomorphism $\varphi : F \xrightarrow{\cong} F'$ in $\mathcal{K}(A)$ that makes the following diagram commutative;*

$$\begin{array}{ccc} F & & \\ & \searrow^{p} & \\ \varphi \downarrow & & M \\ & \nearrow_{p'} & \\ F' & & \end{array}$$

Proof. There is a diagram

$$\begin{array}{ccccccc} F' & \xrightarrow{p'} & M & \xrightarrow{q} & C(p') & \longrightarrow & \Sigma F' \\ & & {\scriptstyle p}\uparrow & & & & \\ & & F & & & & \end{array}$$

whose row is a triangle in $\mathcal{K}(A)$. Since p' is a quasi-isomorphism, it follows that $C(p')$ is acyclic, that is, $H(C(p')) = 0$. Then by Theorem 4.15 we have $qp = 0$. Thus p factors through p', i.e. there is $\varphi : F \to F'$ with $p = p'\varphi$. Then, since $H(p) = H(p')H(\varphi)$, and since both $H(p)$ and $H(p')$ are isomorphism, $H(\varphi)$ is an isomorphism as well, i.e. φ is a quasi-isomorphism. It thus follows from Corollary 4.19 that φ is an isomorphism in $\mathcal{K}(A)$. □

5. The Derived Category $\mathcal{D}(A)$

In this section A is a general dg R-algebra.

Definition 5.1. (1) Let $f : X \to Y$ be a morphism in $\mathcal{K}(A)$. We say f is a *quasi-isomorphism* if $H(f) : H(X) \to H(Y)$ is an isomorphism of graded $H(A)$-modules.

(2) For $X, Y \in \mathcal{K}(A)$, we denote the set of quasi-isomorphisms from X to Y by

$$\mathrm{Qis}(X, Y) = \{f \in \mathrm{Hom}_{\mathcal{K}(A)}(X, Y) |\ f \text{ is a quasi-isomorphism}\} \subset \mathrm{Hom}_{\mathcal{K}(A)}(X, Y).$$

Qis is a bifunctor $\mathcal{K}(A) \times \mathcal{K}(A) \to$ (Sets).

Lemma 5.2. Qis *is a saturated multiplicative system, that is, it satisfies the following conditions:*

(1) If $s, t \in \mathrm{Qis}$ are composable, then $st \in \mathrm{Qis}$. (That is, if $s \in \mathrm{Qis}(Y, Z)$ and $t \in \mathrm{Qis}(X, Y)$ then $st \in \mathrm{Qis}(X, Z)$.)
(2) Let $s, t \in \mathrm{Hom}_{\mathcal{K}(A)}$ be composable and assume the composition $st \in \mathrm{Qis}$. Then $s \in \mathrm{Qis}$ if and only if $t \in \mathrm{Qis}$.
(3) If $s \in \mathrm{Qis}$ then $\Sigma^{\pm} s \in \mathrm{Qis}$.
(4) Any diagram $X \xrightarrow{f} Y' \xleftarrow{s} Y$ in $\mathcal{K}(A)$ where $s \in \mathrm{Qis}$ is embedded into the pull-back diagram;

$$\begin{array}{ccc} X & \xrightarrow{f} & Y' \\ {\scriptstyle t}\uparrow & & \uparrow{\scriptstyle s} \\ X' & \xrightarrow[g]{} & Y \end{array}$$

where $t \in \mathrm{Qis}$, i.e. $ft = sg$.

(The last equality is equivalent to $s^{-1}f = gt^{-1}$ in the fractional notation. This means a left fraction can be described as right fraction.)

Proof. Proofs for (1) (2) and (3) are easy and left to the reader. For (4) we notice from the octahedron axiom TR4 that there is a commutative diagram whose rows and columns are triangles;

$$\begin{array}{ccccc} U & = & U & & \\ \uparrow & & \uparrow & & \\ X & \xrightarrow{f} & Y' & \longrightarrow & Z \\ {\scriptstyle t}\uparrow & & \uparrow{\scriptstyle s} & & \| \\ X' & \xrightarrow[g]{} & Y & \longrightarrow & Z \end{array}$$

Since s is a quasi-isomorphism, we have $H(U) = 0$, hence t is also a quasi-isomorphism. □

Definition 5.3 (derived category)**.** The derived category $\mathcal{D}(A)$ of A is defined to be the localization of $\mathcal{K}(A)$ by Qis;

$$\mathcal{D}(A) = \mathrm{Qis}^{-1}\,\mathcal{K}(A)$$

This means that objects of $\mathcal{D}(A)$ are the same as $\mathcal{K}(A)$ so that they are dg A-modules, while the morphism set is defined as follows for $X, Y \in \mathcal{D}(A)$;

$$\mathrm{Hom}_{\mathcal{D}(A)}(X, Y) = \{s^{-1}f \mid \ X \xrightarrow{f} Y' \xleftarrow{s} Y \ , \ s \in \mathrm{Qis}\}/\approx$$

where $\approx$ is the equivalence relation generated by the following;

$$s^{-1}f \approx (us)^{-1}(uf) \ \text{ for } u \in \mathrm{Qis}\,.$$

By diagrammatic description it means that

$$\left(X \xrightarrow{f} Y' \xleftarrow{s} Y \right) \approx \left(X \xrightarrow{f} Y' \xrightarrow{u} W \xleftarrow{u} Y' \xleftarrow{s} Y \right),$$

and that the morphisms in $\mathcal{D}(A)$ are the equivalence classes $[s^{-1}f]$.

The composition of morphisms is defined as follows: For $u, s \in \mathrm{Qis}$ and g, h, define $[u^{-1}g][s^{-1}h]$ by $[(tu)^{-1}(fh)]$ where $u \in \mathrm{Qis}$ and g is taken so that $gs^{-1} = t^{-1}f$, see Lemma 5.2 (4).

5.4. There is a natural functor $\iota : \mathcal{K}(A) \to \mathcal{D}(A)$ which is the identity mapping on the object sets, and the mapping between Hom-sets are defined by localization;

$$\mathrm{Hom}_{\mathcal{K}(A)}(X, Y) \to \mathrm{Hom}_{\mathcal{D}(A)}(X, Y); \quad f \mapsto \ \iota(f) = [1^{-1}f]$$

for dg A-modules X, Y.

Remark 5.5. Let X be an object in $\mathcal{K}(A)$ and $f : X \to Y$ be a morphism in $\mathcal{K}(A)$.
(1) $X \cong 0$ in $\mathcal{D}(A)$ if and only if $H(X) = 0$.
(2) We can naturally regard f as a morphism $[1^{-1}f]$ in $\mathcal{D}(A)$. Then $[1^{-1}f] = 0$ if and only if $H(f) = 0$.

5.6. J.-L.Verdier proved that the localization of a triangulated category by a multiplicative system again has the naturally induced structure of triangulated category. As a result of his general theory, the derived category $\mathcal{D}(A)$ is a triangulated category, in which the shift functor and the triangles come from $\mathcal{K}(A)$. Thus a triangle in $\mathcal{D}(A)$ is just a diagram that is isomorphic in $\mathcal{D}(A)$ to a triangle in $\mathcal{K}(A)$. Recall that all triangles in $\mathcal{K}(A)$ are given by admissible short exact sequences in $\mathcal{C}(A)$, hence so are the triangles in $\mathcal{D}(A)$.

To give a concrete explanation for it, let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be an admissible short exact sequence in $\mathcal{C}(A)$. Then we have a distinguished triangle $L \longrightarrow M \longrightarrow N \longrightarrow \Sigma L$ in $\mathcal{K}(A)$; see Definition 3.10. If there is a commutative diagram

$$\begin{array}{ccccccc} L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & \Sigma L \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X, \end{array}$$

where the vertical arrows are isomorphisms in $\mathcal{D}(A)$, e.g. quasi-isomorphisms, then the second row is a triangle in $\mathcal{D}(A)$. All the triangles in $\mathcal{D}(A)$ are obtained in this way.

We call a functor between triangulated categories is a *triangle functor* if it preserves the shift and the triangles. Note that the natural functor $\iota : \mathcal{K}(A) \to \mathcal{D}(A)$ defined in 5.4 is a triangle functor.

Exercise 5.7. Try to prove by yourself that $\mathcal{D}(A)$ is a triangulated category.

5.8. Recall that $\mathcal{K}^{sf}(A)$ is a triangulated subcategory of $\mathcal{K}(A)$ consisting of all semi-free dg A-modules. Considering the restriction of the natural functor $\iota : \mathcal{K}(A) \to \mathcal{D}(A)$ on $\mathcal{K}^{sf}(A)$, we have the functor

$$\nu := \iota|_{\mathcal{K}^{sf}(A)} : \mathcal{K}^{sf}(A) \to \mathcal{D}(A).$$

Since the natural embedding functor $\mathcal{K}^{sf} \hookrightarrow \mathcal{K}(A)$ is a triangle functor, ν is also a triangle functor.

Theorem 5.9. *The functor $\nu : \mathcal{K}^{sf}(A) \to \mathcal{D}(A)$ gives category equivalence.*

Proof. We prove that ν is dense, faithful and full.

[ν is dense.] First we have to show that ν yields a surjection onto the set of isomorphism classes of objects of $\mathcal{D}(A)$.

Let $X \in \mathcal{D}(A)$ be any object, that is, X is a dg A-module. Then take a semi-free resolution $F \to X$ of X, and we see that $F \in \mathcal{K}^{sf(A)}$ and $\nu(F) \cong X$ in $\mathcal{D}(A)$.

[ν is faithful.] Consider the mapping between Hom-sets induced by ν:

$$\mathrm{Hom}_{\mathcal{K}^{sf}(A)}(X, Y) \to \mathrm{Hom}_{\mathcal{D}(A)}(X, Y); \ f \mapsto \nu(f) = [1^{-1}f] \tag{5.9.1}$$

where X, Y are semi-free dg A-modules. We show this is injective.

Assume $[1^{-1}f] = 0$ in $\mathcal{D}(A)$. By the meaning of localization this is equivalent to that $tf = 0$ in $\mathcal{K}^{sf}(A)$ for some $t \in$ Qis:

$$X \xrightarrow{f} Y \xrightarrow{t} Y'$$

Now taking the mapping cone $C(t)$ of t, we have a diagram with the row being a triangle;

$$\begin{array}{ccccccc} & & X & & & & \\ & \swarrow & \downarrow f & & & & \\ C(t)[1] & \xrightarrow{u} & Y & \xrightarrow{t} & Y' & \longrightarrow & C(t) \end{array}$$

Since $tf = 0$, there is a morphism $g : X \to C(t)[-1]$ such that $f = ug$. Note that $H(C(t)[-1]) = 0$ because $t \in$ Qis. Thus by Theorem 4.15 we see that $g = 0$ in $\mathcal{K}(A)$, hence $f = ug = 0$. This proves that ν is injective in (5.9.1).

[ν is full.] We show that the mapping ν in (5.9.1) is surjective. Let $[s^{-1}f]$ be an element of $\mathrm{Hom}_{\mathcal{D}(A)}(X, Y)$:

$$X \xrightarrow{f} Y' \xleftarrow{s} Y$$

Taking the mapping cone of s, we have the following diagram;

$$\begin{array}{ccccccc} & & X & & & & \\ & \swarrow & \downarrow f & & & & \\ Y & \xrightarrow{s} & Y' & \xrightarrow{v} & C(s) & \longrightarrow & Y[-1] \end{array}$$

Since $H(C(s)) = 0$ and since X is semi-free, it follows from Theorem 4.15 that $vf = 0$ in $\mathcal{K}(A)$. Then there is a morphism $g : X \to Y$ in $\mathcal{K}^{sf}(A)$ such that $f = sg$. Therefore, since $s^{-1}f = s^{-1}(sg) \approx 1^{-1}g$, we have $\nu(g) = [s^{-1}f]$. $\square$

6. Derived Equivalence

Throughout this section let $\varphi : A \to B$ be a dg R-algebra homomorphism.

6.1. Given a (right) dg B-module X, we may consider X as a dg A-module through φ, which we denote by $X|_A$. More precisely, $X|_A$ is a (right) graded A-module X on which the A-action is defined by $xa = x\varphi(a)$ for $x \in X$, $a \in A$. Further we define the differential on $X|_A$ by $\partial^{X|_A}(x) = \partial^X(x)$ for $x \in X$.

If $f : X \to Y$ is a dg B-module homomorphism, then it defines a dg A-module homomorphism $f|_A : X|_A \to Y|_A$. Therefore we have a natural mapping;

$$\mathrm{Hom}_{\mathcal{C}(B)}(X, Y) \to \mathrm{Hom}_{\mathcal{C}(A)}(X|_A, Y|_A);\ f \mapsto f|_A$$

Summing up the above discussion we obtain the functor;

$$(-)|_A : \mathcal{C}(B) \to \mathcal{C}(A)$$

This is a functor between abelian categories, and it preserves exact sequences.

It is easy to see that the functor $(-)|_A$ preserves the homotopy, i.e. if $f \sim g$ then $f|_A \sim g|_A$. Thus it induces the functor between homotopy categories;

$$(-)|_A : \mathcal{K}(B) \to \mathcal{K}(A)$$

It is also trivial that $(-)|_A$ preserves the quasi-isomorphism, i.e. if f is a qis, then so is $f|_A$. Therefore it induces the functor between derived categories;

$$(-)|_A : \mathcal{D}(B) \to \mathcal{D}(A)$$

Note that the correspondence between the morphism sets;
$\mathrm{Hom}_{\mathcal{D}(B)}(X,Y) \to \mathrm{Hom}_{\mathcal{D}(A)}(X|_A, Y|_A)$ is given by $[s^{-1}f] \mapsto [(s|_A)^{-1}(f|_A)]$.

Exercise 6.2. Give precise proofs for all statements in the previous paragraph.

Exercise 6.3. Prove that the functors $(-)|_A$ (of all three types) preserve the coproducts (or direct sums) and the products.

$$\left(\coprod_\lambda X_\lambda\right)\Bigg|_A \cong \coprod_\lambda (X_\lambda|_A), \quad \left(\prod_\lambda X_\lambda\right)\Bigg|_A \cong \prod_\lambda (X_\lambda|_A)$$

6.4 (Caution)**.** Let $F \in \mathcal{K}^{sf}(B)$ be a semi-free dg B-module. Notice that $F|_A$ is not necessarily a semi-free dg A-module. Therefore the functor $(-)|_A : \mathcal{K}^{sf}(A) \to \mathcal{K}^{sf}(B)$ is not defined in general.

If $B|_A$ is semi-free, then $F|_A$ is semi-free for $F \in \mathcal{K}^{sf}(B)$. (Prove it!)

Definition 6.5. Let $X \in \mathcal{C}(A)$ be a right dg A-module and $Y \in \mathcal{C}(A^{op})$ a left dg A-module. Then the tensor product over R is the graded R-module;

$$X \otimes_R Y = \bigoplus_{n\in\mathbb{N}} \left(\bigoplus_{i+j=n} X_i \otimes_R Y_j \right)$$

with the differential $\partial^{X\otimes_R Y}(x \otimes y) = \partial^X(x) \otimes y + (-1)^{|x|} x \otimes \partial^Y(y)$ for $x \in X$, $y \in Y$. Then $\left(X \otimes_R Y,\ \partial^{X\otimes_R Y}\right) \in \mathcal{C}(R)$.

The tensor product over A is defined as the graded R-module;

$$X \otimes_A Y = X \otimes_R Y / (xa \otimes y - x \otimes ay \mid x \in X, y \in Y, a \in A)$$

with the differential $\partial^{X\otimes_A Y}$ that is naturally induced from $\partial^{X\otimes_R Y}$.
Then $\left(X \otimes_A Y,\ \partial^{X\otimes_A Y}\right) \in \mathcal{C}(R)$.

Remark 6.6. Let F be a semi-free right (resp. left) dg A-module.
If $X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$ is a triangle in $\mathcal{K}(A^{op})$ (resp. $\mathcal{K}(A)$), then
$F \otimes_A X \longrightarrow F \otimes_A Y \longrightarrow F \otimes_A Z \longrightarrow \Sigma(F \otimes_A X)$
(resp. $X \otimes_A F \longrightarrow Y \otimes_A F \longrightarrow Z \otimes_A F \longrightarrow \Sigma(X \otimes_A F)$) is a triangle in $\mathcal{K}(R)$.

Lemma 6.7. *Let F be a right semi-free dg A-module. Then the functor $F \otimes_A (-) : \mathcal{C}(A^{op}) \to \mathcal{C}(R)$ preserves quasi-isomorphisms, i.e. if $f : X \to Y$ in $\mathcal{C}(A^{op})$ is a quasi-isomorphism, then $F \otimes_A f : F \otimes_A X \to F \otimes_A Y$ in $\mathcal{C}(R)$ is also a quasi-isomorphism.*

Similarly if G is a left semi-free dg A-module, then $(-) \otimes_A G : \mathcal{C}(A) \to \mathcal{C}(R)$ preserves quasi-isomorphisms.

Proof. Considering the mapping cone $C(f)$, we only have to prove the following;

If $H(X) = 0$, then $H(F \otimes_A X) = 0$.

The proof of this claim will be done in essentially the same way as the proof of Theorem 4.15. Let $X \in \mathcal{C}(A^{op})$ and assume that $H(X) = 0$. We consider

$$\mathcal{U} = \mathcal{U}_X = \{F \in \mathcal{K}(A) \mid H(F \otimes_A X) = 0\}.$$

Then The following properties for $\mathcal{U}$ are easily verified:
(1) $A \in \mathcal{U}$
(2) $\mathcal{U}$ is closed under shift functor; $Y \in \mathcal{U}$ if and only if $\Sigma Y \in \mathcal{U}$.

(3) $\mathcal{U}$ is closed under taking triangles; if there is a triangle $Y \to Z \to W \to \Sigma Y$ in $\mathcal{K}(A)$ and if two of $Y, Z, W \in \mathcal{U}$ then all belong to $\mathcal{U}$.
(4) $\mathcal{U}$ is closed under coproducts; if $Y_j \in \mathcal{U}$ for $j \in J$ then $\coprod_{j\in J} Y_j \in \mathcal{U}$.

Then it follows from Lemma 4.14 that $\mathcal{K}^{sf}(A) \subset \mathcal{U}$, hence $H(F \otimes_A X) = 0$ for a semi-free dg A-module F. □

Exercise 6.8. Assume A is a commutative dg R-algebra. Let F be a right semi-free A-module and G be a left semi-free dg A-module. Then prove that $F \otimes_A G$ is also a semi-free dg A-module. (Hint; Using Exercise 2.5, we see all left dg modules are right dg modules. If F and G have semi-free bases $\{e_\lambda\}$ and $\{e'_\mu\}$ respectively, then $F \otimes_A G$ has a semi-free basis $\{e_\lambda \otimes e'_\mu\}$.)

Definition 6.9. Let $X \in \mathcal{D}(A)$ and $Y \in \mathcal{D}(A^{op})$ and assume that $F \to X$, $G \to Y$ are semi-free resolutions. Then it follows from Lemma 6.7 that the following are quasi-isomorphisms;

$$F \otimes_A Y \leftarrow F \otimes_A G \to X \otimes_A G$$

The *derived tensor product* $X \otimes^{\mathbf{L}}_A Y \in \mathcal{D}(R)$ is defined as the above object in $\mathcal{D}(R)$ which is uniquely determined up to isomorphism, and is independent of semi-free resolutions.

If W is a dg (A, B)-bimodule, then $X \otimes^{\mathbf{L}}_A W \in \mathcal{D}(B)$ is defined to be a dg B-module $F \otimes_A W$.

If $f : X \to Y$ is a morphism in $\mathcal{D}(A)$, then we take a commutative diagram

$$\begin{array}{ccc} F & \longrightarrow & X \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle f} \\ G & \longrightarrow & Y \end{array}$$

where F and G are semi-free resolutions of X and Y respectively. Then we define the morphism $f \otimes^{\mathbf{L}}_A Z$ in $\mathcal{D}(A)$ to be $\nu(\varphi \otimes_A Z)$ for $Z \in \mathcal{D}(A^{op})$. Thus the functor $(-) \otimes^{\mathbf{L}}_A Z : \mathcal{D}(A) \to \mathcal{D}(R)$ is defined for $Z \in \mathcal{D}(A^{op})$. .

In case that W is a dg (A, B)-bimodule, $(-) \otimes^{\mathbf{L}}_A W$ is the functor $\mathcal{D}(A) \to \mathcal{D}(B)$.

6.10. Let $F \in \mathcal{C}(A)$ be a semi-free dg A-module. Recall from Lemma 4.4 that F is free as an underlying graded A-module with basis $\{e_\lambda | \lambda \in \Lambda\}$ indexed by a well-ordered set Λ and satisfies; $\partial^F(e_\lambda) = \sum_{\mu<\lambda} e_\mu a_{\mu\lambda}$ for $\lambda \in \Lambda$, where $a_{\mu\lambda} \in A$.

Then the tensor product $F \otimes_A B$ is the dg B-module that is free as an underlying graded B-module with basis $\{e_\lambda \otimes 1 | \lambda \in \Lambda\}$ and satisfies;

$$\partial^{F\otimes_A B}(e_\lambda \otimes 1) = \sum_{\mu<\lambda} (e_\mu \otimes 1)\varphi(a_{\mu\lambda})$$

for $\lambda \in \Lambda$. Therefore $F \otimes_A B$ is a semi-free dg B-module.

If $f : F \to G$ is a dg A-module homomorphism of semi-free dg A-modules, then the dg B-module homomorphism $f \otimes_A B : F \otimes_A B \to G \otimes_A B$ is naturally defined (as is the dg B-module homomorphism that maps the base element $e_\lambda \otimes 1$ to $f(e_\lambda) \otimes 1$). If $f \sim 0$ via a homotopy h, then it is easy to see that $f \otimes_A B \sim 0$ as a dg B-module homomorphism via $h \otimes_A B$. Hence the homotopy class $[f \otimes_A B]$ is uniquely determined by the homotopy class $[f]$.

In such a way we obtain the functor;

$$(-) \otimes_A B : \mathcal{K}^{sf}(A) \to \mathcal{K}^{sf}(B).$$

Exercise 6.11. Show that $(-)\otimes_A B$ is a triangle functor. (Hint; The tensor product of an admissible short exact sequence of dg A-modules with B is admissible over B.)

Definition 6.12 (derived tensor product)**.** The derived tensor functor $(-)\otimes^{\mathbf{L}}_A B : \mathcal{D}(A) \to \mathcal{D}(B)$ is defined as the functor that makes the following diagram commutative;

$$\begin{array}{ccc} \mathcal{D}(A) & \xrightarrow{(-)\otimes^{\mathbf{L}}_A B} & \mathcal{D}(B) \\ {\scriptstyle \nu_A}\uparrow & & \uparrow{\scriptstyle \nu_B} \\ \mathcal{K}^{sf}(A) & \xrightarrow{(-)\otimes_A B} & \mathcal{K}^{sf}(B) \end{array}$$

(Note that the vertical arrows are category equivalences and the lower right arrow is defined so that $(-)\otimes^{\mathbf{L}}_A B$ is uniquely defined.) Precisely saying if $X \in \mathcal{D}(A)$ then take a semi-free resolution $F \to X$ and we obtain $X \otimes^{\mathbf{L}}_A B = \nu_B(F \otimes_A B)$. If $f : X \to Y$ is a morphism in $\mathcal{D}(A)$, then we take a commutative diagram

$$\begin{array}{ccc} F & \longrightarrow & X \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle f} \\ G & \longrightarrow & Y \end{array}$$

where F and G are semi-free resolutions of X and Y respectively. Then we have $f \otimes^{\mathbf{L}}_A B = \nu_B(\varphi \otimes_A B)$.

Exercise 6.13. Prove $A \otimes_A B \cong B$ in $\mathcal{K}^{sf}(B)$. Prove $A \otimes^{\mathbf{L}}_A B \cong B$ in $\mathcal{D}(B)$.

Exercise 6.14. Prove $(\coprod_\lambda X_\lambda) \otimes^{\mathbf{L}}_A B \cong \coprod_\lambda \left(X_\lambda \otimes^{\mathbf{L}}_A B\right)$.

6.15. Let M be a dg B-module, and let F be a semi-free dg A-module with semi-free basis $\{e_\lambda\}_{\lambda\in\Lambda}$ where the differential is described as $\partial^F(e_\lambda) = \sum_{\mu<\lambda} e_\mu a_{\mu\lambda}$ with $a_{\mu\lambda} \in A$.

Recall from Remark 3.2 that $\mathrm{Hom}_{\mathrm{gr}A\text{-mod}}(F, M|_A)_n$ is the set of all graded A-module homomorphism from F to $M|_A$ of degree n, and

$$\mathrm{Hom}_A(F, M|_A) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{gr}A\text{-mod}}(F, M|_A)_n$$

is a chain complex of R-modules with the differential that is defined by

$$\partial^{\mathrm{Hom}}(f) := \partial^M f - (-1)^n f\partial^F \in \mathrm{Hom}_{\mathrm{gr}A\text{-mod}}(F, M|_A)_{n-1}$$

for $f \in \mathrm{Hom}_{\mathrm{gr}A\text{-mod}}(F, M|_A)_n$. Since F is A-free with basis $\{e_\lambda\}_{\lambda\in\Lambda}$, an element $f \in \mathrm{Hom}_{\mathrm{gr}A\text{-mod}}(F, M|_A)_n$ is determined by the images $f(e_\lambda) \in M_{|e_\lambda|+n}$ for $\lambda \in \Lambda$. Hence $\mathrm{Hom}_{\mathrm{gr}A\text{-mod}}(F, M|_A)_n \cong \prod_\lambda M_{|e_\lambda|+n}$ as R-modules, and the differential ∂^{Hom} is described as

$$\prod_\lambda M_{|e_\lambda|+n} \ni (m_\lambda)_\lambda \mapsto \left(\partial^M(m_\lambda) - \sum_{\mu<\lambda} m_\mu a_{\mu\lambda}\right)_\lambda \in \prod_\lambda M_{|e_\lambda|+n-1}$$

Since $F \otimes_A B$ is a graded free B-module with basis $\{e_\lambda \otimes 1\}_{\lambda\in\Lambda}$ and the differential is $\partial^{F\otimes_A B}(e_\lambda \otimes 1) = \sum_{\mu<\lambda}(e_\mu \otimes 1)a_{\mu\lambda}$ with the same $a_{\mu\lambda} \in A$ as above, we can see that the chain complex $\mathrm{Hom}_{\mathrm{gr}B\text{-mod}}(F \otimes_A B, M)$ is identical with the chain complex above. Therefore we have a chain isomorphism of R-chain complexes;

$$\mathrm{Hom}_A(F, M|_A) \cong \mathrm{Hom}_B(F \otimes_A B, M) \qquad (6.15.1)$$

Now taking the cycles of these complexes we have a natural isomorphism of R-modules (cf. Remark 3.2);

$$\mathrm{Hom}_{\mathcal{C}(B)}(F \otimes_A B, M) \cong \mathrm{Hom}_{\mathcal{C}(A)}(F, M|_A). \tag{6.15.2}$$

Exercise 6.16. Under the above notation, show the equality;

$$\sum_{\nu<\mu<\lambda} a_{\nu\mu}a_{\mu\lambda} + (-1)^{|e_\nu|}a_{\nu\lambda} = 0$$

holds for all $\nu < \lambda$. (Hint; $\partial^F(\partial^F(e_\lambda)) = 0$.)

Taking the homology of both sides in (6.15.1) we also have the following lemma.

Lemma 6.17. *Let F be a semi-free dg A-module and let M be a dg B-module. Then there is a natural isomorphism of R-modules;*

$$\mathrm{Hom}_{\mathcal{K}(B)}(F \otimes_A B, M) \cong \mathrm{Hom}_{\mathcal{K}(A)}(F, M|_A).$$

The following theorem is a direct consequence of this lemma. It is called the adjoint formula (or the adjunction formula).

Theorem 6.18. *For any $X \in \mathcal{D}(A)$ and $Y \in \mathcal{D}(B)$, we have a natural isomorphism of R-modules;*

$$\mathrm{Hom}_{\mathcal{D}(B)}(X \otimes^{\mathbf{L}}_A B, Y) \cong \mathrm{Hom}_{\mathcal{D}(A)}(X, Y|_A).$$

In other words,

$$(-) \otimes^{\mathbf{L}}_A B \ : \mathcal{D}(A) \rightleftarrows \mathcal{D}(B) : \ (-)|_A$$

(or $((-) \otimes_A B,\ (-)|_A)$) is an adjoint pair.

Proof. Replacing X and Y with their semi-free resolutions, we can easily prove the formula from the previous lemma. □

The following is the main theorem of this section, which asserts that "a quasi-isomorphism of dg algebras induces derived equivalence".

Theorem 6.19. *Let $\varphi : A \to B$ be a dg R-algebra homomorphism. Assume that φ is a quasi-isomorphism, i.e. $H(\varphi) : H(A) \xrightarrow{\cong} H(B)$ is an isomorphism of graded R-algebras. Then the functors $(-)\otimes^{\mathbf{L}}_A B : \mathcal{D}(A) \to \mathcal{D}(B)$ and $(-)|_A : \mathcal{D}(B) \to \mathcal{D}(A)$ give equivalences of categories. They are triangle equivalences, and quasi-inverses each other.*

Proof. Assume in the following that the dg R-algebra homomorphism $\varphi : A \to B$ is a quasi-isomorphism. We prove Theorem 6.19 step-by-step.

(i) For a semi-free dg A-module F, a dg A-module homomorphism;

$$f_F : F \to (F \otimes_A B)|_A; \quad x \mapsto f(x) = x \otimes 1$$

is a qis.

In fact, $\varphi : A \to B$ is a qis as left dg A-module homomorphism. It then follows from Lemma 6.7 that $1_F \otimes \varphi : F = F \otimes_A A \to F \otimes_A B$ is also a qis of dg R-modules. Note that this maps $x \in F$ to $x \otimes 1$, hence it is a right dg A-module homomorphism and $1_F \otimes \varphi = f_F$. Since we have shown $H(f_F) : H(F) \to H((F \otimes_A B)|_A)$ is a bijection, f_F is a qis as a right dg A-module homomorphism.

(ii) For any $X \in \mathcal{D}(A)$, the natural morphism in $\mathcal{D}(A)$;

$$\alpha : X \to (X \otimes^{\mathbf{L}}_A B)|_A; \quad x \mapsto \alpha(x) = x \otimes 1$$

is an isomorphism in $\mathcal{D}(A)$.

Let $p : F \to X$ be a semi-free resolution in $\mathcal{C}(A)$. Then there is a commutative diagram in $\mathcal{D}(A)$;

$$\begin{array}{ccc} F & \xrightarrow{f_F} & (F \otimes_A B)|_A \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle (p\otimes 1_B)|_A} \\ X & \xrightarrow{\alpha} & (X \otimes^{\mathbf{L}}_A B)|_A \end{array}$$

where f_F is a qis by step (i), and vertical arrows are isomorphisms in $\mathcal{D}(A)$. Therefore α is an isomorphism as well.

(iii) For any $Y \in \mathcal{D}(B)$, the natural morphism in $\mathcal{D}(B)$;

$$\beta : (Y|_A) \otimes^{\mathbf{L}}_A B \to Y; \quad y \otimes b \mapsto yb$$

is an isomorphism in $\mathcal{D}(B)$.

For the proof of (iii), we first remark that, for any dg B-module N, there is a natural mapping $\mu_N : N|_A \otimes_A B \to N$ that is defined as $\mu_N(x \otimes b) = xb$ for $x \otimes b \in N|_A \otimes_A B$. Note μ_N is a (right) dg B-module homomorphism.

Given $Y \in \mathcal{D}(B)$, we take a semi-free resolution $p : G \to Y$ as a dg B-module, and take a semi-free resolution $q : F \to G|_A$ as a dg A-module. (Notice that $G|_A$ is not necessarily semi-free.) Note that the composition

$$(p|_A) \circ q : F \xrightarrow{q} G|_A \xrightarrow{p|_A} Y|_A$$

is a semi-free resolution of $Y|_A$. Then there is a commutative diagram in $\mathcal{C}(A)$;

$$\begin{array}{ccccc} & & F & & \\ & \swarrow^{f_F} & & \searrow^{q} & \\ (F \otimes_A B)|_A & \xrightarrow[(q\otimes 1)|_A]{} & (G|_A \otimes_A B)|_A & \xrightarrow[\mu_{G|_A}]{} & G|_A \end{array}$$

This is actually commutative, since

$$\left(\mu_{G|_A} \circ (q \otimes 1)|_A \circ f_F\right)(x) = \mu_{G|_A} \circ (q \otimes 1)|_A(x \otimes 1) = \mu_{G|_A}(q(x) \otimes 1) = q(x)$$

for $x \in F$. Since f_F and q are qis's, the composition $\mu_{G|_A} \circ (q \otimes 1)|_A$ is also a qis. Now consider the right dg B-module homomorphism $\psi : F \otimes_A B \to G$ defined by $\psi(x \otimes b) = \mu_G(q(x) \otimes b) = q(x)b$ for $x \otimes b \in F \otimes_A B$. Then we can easily see the equality $\psi|_A = \mu_{G|_A} \circ (q \otimes 1)|_A$, hence $\psi|_A$ is a qis, that is, $H(\psi|_A) = H(\psi)$ is a bijection. Therefore ψ is a qis as well, which induces the isomorphism in $\mathcal{D}(B)$;

$$\beta : Y|_A \otimes^{\mathbf{L}}_A B \xrightarrow{\simeq} Y$$

(iv) The functors $(-) \otimes^{\mathbf{L}}_A B$ and $(-)|_A$ are dense functors.

This is a consequence of steps (ii) and (iii).

(v) The functor $(-) \otimes^{\mathbf{L}}_A B$ is a full and faithful functor; i.e.

$$\mathrm{Hom}_{\mathcal{D}(A)}(X, X') \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{D}(B)}(X \otimes^{\mathbf{L}}_A B, X' \otimes^{\mathbf{L}}_A B) \ \text{ for } X, X' \in \mathcal{D}(A).$$

This is also a consequence of (ii), since the mapping between Hom sets is the composition

$$\mathrm{Hom}_{\mathcal{D}(A)}(X, X') \to \mathrm{Hom}_{\mathcal{D}(A)}(X, (X' \otimes_A B)|_A) \to \mathrm{Hom}_{\mathcal{D}(B)}(X \otimes^{\mathbf{L}}_A B, X' \otimes^{\mathbf{L}}_A B)$$

where the first map is $\mathrm{Hom}_{\mathcal{D}(A)}(X, \alpha)$, and the second is the adjoint formula, both of which are bijective.

(vi) The functor $(-)|_A$ is a full and faithful functor; i.e.

$$\mathrm{Hom}_{\mathcal{D}(B)}(Y, Y') \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{D}(A)}(Y|_A, Y'|_A) \quad \text{for } Y, Y' \in \mathcal{D}(B).$$

We omit the proof and we leave it to the reader, since it is similar to (v) but using (iii).

By $(iv), (v), (vi)$ both functors give equivalences of categories. Finally, since the pair $\left((-) \otimes^{\mathbf{L}}_A B,\ (-)|_A\right)$ is an adjoint pair, and since both functors are equivalences, they are quasi-inverses each other. □

Corollary 6.20. *Let $R \twoheadrightarrow S$ be a surjective ring homomorphism of commutative Noetherian rings. Let $A \to S$ be either a Tate resolution or an Avramov resolution of S over R. (See Theorems 1.21, 1.24.) Then there is a category equivalence $\mathcal{D}(A) \simeq \mathcal{D}(S)$.*

Note that the derived equivalence $\mathcal{D}(A) \simeq \mathcal{D}(S)$ means that all homological properties over S hold over A as well. In this sense the homological algebra over S is identical with that over A.

7. Compact Objects

In this section $\mathcal{C}$ is an additive category[5] which admits infinite coproducts.

Definition 7.1. An object $C \in \mathcal{C}$ is said to be *compact* if for any set $\{X_\lambda\}$ of objects in $\mathcal{C}$, the natural map

$$\coprod_\lambda \mathrm{Hom}_{\mathcal{C}}(C, X_\lambda) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(C, \coprod_\lambda X_\lambda)$$

is bijection, where the coproduct in the left-hand side is coproduct (direct sum) of abelian groups.

The case of module categories: In the meantime we consider compact objects in module categories. In this subsection, $\mathcal{C} = \mathrm{Mod}\text{-}A$ denotes the category of right A-modules where A is a (not necessarily commutative) ring.

7.2. An object $C \in \mathrm{Mod}\text{-}A$ is compact if and only if, for any subset $\{X_\lambda\}_{\lambda \in \Lambda} \subset \mathrm{Mod}\text{-}A$ and for any homomorphism $f : C \to \coprod_\lambda X_\lambda = \bigoplus_\lambda X_\lambda$, the image $f(C)$ is contained in finite coproduct $\bigoplus_{i=1}^n X_{\lambda_i}$ for some $\lambda_1, \dots, \lambda_n \in \Lambda$.

7.3. Every finitely generated A-module is compact in $\mathrm{Mod}\text{-}A$.
This is a result of the previous paragraph.

[5]A category is said to be additive if (i) Hom sets are abelian groups and the compositions of morphisms are abelian group homomorphisms, (ii) it admits finite products and finite coproducts that coincide each other.

7.4. Let $L \twoheadrightarrow M$ be an epimorphism in Mod-A. If L is compact, then so is M.

This also follows from 7.2. Thus the compact property inherits to quotients, but not to submodules in general.

7.5. Let A be (right) Noetherian and let $C \in \text{Mod-}A$ be compact. Then we have

$$\operatorname{Ext}^i_A(C, \bigoplus_\lambda X_\lambda) \cong \bigoplus_\lambda \operatorname{Ext}^i_A(C, X_\lambda)$$

for all $i \in \mathbb{N}$ and any set $\{X_\lambda\} \subset \text{Mod-}A$.

Proof. Take injective resolutions $X_\lambda \to I^\bullet_\lambda$ for each λ. Then $\bigoplus_\lambda X_\lambda \to \bigoplus_\lambda I^\bullet_\lambda$ is also an injective resolution, since a direct sum of injective modules over a Noetherian ring is injective. Since C is compact, we have a chain isomorphism $\operatorname{Hom}_A(C, \bigoplus_\lambda I^\bullet_\lambda) \cong \bigoplus_\lambda \operatorname{Hom}_A(C, I^\bullet_\lambda)$. Taking the cohomology of these chain complexes, we have the assertion. $\square$

7.6. Let $N \hookrightarrow L$ be a monomorphism in Mod-A. Assume A is a right Noetherian ring. If L is compact, then so is N.

Proof. Consider the short exact sequence $0 \to N \to L \to M \to 0$, where L and $M = L/N$ are compact by 7.4. Then for any set $\{X_\lambda\} \subset \text{Mod-}A$, we have a commutative diagram by 7.5;

$$\begin{array}{ccccccccc}
\bigoplus_\lambda \operatorname{Hom}_A(M, X_\lambda) & \to & \bigoplus_\lambda \operatorname{Hom}_A(L, X_\lambda) & \to & \bigoplus_\lambda \operatorname{Hom}_A(N, X_\lambda) & \to & \bigoplus_\lambda \operatorname{Ext}^1_A(M, X_\lambda) & \to & \bigoplus_\lambda \operatorname{Ext}^1_A(L, X_\lambda) \\
\downarrow \cong & & \downarrow \cong & & \downarrow & & \downarrow \cong & & \downarrow \cong \\
\operatorname{Hom}_A(M, \bigoplus_\lambda X_\lambda) & \to & \operatorname{Hom}_A(L, \bigoplus_\lambda X_\lambda) & \to & \operatorname{Hom}_A(N, \bigoplus_\lambda X_\lambda) & \to & \operatorname{Ext}^1_A(M, \bigoplus_\lambda X_\lambda) & \to & \operatorname{Ext}^1_A(L, \bigoplus_\lambda X_\lambda)
\end{array}$$

The assertion follows from 5-lemma. $\square$

7.7. Assume A is a right Noetherian ring. Then an A-module M is compact in Mod-A if and only if M is finitely generated.

Proof. Suppose M is compact but not finitely generated, and take an infinite sequence $x_1, x_2, \ldots$ of elements of M so that $x_{i+1} \notin (x_1, \ldots, x_i)A =: M_i$ for all $i \in \mathbb{N}$. Set $N = \bigcup_{i \in \mathbb{N}} M_i$ and notice that N is compact, since N is a submodule of compact M. Consider the natural map $f : N \to \bigoplus_{i \in \mathbb{N}} N/M_i$. Since N is compact, the image $f(N)$ is contained in a finite direct sum. It means that $N/M_i = (0)$ for sufficiently large i, hence $N = M_i = M_{i+1}$ a contradiction. $\square$

Exercise 7.8. Let Top be the category of topological spaces with continuous functions as morphisms. What are the compact objects in Top? Are compact topological spaces compact in the sense of Definition 7.1? (Maybe No!)

The case of triangulated categories: In this subsection, let $\mathcal{T}$ be a triangulated category that admits any coproducts, e.g. $\mathcal{T} = \mathcal{K}(A), \mathcal{K}^{sf}(A)$ or $D(A)$.

Lemma 7.9. *Let* $X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$ *be a triangle in* $\mathcal{T}$.

(1) *If* X, Y *are compact in* $\mathcal{T}$, *then so is* Z.
(2) *If* X *is compact in* $\mathcal{T}$, *then so are* $\Sigma^n X$ *for all* $n \in \mathbb{Z}$.
(3) *If* X *is compact in* $\mathcal{T}$, *then any direct summand of* X *is compact.*

Proof. For any set of objects $\{X_\lambda\} \subset \mathcal{T}$, note that

$$\mathrm{Hom}_{\mathcal{T}}(\Sigma^{-1}X, X_\lambda) \cong \mathrm{Hom}_{\mathcal{T}}(X, \Sigma X_\lambda) \text{ and } \coprod_\lambda (\Sigma X_\lambda) \cong \Sigma\left(\coprod_\lambda X_\lambda\right).$$

The assertion (2) follows from these isomorphisms. For (1) note that there is a commutative diagram of abelian groups with exact rows;

$$\begin{array}{ccccccccc}
\coprod_\lambda \mathrm{Hom}_{\mathcal{T}}(\Sigma Y, X_\lambda) & \to & \coprod_\lambda \mathrm{Hom}_{\mathcal{T}}(\Sigma X, X_\lambda) & \to & \coprod_\lambda \mathrm{Hom}_{\mathcal{T}}(Z, X_\lambda) & \to & \coprod_\lambda \mathrm{Hom}_{\mathcal{T}}(Y, X_\lambda) & \to & \coprod_\lambda \mathrm{Hom}_{\mathcal{T}}(X, X_\lambda) \\
\downarrow \cong & & \downarrow \cong & & \downarrow & & \downarrow \cong & & \downarrow \cong \\
\mathrm{Hom}_{\mathcal{T}}(\Sigma Y, \coprod_\lambda X_\lambda) & \to & \mathrm{Hom}_{\mathcal{T}}(\Sigma X, \coprod_\lambda X_\lambda) & \to & \mathrm{Hom}_{\mathcal{T}}(Z, \coprod_\lambda X_\lambda) & \to & \mathrm{Hom}_{\mathcal{T}}(Y, \coprod_\lambda X_\lambda) & \to & \mathrm{Hom}_{\mathcal{T}}(X, \coprod_\lambda X_\lambda)
\end{array}$$

which together with the 5-lemma proves (2). The proof of (3) is left to the reader. □

Definition 7.10. The full subcategory of $\mathcal{T}$ consisting of all compact objects is denoted by $\mathcal{T}^c$:

$$\mathcal{T}^c = \{C \in \mathcal{T} \mid C \text{ is compact}\} \subset \mathcal{T}$$

The previous lemma states that $\mathcal{T}^c$ is a thick subcategory of $\mathcal{T}$ (i.e. = a triangulated subcategory that is closed under direct summands).

Remark 7.11. Recall from Theorem 5.9 that there is a category equivalence $\nu : \mathcal{K}^{sf}(A) \to \mathcal{D}(A)$. It is not difficult to see that the functor ν preserves coproducts; $\nu\left(\coprod_\lambda X_\lambda\right) \cong \coprod_\lambda \nu(X_\lambda)$. Therefore we can show that $F \in \mathcal{K}^{sf}(A)$ is compact if and only if $\nu(F) \in \mathcal{D}(A)$ is compact. In particular, we have a category equivalence; $\left(\mathcal{K}^{sf}(A)\right)^c \simeq \mathcal{D}(A)^c$.

Definition 7.12. A semi-free dg A-module $F \in \mathcal{K}^{sf}(A)$ is called *perfect* if it has a finite semi-free basis, more precisely saying, F satisfies one of the following equivalent conditions;

(1) There is a finite sequence of dg submodules;

$$0 = F_{-1} \subset F_0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n = F$$

such that each F_i/F_{i-1} is a finite free dg A-module for $0 \leqslant i \leqslant n$.

(2) There is a finite free basis $\{e_1, e_2, \ldots, e_m\}$ of F as an underlying free A-module which satisfies $\partial^F(e_1) = 0$ and $\partial^F(e_i) \in \sum_{j=1}^{i-1} e_j A$ for $2 \leqslant i \leqslant m$.

Lemma 7.13. *If $F \in \mathcal{K}^{sf}(A)$ is perfect, then F is compact in $\mathcal{K}^{sf}(A)$.*

Proof. Remark that the free dg A-module A is compact in $\mathcal{K}^{sf}(A)$, since $\mathrm{Hom}_{\mathcal{K}(A)}(A, X) \cong H_0(X)$ and H_0 commutes with direct sums. It then follows from Lemma 7.9 that all perfect dg modules are compact. □

Remark 7.14. A direct summand of a perfect dg A-module is not necessarily perfect. So the subcategory consisting of all perfect modules is not a thick subcategory.

For example, let $R = k[x]/(x(x-1))$ be a commutative ring where k is field. Note that $R \cong R/(x) \times R/(x-1)$ as a ring, hence as an R-module. Therefore $R/(x)$ is a direct summand of R, but it is not perfect in $\mathcal{D}(R)$ by the following reason: The R-module $R/(x)$ has an R-free resolution;

$$\cdots \longrightarrow R \xrightarrow{x-1} R \xrightarrow{x} R \xrightarrow{x-1} R \xrightarrow{x} R \longrightarrow R/(x) \longrightarrow 0$$

Thus considering a semi-free dg R-module $F = \bigoplus_{i=0}^{\infty} e_i R$ with $\partial(e_i) = e_{i-1}x$ for odd i and $\partial(e_i) = e_{i-1}(x-1)$ for even i, we see that there is a qis $F \to R/(x)$ in $\mathcal{C}(R)$, hence $F \cong R/(x)$ in $\mathcal{D}(R)$. Thus F is a direct summand of the free module R as an object in $\mathcal{K}^{sf}(R)$. However F requires infinite elements as its free basis, and F is not perfect. (The point is that $R/(x)$ is a projective R-module but does not have a finite R-free resolution.)

This example also shows that 'compact' does not necessarily imply 'perfect'. Actually F is compact but not perfect in $\mathcal{K}^{sf}(R)$.

Theorem 7.15. *A semi-free dg A-module is compact in $\mathcal{K}^{sf}(A)$ if and only if it is a direct summand of a perfect dg A-module.*

Proof. The "if" part follows from Lemmas 7.13 and 7.9.

To prove the "only if" part, let $C \in \mathcal{K}^{sf}(A)$ be a compact object. Since C is semi-free, it has a semi-free filtration;

$$0 = F_{-1} \subset F_0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n \subset \cdots \subset C = \bigcup_{i \in \mathbb{N}} F_i,$$

where each F_n/F_{n-1} is a free dg A-module. First of all we claim:

(1) There is an integer $n \in \mathbb{N}$ such that C is a direct summand of F_n in $\mathcal{K}^{sf}(A)$.

To prove this let $p_n : C \twoheadrightarrow C/F_n$ be the natural projection for $n \in \mathbb{N}$, and let $p : C \to \coprod_{i \in \mathbb{N}} C/F_n$ be a morphism in $\mathcal{C}(A)$ defined by $p(x) = (p_n(x))_{n \in \mathbb{N}}$ for $x \in C$. Since, for each $x \in C$, there is an integer $n_0 \in \mathbb{N}$ such that $p_n(x) = 0$ for $n \geqslant n_0$, the dg A-module homomorphism p is well-defined. Since C is compact, we have an isomorphism

$$\mathrm{Hom}_{\mathcal{K}^{sf}(A)}(C, \coprod_{i \in \mathbb{N}} C/F_n) \cong \coprod_{i \in \mathbb{N}} \mathrm{Hom}_{\mathcal{K}^{sf}(A)}(C, C/F_n),$$

which corresponds p to $(p_n)_{n \in \mathbb{N}}$. Therefore $p_n = 0$ in $\mathcal{K}^{sf}(A)$ for large integer n. Thus the triangle $\Sigma^{-1}(C/F_n) \to F_n \to C \xrightarrow{p_n} C/F_n$ splits, and C is a direct summand of F_n in $\mathcal{K}^{sf}(A)$.

As the second step of proof we prove the following slightly general assertion.

(2) Let $0 = F_{-1} \subset F_0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n$ be part of semi-free filtration of any dg A-module, and let $f : C \to F_n$ be any morphism in $\mathcal{K}^{sf}(A)$. Then there is a sequence of perfect modules;

$$0 = F'_{-1} \subset F'_0 \subset F'_1 \subset \cdots \subset F'_{n-1} \subset F'_n$$

where

- (i) each F'_i is a dg submodule of F_i and the inclusion mapping $F'_i \hookrightarrow F_i$ is an admissible mono for $1 \leqslant i \leqslant n$,
- (ii) the sequence is a semi-free filtration of F'_n,
- (iii) each F'_i is a graded finite free A-module as an underlying graded A-module,

and f factors through the natural inclusion $i_n : F'_n \hookrightarrow F_n$, that is, there is a morphism $f' : C \to F'_n$ with $f = i_n \circ f'$.

If claim (2) is proved, then the splitting monomorphism $C \hookrightarrow F_n$ whose existence has been proved in claim (1) factors through a perfect dg A-module F'_n, hence C is a direct summand of F'_n. And we complete proving the theorem.

Now we prove claim (2) by induction on n.

If $n = 1$, then F_1 is a free dg A-module, that is, $F_1 = \coprod_{\lambda \in \Lambda} A[a_\lambda]$ with $a_\lambda \in \mathbb{Z}$. Since C is compact in $\mathcal{K}^{sf}(A)$, we have $\mathrm{Hom}_{\mathcal{K}^{sf}(A)}(C, \coprod_\lambda A[a_\lambda]) \cong \coprod_\lambda \mathrm{Hom}_{\mathcal{K}^{sf}(A)}(C, A[a_\lambda])$, which means that the morphism $f : C \to F_1$ factors through a finite free module $F_1' = \coprod_{i=1}^m A[a_{\lambda_i}]$ for some $\lambda_1, \ldots, \lambda_m \in \Lambda$, and so we are done.

Assume $n \geqslant 2$. Since F_n/F_{n-1} is a free dg module, as in the same way as the $n = 1$ case, we can show that the morphism $C \xrightarrow{f} F_n \twoheadrightarrow F_n/F_{n-1}$ factors through a finite free sumbodule of F_n/F_{n-1}, which we denote by F_n'/F_{n-1} for a direct summand $F_n' \subset F_n$. Replacing F_n with F_n', we may assume F_n/F_{n-1} is a finite free dg A-module. Take the mapping cone of $C \xrightarrow{f} F_n \twoheadrightarrow F_n/F_{n-1}$ we have a commutative diagram

$$\begin{array}{ccccccc}
C' & \longrightarrow & C & \longrightarrow & F_n/F_{n-1} & \longrightarrow & \Sigma C' \\
\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \| & & \downarrow{\scriptstyle \Sigma f'} \\
F_{n-1} & \longrightarrow & F_n & \longrightarrow & F_n/F_{n-1} & \longrightarrow & \Sigma F_{n-1}
\end{array}$$

where the rows are triangles in $\mathcal{K}^{sf}(A)$ and f' is the induced morphism by f. Since both C and F_n/F_{n-1} are compact, C' is also compact. Then we can apply the induction hypothesis to f', and we find a sequence $0 = F_{-1}' \subset F_0' \subset F_1' \subset \cdots \subset F_{n-2}' \subset F_{n-1}'$, where there is an admissible mono $F_{n-1}' \hookrightarrow F_{n-1}$ and f' factors through F_{n-1}'. Now take a dg submodule $F_n' \subset F_n$ so that the inclusion $F_n' \hookrightarrow F_n$ is an admissible mono and there is a short exact seqeunce $0 \to F_{n-1}' \to F_n' \to F_n/F_{n-1} \to 0$. Then it is obvious that f factors through F_n and the proof is completed. $\square$

8. Generators

Definition 8.1. Let $\mathcal{T}$ be a triangulated category that admits infinite coproducts.

(a) A full subcategory $\mathcal{L} \subset \mathcal{T}$ is called *localizing* if it is a triangulated subcategory and closed under coproducts; more precisely saying, $\mathcal{L}$ satisfies the following conditions:
  (1) $X \in \mathcal{L}$ if and only if $\Sigma X \in \mathcal{L}$.
  (2) Let $X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$ be a triangle in $\mathcal{T}$. If $X, Y \in \mathcal{L}$ then $Z \in \mathcal{L}$.
  (3) If $\{X_\lambda\}_\lambda \subset \mathcal{L}$ then $\coprod_\lambda X_\lambda \in \mathcal{L}$.

(b) Let $T \in \mathcal{T}$ be an object. We denote by $\mathrm{Loc}(T)$ the smallest localizing subcategory of $\mathcal{T}$ that contains T. (Since the intersection of localizing subcategories is again localizing, the smallest one exists.) The subcategory $\mathrm{Loc}(T)$ is called a localizing subcategory of $\mathcal{T}$ generated by T.

Remark 8.2. Note that a localizing subcategory is a thick subcategory, i.e. closed under direct summands.

In fact, assume $\mathcal{L}$ is a localizing subcateogry of $\mathcal{T}$ and assume $X = X_1 \amalg X_2 \in \mathcal{L}$ where $X_1, X_2 \in \mathcal{T}$. Since $\mathcal{L}$ is closed under coproducts, the infinite sum $Y := X \amalg X \amalg X \amalg \cdots$ belongs to $\mathcal{L}$. Note that there is an isomorphism

$$Y = (X_1 \amalg X_2) \amalg (X_1 \amalg X_2) \amalg \cdots \cong X_1 \amalg (X_2 \amalg X_1) \amalg (X_2 \amalg X_1) \amalg \cdots = X_1 \amalg Y,$$

and hence a triangle $Y \to Y \to X_1 \to \Sigma Y$. It follows that $X_1 \in \mathcal{L}$, since $\mathcal{L}$ is closed under making triangles.

Definition 8.3. Let $\mathcal{T}$ be a triangulated category that admits infinite coproducts. Let $\{X_i\}_{i\in\mathbb{N}}$ be a set of objects indexed by $\mathbb{N}$ and assume there are morphisms $f_i : X_i \to X_{i+1}$ for all $i \in \mathbb{N}$. Then the colimit of $\{X_i, f_i\}_{i\in\mathbb{N}}$ denoted by $\operatorname{colim} X_i$ is defined as the mapping cone of the morphism

$$\coprod_{i\in\mathbb{N}} X_i \xrightarrow{\Phi} \coprod_{i\in\mathbb{N}} X_i \; ; \; (x_1, x_2, x_3, \dots) \mapsto (x_1,\ x_2 - f_1(x_1),\ x_3 - f_2(x_2), \dots) .$$

There is a triangle in $\mathcal{T}$;

$$\Sigma^{-1}(\operatorname{colim} X_i) \longrightarrow \coprod_{i\in\mathbb{N}} X_i \xrightarrow{\Phi} \coprod_{i\in\mathbb{N}} X_i \longrightarrow \operatorname{colim} X_i \tag{8.3.1}$$

Lemma 8.4. *A localizing subcategory is closed under taking colimits.*

Proof. The lemma asserts that if $\mathcal{L} \subset \mathcal{T}$ is a localizing subcategory, then $\operatorname{colim} X_i \in \mathcal{L}$ whenever $\{X_i\} \subset \mathcal{L}$. This is true; because $\coprod_{i\in\mathbb{N}} X_i \in \mathcal{L}$, and then the middle two terms in the triangle (8.3.1) belong to $\mathcal{L}$, hence so does $\operatorname{colim} X_i$. □

Remark 8.5. Let $F \in \mathcal{K}^{sf}(A)$ be a semi-free dg A-module and let $F_0 \subset F_1 \subset \cdots \subset F_n \subset \cdots$ be a semi-free filtration of F. Then it follows from Lemma 4.10 that $F = \operatorname{colim} F_n$.

The following lemma will be useful later.

Lemma 8.6. *Let $\mathcal{T}$ be a triangulated category that admits infinite coproducts and $C \in \mathcal{T}$ be an object. Given a sequence of morphisms in $\mathcal{T}$;*

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \longrightarrow X_n \xrightarrow{f_n} X_{n+1} \longrightarrow \cdots ,$$

we get an inductive system of abelian groups;

$$\operatorname{Hom}_{\mathcal{T}}(C, X_1) \xrightarrow{(f_1)_*} \operatorname{Hom}_{\mathcal{T}}(C, X_2) \xrightarrow{(f_2)_*} \operatorname{Hom}_{\mathcal{T}}(C, X_3) \xrightarrow{(f_3)_*} \cdots ,$$

where $(f_i)_*$ denotes $\operatorname{Hom}_{\mathcal{T}}(C, f_i)$ for $i \in \mathbb{N}$. If C is compact, then there is an isomorphism of abelian groups;

$$\operatorname{Hom}_{\mathcal{T}}(C, \operatorname{colim} X_i) \cong \varinjlim \operatorname{Hom}_{\mathcal{T}}(C, X_i),$$

where the right-hand side is the inductive limit of the above inductive system.

Proof. Set $L = \operatorname{colim} X_i$. It follows from the definition of colimits that there is a long exact sequence of abelian groups;

$$\cdots \to \operatorname{Hom}_{\mathcal{T}}(C, \textstyle\coprod_i X_i) \to \operatorname{Hom}_{\mathcal{T}}(C, \coprod_i X_i) \to \operatorname{Hom}_{\mathcal{T}}(C, L) \to \operatorname{Hom}_{\mathcal{T}}(C, \Sigma \coprod_i X_i) \to \cdots$$

Since C is compact, the long exact sequence is described as:

$$\cdots \to \textstyle\coprod_i \operatorname{Hom}_{\mathcal{T}}(C, X_i) \xrightarrow{\Phi_*} \coprod_i \operatorname{Hom}_{\mathcal{T}}(C, X_i) \to \operatorname{Hom}_{\mathcal{T}}(C, L) \to \coprod_i \operatorname{Hom}_{\mathcal{T}}(C, \Sigma X_i) \xrightarrow{\Phi_*} \cdots ,$$

where Φ_* maps $(a_i)_i$ to $(a_1,\ a_2 - f_1 \circ a_1,\ a_3 - f_2 \circ a_2, \dots)$. We show that Φ_* is an injective mapping. In fact, if $\Phi_*((a_i)_i) = 0$, then $a_1 = 0$, $a_2 = f_1 \circ a_1 = 0$, $a_3 = f_2 \circ a_2 = 0$, …, hence $(a_i)_i = 0$. Therefore the above long sequence is just a collection of short exact sequences of abelian groups;

$$0 \to \textstyle\coprod_i \operatorname{Hom}_{\mathcal{T}}(C, \Sigma^n X_i) \xrightarrow{\Phi_*} \coprod_i \operatorname{Hom}_{\mathcal{T}}(C, \Sigma^n X_i) \to \operatorname{Hom}_{\mathcal{T}}(C, \Sigma^n L) \to 0 \ \ (n \in \mathbb{Z}).$$

Hence from the definition of inductive limit, just looking at the case $n = 0$, we have $\operatorname{Hom}_{\mathcal{T}}(C, L) \cong \varinjlim \operatorname{Hom}_{\mathcal{T}}(C, X_i)$. □

Definition 8.7. Let $\mathcal{T}$ be a triangulated category that admits infinite coproducts.

(1) An object $G \in \mathcal{T}$ is said to be a *classical generator* of $\mathcal{T}$ if $\mathrm{Loc}(G) = \mathcal{T}$.
(2) An object $G \in \mathcal{T}$ is said to be a *generator* of $\mathcal{T}$ if it satisfies the condition;
($\sharp$) For $X \in \mathcal{T}$, if $\mathrm{Hom}_{\mathcal{T}}(\Sigma^n G, X) = 0$ for all $n \in \mathbb{Z}$, then $X = 0$ in $\mathcal{T}$.

Lemma 8.8. *All classical generators are generators.*

Proof. Let $G \in \mathcal{T}$ be a classical generator of $\mathcal{T}$, and assume that $\mathrm{Hom}_{\mathcal{T}}(\Sigma^n G, X) = 0$ $(\forall n \in \mathbb{Z})$ for $X \in \mathcal{T}$. We want to show $X = 0$. Consider the full subcategory of $\mathcal{T}$;

$$\mathcal{U}_X = \{Y \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(\Sigma^n Y, X) = 0 \ (\forall n \in \mathbb{Z})\}$$

It is easy to see that $\mathcal{U}_X \subset \mathcal{T}$ is localizing and $G \in \mathcal{U}_X$. Hence $\mathcal{U}_X = \mathrm{Loc}(G) = \mathcal{T}$, in particular, $\mathrm{Hom}_{\mathcal{T}}(X, X) = 0$. Therefore $X = 0$ in $\mathcal{T}$. □

Exercise 8.9. In general, generators are not necessarily classical generators. Give an example.

The most profound result for generation problem of triangulated categories is the following theorem, whose proof will be given in the proof of Brown representation theorem in the next section.

Theorem 8.10. *Let $\mathcal{T}$ be a triangulated category that admits infinite coproducts. Then every compact generator of $\mathcal{T}$ is a classical generator.*

In the theorem 'compact generator' means an object of $\mathcal{T}$ that is a generator and compact simultaneously.

9. Brown representation theorem

Definition 9.1. Let $\mathcal{T}$ be a triangulated category that admits infinite coproducts. Then $\mathcal{T}$ is said to be "compactly generated" if there is a compact generator in $\mathcal{T}$.

Remark 9.2. A dg R-algebra A is compact as an object in $\mathcal{K}(A)$. (Just note $\mathrm{Hom}_{\mathcal{K}(A)}(A, -) = H_0(-)$.) We have shown in Theorem 4.14 that $\mathrm{Loc}(A) = \mathcal{K}^{sf}(A)$. Hence $\mathcal{K}^{sf}(A)$ is compactly generated. By Remark 7.11 we may say that $\mathcal{D}(A)$ is compactly generated.

Exercise 9.3. Prove or disprove that $\mathcal{K}(A)$ is compactly generated.

Definition 9.4. Let $\mathcal{T}$ be a triangulated category. An additive functor $F : \mathcal{T}^{op} \to (Ab)$ (i.e. a contravariant functor from $\mathcal{T}$ to the category of abelian groups) is called a *cohomological functor* if it satisfies the following condition:

($\natural$) For any triangle $X \xrightarrow{a} Y \xrightarrow{b} Z \xrightarrow{c} \Sigma X$ in $\mathcal{T}$, the induced sequence

$$\cdots \xrightarrow{F(\Sigma a)} F(\Sigma X) \xrightarrow{F(c)} F(Z) \xrightarrow{F(b)} F(Y) \xrightarrow{F(a)} F(X) \xrightarrow{F(\Sigma^{-1}c)} F(\Sigma^{-1}Z) \longrightarrow \cdots$$

is a long exact sequence of abelian groups.

For example, for an object $T \in \mathcal{T}$, the functor $\mathrm{Hom}_{\mathcal{T}}(-, T)$ is a cohomological functor on $\mathcal{T}$. This type of cohomological functor is said to be *representable* or *represented by an object.*

A landmark result is the following Brown's theorem, whose proof is given later.

Theorem 9.5 (Brown representation theorem)**.** *Let $\mathcal{T}$ be a triangulated category that admits infinite coproducts. Assume $\mathcal{T}$ is compactly generated. Let $F : \mathcal{T}^{op} \to (Ab)$ be a cohomological functor and we assume that F sends coproducts to products, i.e.*

$$F\left(\coprod_{\lambda} X_{\lambda}\right) = \prod_{\lambda} F(X_{\lambda}) \text{ for any } \{X_{\lambda}\}_{\lambda} \subset \mathcal{T}. \tag{9.5.1}$$

Then F is represented by an object of $\mathcal{T}$, i.e. there is an object $T \in \mathcal{T}$ and an isomorphism of functors; $F \cong \mathrm{Hom}_{\mathcal{T}}(-, T)$.

The following corollary has been announced in the previous section. This is actually a direct consequence of Brown theorem as we see below. However in the next section we shall prove this corollary independently.

Corollary 9.6. *Let $\mathcal{T}$ be a triangulated category that admits infinite coproducts. Then every compact generator of $\mathcal{T}$ is a classical generator.*

Proof. Let $C \in \mathcal{T}$ be a compact generator. We want to show that $\mathrm{Loc}(C) = \mathcal{T}$. For this purpose let $X \in \mathcal{T}$ be any object. Then the functor $F := \mathrm{Hom}_{\mathcal{T}}(-, X)|_{\mathrm{Loc}(C)}$ is a cohomological functor on $\mathrm{Loc}(C)$. Trivially from the definition $\mathrm{Loc}(C)$ is compactly generated. It is also clear that $F(\coprod_{\lambda} X_{\lambda}) = \prod_{\lambda} F(X_{\lambda})$ for any set $\{X_{\lambda}\}_{\lambda} \subset \mathrm{Loc}(C)$. Therefore by Brown representation theorem, there is an object $Y \in \mathrm{Loc}(C)$ that satisfies $\mathrm{Hom}_{\mathcal{T}}(-, X)|_{\mathrm{Loc}(C)} \cong \mathrm{Hom}_{\mathrm{Loc}(C)}(-, Y)$ as functors on $\mathrm{Loc}(C)$. By Yoneda lemma, there is a morphism $f : Y \to X$ in $\mathcal{T}$ such that $\mathrm{Hom}_{\mathcal{T}}(-, f)|_{\mathrm{Loc}(C)}$ is an isomorphism of functors. Let $Z = C(f)$ be the mapping cone of f, i.e. $Y \xrightarrow{f} X \longrightarrow Z \longrightarrow \Sigma Y$ is a triangle in $\mathcal{T}$. Then we have $\mathrm{Hom}_{\mathcal{T}}(-, Z)|_{\mathrm{Loc}(C)} = 0$. In particular we have $\mathrm{Hom}_{\mathcal{T}}(\Sigma^n C, Z) = 0$. Since C is a generator, it follows $Z = 0$, hence $X \cong Y$, and thus $X \in \mathrm{Loc}(C)$[6]. □

9.7 (Alternative proof for Theorem 4.20)**.** By using Brown representation theorem we can easily prove Theorem 4.20 which guarantees the existence of semi-free resolution.

First of all recall that $\mathrm{Loc}(A) = \mathcal{K}^{sf}(A) \subset \mathcal{K}(A)$, and so that $\mathcal{K}^{sf}(A)$ is compactly generated. Now let $X \in \mathcal{K}(A)$ be any object, and consider the functor $\mathrm{Hom}_{\mathcal{K}(A)}(-, X)|_{\mathcal{K}^{sf}(A)}$. It is easy to see that this is a cohomological functor on $\mathcal{K}^{sf}(A)$ that satisfies the condition (9.5.1). Hence it follows from Brown representation theorem that $\mathrm{Hom}_{\mathcal{K}(A)}(-, X)|_{\mathcal{K}^{sf}(A)} \cong \mathrm{Hom}_{\mathcal{K}^{sf}(A)}(-, F)$ for some $F \in \mathcal{K}^{sf}(A)$, where by Yoneda lemma this isomorphism is induced by a morphism $f : F \to X$. Noting that $\mathrm{Hom}_{\mathcal{K}(A)}(\Sigma^n A, Y) = H_n(Y)$ for $Y \in \mathcal{K}(A)$ and $n \in \mathbb{Z}$, we see that $H_n(f) : H_n(F) \to H_n(X)$ is an isomorphism for each $n \in \mathbb{Z}$. Hence $f : F \to X$ is a quasi-isomorphism and $F \in \mathcal{K}^{sf}(A)$. □

The following theorem is crucial for the proof of Brown representation theorem.

Theorem 9.8. *Let $\mathcal{T}$ be a triangulated category admitting infinite coproduct and $C \in \mathcal{T}$ be a compact object. Further let $F : \mathcal{T}^{op} \to (Ab)$ be a cohomological functor that sends coproducts to products as in* (9.5.1)*. Then there are an object $L \in \mathrm{Loc}(C)$ and a morphism of functors $\varphi : \mathrm{Hom}_{\mathcal{T}}(-, L) \to F$ such that restriction of φ to $\mathrm{Loc}(C)$ is an isomorphism;*

$$\varphi|_{\mathrm{Loc}(C)} : \mathrm{Hom}_{\mathcal{T}}(-, L)|_{\mathrm{Loc}(C)} \xrightarrow{\simeq} F|_{\mathrm{Loc}(C)}$$

[6] In this note every full subcategory is assumed to be closed under isomorphism.

We first show how this theorem implies Theorems 9.5 and 8.10.

9.9 (Proof of Theorem 9.8 $\Rightarrow$ Theorem 8.10)**.** Let $\mathcal{T}$ be a triangulated category admitting infinite coproduct and $C \in \mathcal{T}$ be a compact generator. We want to show that C is a classical generator, i.e. $\mathrm{Loc}(C) = \mathcal{T}$.

Let $X \in \mathcal{T}$ be any object. Since C is compact, we can apply Theorem 9.8 to the functor $\mathrm{Hom}_{\mathcal{T}}(-, X) : \mathcal{T}^{op} \to (Ab)$, and we find an object $L \in \mathrm{Loc}(C)$ and a morphism of functors $\varphi : \mathrm{Hom}_{\mathcal{T}}(-, L) \to \mathrm{Hom}_{\mathcal{T}}(-, X)$ such that $\varphi|_{\mathrm{Loc}(C)}$ is an isomorphism. By Yoneda Lemma there is a morphism $f : L \to X$ in $\mathcal{T}$ that satisfies $\varphi = \mathrm{Hom}_{\mathcal{T}}(-, f)$. Let Y be the mapping cone of f, so there is a triangle in $\mathcal{T}$; $L \xrightarrow{f} X \to Y \to \Sigma L$. Then we have $\mathrm{Hom}_{\mathcal{T}}(-, Y)|_{\mathrm{Loc}(C)} = 0$, and in particular $\mathrm{Hom}_{\mathcal{T}}(\Sigma^n C, Y) = 0$ for all $n \in \mathbb{Z}$. Since C is a generator, it induces that $Y = 0$ hence $X \cong L$ in $\mathcal{T}$. Thus $X \in \mathrm{Loc}(C)$. □

9.10 (Proof of Theorem 9.8 $\Rightarrow$ Brown Representation Theorem)**.** Let $\mathcal{T}$ be a triangulated category admitting infinite coproduct, and let $F : \mathcal{T}^{op} \to (Ab)$ be a cohomological functor that sends coproducts to products. Assume $\mathcal{T}$ is compactly generated, and $C \in \mathcal{T}$ is a compact generator. We want to show that F is representable. By Theorem 9.8 we have an object $L \in \mathrm{Loc}(C) \subset \mathcal{T}$ and a morphism $\varphi : \mathrm{Hom}_{\mathcal{T}}(-, L) \to F$ where the restriction $\varphi|_{\mathrm{Loc}(C)}$ is an isomorphism. Note however that we have proved that $\mathcal{T} = \mathrm{Loc}(C)$ for the compact generator C. Thus φ itself is an isomorphism, and F is represented by $L \in \mathcal{T}$. □

The proof of Theorem 9.8 will be a little long. Before proceeding to the proof the reader is recommended to solve the following exercise:

Exercise 9.11 (Yoneda Lemma)**.** Let $\mathcal{C}$ be any additive category and $F : \mathcal{C}^{op} \to (Ab)$ be any additive functor. Prove the following map is an isomorphism of abelian groups for any object $X \in \mathcal{C}$:

$$\mathrm{Hom}(\mathrm{Hom}_{\mathcal{C}}(-, X), F) \to F(X);\ f \mapsto \overline{f} := f(X)(id_X),$$

where the left-hand side is the set of all morphisms of functors (=natural transformations).

9.12 (Proof of Theorem 9.8)**.** We consider the additive closure $\mathrm{Add}(C)$ of C that is defined to be the smallest full subcategory of $\mathcal{T}$ satisfying the following conditions:

(i) $C \in \mathrm{Add}(C)$
(ii) Let $X \in \mathcal{T}$. Then $X \in \mathrm{Add}(C)$ if and only if $\Sigma X \in \mathrm{Add}(C)$.
(iii) Let $\{X_\lambda\}_\lambda \subset \mathcal{T}$ be a set of objects. Then $\coprod_\lambda X_\lambda \in \mathrm{Add}(C)$ if and only if $X_\lambda \in \mathrm{Add}(C)$ for all λ.

Note that an object of $\mathrm{Add}(C)$ is a direct summand of a direct sum of copies of objects in $\{\Sigma^n C \mid n \in \mathbb{Z}\}$, and that $\mathrm{Add}(C) \subset \mathrm{Loc}(C) \subset \mathcal{T}$. To begin with, we show the following claim:

(I) Let $F : \mathcal{T}^{op} \to (Ab)$ be any additive functor that sends coproducts to products. Then there are an object $U \in \mathrm{Add}(C)$ and a morphism of functors $p : \mathrm{Hom}_{\mathcal{T}}(-, U) \to F$ such that

$$\mathrm{Hom}_{\mathcal{T}}(-, U)|_{\mathrm{Add}(C)} \xrightarrow{p|_{\mathrm{Add}(C)}} F|_{\mathrm{Add}(C)}$$

is an epimorphism as a morphism of functors, i.e. a surjective map when an object of $\mathrm{Add}(C)$ is substituted.

(Proof of (I)): To prove the claim, recall from Yoneda Lemma that there is an isomorphism of abelian groups

$$\mathrm{Hom}(\mathrm{Hom}_{\mathcal{T}}(-, \Sigma^n C), F) \xrightarrow{\simeq} F(\Sigma^n C);\ p \ \mapsto\ \overline{p} = p(\Sigma^n C)(id_{\Sigma^n C}).$$

We consider a generating set of $\bigoplus_{n\in\mathbb{Z}} F(\Sigma^n C)$ as an abelian group and we denote it by $\{\xi_\lambda \in F(\Sigma^{n_\lambda} C) \mid \lambda \in \Lambda\}$. Then consider the product element of all of these generators;

$$(\xi_\lambda)_\lambda \in \prod_\lambda F(\Sigma^{n_\lambda} C) \cong F(\coprod_\lambda \Sigma^{n_\lambda} C),$$

and take a morphism $p : \mathrm{Hom}_{\mathcal{T}}(-, \coprod_\lambda \Sigma^{n_\lambda} C) \to F$ that corresponds to $(\xi_\lambda)_\lambda$ under Yoneda bijection. Then setting $U = \coprod_\lambda \Sigma^{n_\lambda} C$, we see that $U \in \mathrm{Add}(C)$ and $p : \mathrm{Hom}_{\mathcal{T}}(-, U) \to F$ has been defined. Note that, for any $\lambda \in \Lambda$, the mapping $p(\Sigma^{n_\lambda} C) : \mathrm{Hom}_{\mathcal{T}}(\Sigma^{n_\lambda} C, U) \to F(\Sigma^{n_\lambda} C)$ sends the natural splitting monomorphism $\Sigma^{n_\lambda} C \hookrightarrow U$ to ξ_λ. Therefore, for any $n \in \mathbb{Z}$, the image of the mapping $p(\Sigma^n C) : \mathrm{Hom}_{\mathcal{T}}(\Sigma^n C, U) \to F(\Sigma^n C)$ contains all the generators, hence it is a surjective mapping. Now consider the full subcategory; $\mathcal{L} := \{X \in \mathcal{T} \mid p(X) \text{ is surjective}\}$, and we have shown $\Sigma^n C \in \mathcal{L}$ for all $n \in \mathbb{Z}$. Let $\{X_\lambda\}_\lambda \in \mathcal{T}$ be a set of objects. Since both $\mathrm{Hom}_{\mathcal{T}}(-, U)$ and F send coproducts to products, we can see that $p(\coprod_\lambda X_\lambda) \cong \prod_\lambda p(X_\lambda)$. Thus $\coprod_\lambda X_\lambda \in \mathcal{L}$ if and only if $X_\lambda \in \mathcal{L}$ for all λ. Hence all direct summands of direct sums of $\Sigma^n C$ $(n \in \mathbb{Z})$ belong to $\mathcal{L}$, and thus $\mathrm{Add}(C) \subset \mathcal{L}$. It follows that $p|_{\mathrm{Add}(C)}$ is an epimorphism. This proves the claim (I).

Next, we show the following claim:

(II) Let $F : \mathcal{T}^{op} \to (Ab)$ be any additive functor that sends coproducts to products. Then there is a sequence of morphisms $\mathrm{Hom}_{\mathcal{T}}(-, V) \xrightarrow{q} \mathrm{Hom}_{\mathcal{T}}(-, U) \xrightarrow{p} F$ with $q \circ p = 0$ and $U, V \in \mathrm{Add}(C)$ such that its restriction to $\mathrm{Add}(C)$ yields an exact sequence of abelian groups;

$$\mathrm{Hom}_{\mathcal{T}}(-, V)|_{\mathrm{Add}(C)} \longrightarrow \mathrm{Hom}_{\mathcal{T}}(-, U)|_{\mathrm{Add}(C)} \longrightarrow F|_{\mathrm{Add}(C)} \longrightarrow 0.$$

(Proof of (II)): The construction of p has been done in (I). Let F' be the kernel of p, so that there is an exact sequence of functors; $0 \to F' \to \mathrm{Hom}_{\mathcal{T}}(-, U) \xrightarrow{p} F$. Since both $\mathrm{Hom}_{\mathcal{T}}(-, U)$ and F are additive functors that send coproducts to products, it is easy to see that so is F'. Then applying claim (I) to F', we find $V \in \mathrm{Add}(C)$ and a morphism $\mathrm{Hom}_{\mathcal{T}}(-, V) \to F'$ whose restriction to $\mathrm{Add}(C)$ is surjective. It is enough to set q as the composition $\mathrm{Hom}_{\mathcal{T}}(-, V) \to F' \hookrightarrow \mathrm{Hom}_{\mathcal{T}}(-, U)$.

(III) Let $F : \mathcal{T}^{op} \to (Ab)$ be any triangle functor that sends coproducts to products. By induction on n, we construct the following diagram (9.12.1) in the functor category satisfying the following conditions;

(a) the right squares are commutative and $p_n \circ q_n = 0$ for all $n \geqslant 0$,
(b) $U_0 \in \mathrm{Add}(C)$ and $V_n \in \mathrm{Add}(C)$ for all $n \geqslant 0$,
(c) there are triangles for all $n \geqslant 0$;

$$V_n \xrightarrow{f_n} U_n \xrightarrow{i_n} U_{n+1} \longrightarrow \Sigma V_n\ ,$$

(d) $(i_n)_* = \mathrm{Hom}_{\mathcal{T}}(-, i_n)$ and $q_n = \mathrm{Hom}_{\mathcal{T}}(-, f_n)$,

(e) each row induces the exact sequence by restricting it to $\mathrm{Add}(C)$;

$$\mathrm{Hom}_{\mathcal{T}}(-,V_n)|_{\mathrm{Add}(C)} \longrightarrow \mathrm{Hom}_{\mathcal{T}}(-,U_n)|_{\mathrm{Add}(C)} \longrightarrow F|_{\mathrm{Add}(C)} \longrightarrow 0$$

Note from (b) and (c) that $U_n \in \mathrm{Loc}(C)$ for all $n \geqslant 0$.

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathcal{T}}(-,V_0) & \xrightarrow{q_0} & \mathrm{Hom}_{\mathcal{T}}(-,U_0) & \xrightarrow{p_0} & F \\
 & & \downarrow (i_0)_* & & \| \\
\mathrm{Hom}_{\mathcal{T}}(-,V_1) & \xrightarrow{q_1} & \mathrm{Hom}_{\mathcal{T}}(-,U_1) & \xrightarrow{p_1} & F \\
 & & \downarrow (i_1)_* & & \| \\
 & & \vdots & & \vdots \\
 & & \downarrow & & \| \\
\mathrm{Hom}_{\mathcal{T}}(-,V_{n-1}) & \xrightarrow{q_{n-1}} & \mathrm{Hom}_{\mathcal{T}}(-,U_{n-1}) & \xrightarrow{p_{n-1}} & F \\
 & & \downarrow (i_{n-1})_* & & \| \\
\mathrm{Hom}_{\mathcal{T}}(-,V_n) & \xrightarrow{q_n} & \mathrm{Hom}_{\mathcal{T}}(-,U_n) & \xrightarrow{p_n} & F \\
 & & \downarrow & & \| \\
 & & \vdots & & \vdots
\end{array} \tag{9.12.1}$$

(Proof of (Ⅲ)): The proof goes in a similar way to 4.22. The first row of (9.12.1) has been constructed in the claim (Ⅱ).

Now assume we have constructed the diagram (9.12.1) up to the $(n-1)$st row;

$$\mathrm{Hom}_{\mathcal{T}}(-,V_{n-1}) \xrightarrow{q_{n-1}} \mathrm{Hom}_{\mathcal{T}}(-,U_{n-1}) \xrightarrow{p_{n-1}} F$$

By Yoneda Lemma there is a morphism $f_{n-1} : V_{n-1} \to U_{n-1}$ in $\mathcal{T}$ such that $q_{n-1} = \mathrm{Hom}_{\mathcal{T}}(-, f_{n-1})$. Take the mapping cone of f_{n-1} and we set $U_n := C(f_{n-1})$. Then we have a triangle $V_{n-1} \xrightarrow{f_{n-1}} U_{n-1} \xrightarrow{i_{n-1}} U_n \longrightarrow \Sigma V_{n-1}$. Since F is a cohomological functor, it induces an exact sequence of abelian groups $F(U_n) \xrightarrow{F(i_{n-1})} F(U_{n-1}) \xrightarrow{F(f_{n-1})} F(V_{n-1})$. Then by Yoneda Lemma we have an exact sequence;

$$\mathrm{Hom}(\mathrm{Hom}_{\mathcal{T}}(-,U_n),F) \xrightarrow{(i_{n-1})_*} \mathrm{Hom}(\mathrm{Hom}_{\mathcal{T}}(-,U_{n-1}),F) \xrightarrow{\circ q_{n-1}} \mathrm{Hom}(\mathrm{Hom}_{\mathcal{T}}(-,V_{n-1}),F).$$

Since $p_{n-1} \in \mathrm{Hom}(\mathrm{Hom}_{\mathcal{T}}(-,U_{n-1}),F)$ goes to zero by the above right mapping, there is $p_n \in \mathrm{Hom}(\mathrm{Hom}_{\mathcal{T}}(-,U_n),F)$ that satisfies $p_n \circ (i_{n-1})_* = p_{n-1}$. Since $(p_{n-1})|_{\mathrm{Add}(C)}$ is surjective by the induction hypothesis, it is obvious that $(p_n)|_{\mathrm{Add}(C)}$ is surjective as well. Now take the kernel functor $G_n = \mathrm{Ker}(p_n)$. Since both F and $\mathrm{Hom}_{\mathcal{T}}(-,U_{n-1})$ are cohomological functors which send coproducts to products, so is G_n. Then applying claim (I) to G_n, we find $V_n \in \mathrm{Add}(C)$ and a morphism $\mathrm{Hom}_{\mathcal{T}}(-,V_n) \to G_n$ whose restriction to $\mathrm{Add}(C)$ is surjective. It is enough to set q_n as the composition $\mathrm{Hom}_{\mathcal{T}}(-,V_n) \to G_n \hookrightarrow \mathrm{Hom}_{\mathcal{T}}(-,U_n)$. Thus we have the nth row of (9.12.1).

(Ⅳ) In the claim (Ⅲ) we have constructed a sequence $U_0 \xrightarrow{i_0} U_1 \xrightarrow{i_1} U_2 \xrightarrow{i_2} \cdots$, where $U_n \in \mathrm{Loc}(C)$ for all $n \in \mathbb{N}$. Now we denote by L the colimit of $\{U_n, i_n\}$; $L := \mathrm{colim}\, U_n$, see Definition 8.3. As the final claim we are going to prove that L meets all the requirements in Theorem 9.8.

(Proof of (IV)): We see from Lemma 8.4 that $L \in \mathrm{Loc}(C)$, and we have a triangle in $\mathcal{T}$;

$$\Sigma^{-1}L \longrightarrow \coprod_{n=0}^{\infty} U_n \xrightarrow{\Phi} \coprod_{n=0}^{\infty} U_n \xrightarrow{\Psi} L\ ,$$

where Φ maps $(x_0, x_1, x_2, \dots)$ to $(x_0, x_1 - i_0(x_0), x_2 - i_1(x_1), \dots)$. Since F is a cohomological functor, it induces an exact sequence of abelian groups;

$$\begin{array}{ccccc} F(L) & \xrightarrow{F(\Psi)} & F\left(\coprod_n U_n\right) & \xrightarrow{F(\Phi)} & F\left(\coprod_n U_n\right) \\ & & \downarrow\cong & & \downarrow\cong \\ & & \prod_n F(U_n) & \longrightarrow & \prod_n F(U_n) \\ & & \downarrow\cong & & \downarrow\cong \\ & & \prod_n \mathrm{Hom}(\mathrm{Hom}_{\mathcal{T}}(-,U_n),F) & \xrightarrow{\Phi^*} & \prod_n \mathrm{Hom}(\mathrm{Hom}_{\mathcal{T}}(-,U_n),F) \end{array}$$

where the first vertical isomorphisms are obtained by the assumption, the second are Yoneda bijections, and Φ^* is a morphism induced by $F(\Phi)$. By definition Φ^* is given by mapping $(f_0, f_1, f_2, \dots) \in \prod_n \mathrm{Hom}(\mathrm{Hom}_{\mathcal{T}}(-,U_n),F)$ to $(f_0 - f_1 \circ (i_0)_*,\ f_1 - f_2 \circ (i_1)_*, \dots)$.

The morphisms $p_n : \mathrm{Hom}_{\mathcal{T}}(-,U_n) \to F$ in the diagram (9.12.1) form the element $(p_n)_n \in \prod_n \mathrm{Hom}(\mathrm{Hom}_{\mathcal{T}}(-,U_n),F)$ which satisfies $\Phi^*((p_n)_n) = 0$. Hence there is an element $\varphi \in \mathrm{Hom}(\mathrm{Hom}_{\mathcal{T}}(-,L),F) \cong F(L)$ such that $F(\Psi)(\varphi) = (p_n)_n$. In such a way we obtain a morphism $\varphi : \mathrm{Hom}_{\mathcal{T}}(-,L) \to F$ of functors. To complete the proof of Theorem 9.8 it remains to show that the restriction of φ to the subcategory $\mathrm{Loc}(C)$ gives an isomorphism. Considering the full subcategory $\mathcal{L} = \{X \in \mathcal{T} \mid \varphi(\Sigma^n X)$ is an isomorphism for all $n \in \mathbb{Z}\}$, we easily see that $\mathcal{L} \subset \mathcal{T}$ is a localizing subcategory. Hence to prove $\mathrm{Loc}(C) \subset \mathcal{L}$, it is enough to show that $\varphi(C')$ is an isomorphism when C' is one of $\Sigma^n C$ $(n \in \mathbb{Z})$.

By the above choice of φ, there are morphisms $j_n : U_n \to L$ for all $n \geqslant 0$ such that $j_{n+1} \circ i_n = j_n$ and the following diagram in $\mathcal{T}$ is commutative for all $n \geqslant 0$:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{T}}(-,U_n) & \xrightarrow{p_n} & F \\ {\scriptstyle \mathrm{Hom}(-,j_n)}\downarrow & & \| \\ \mathrm{Hom}_{\mathcal{T}}(-,L) & \xrightarrow{\varphi} & F \end{array}$$

Since C' is compact by the assumption, it follows from Lemma 8.6 that $\mathrm{Hom}_{\mathcal{T}}(C',L) = \varinjlim \mathrm{Hom}_{\mathcal{T}}(C',U_n)$ and hence we have $\varphi(C') = \varinjlim p_n(C')$ by the above commutative diagram.

Since all the $p_n(C') : \mathrm{Hom}_{\mathcal{T}}(C',U_n) \to F(C')$ are surjective by the construction in (III), it follows that $\varphi(C') = \varinjlim p_n(C')$ is also a surjective mapping. It remains to prove that $\varphi(C') : \mathrm{Hom}_{\mathcal{T}}(C',L) \to F(C')$ is injective. For this aim let α be an element of $\mathrm{Hom}_{\mathcal{T}}(C',L)$ and assume $\varphi(C')(\alpha) = 0$. Since $\mathrm{Hom}_{\mathcal{T}}(C',L)$ is the inductive limit of $\mathrm{Hom}_{\mathcal{T}}(C',U_n)$, we find an element $\alpha_n \in \mathrm{Hom}_{\mathcal{T}}(C',U_n)$ for a large number n that satisfies $\alpha = j_n \circ \alpha_n$.

$$\begin{array}{ccc} \alpha_n \in \mathrm{Hom}_{\mathcal{T}}(C', U_n) & \xrightarrow{p_n(C')} & F(C') \\ {\scriptstyle \mathrm{Hom}(C', j_n)} \downarrow & & \| \\ \alpha \in \mathrm{Hom}_{\mathcal{T}}(C', L) & \xrightarrow{\varphi(C')} & F(C') \end{array}$$

Hence $\varphi(C')(\alpha) = 0$ implies $p_n(C')(\alpha_n) = 0$. On the other hand, from condition (e) in (Ⅲ), we have an exact sequence of abelian groups;

$$\mathrm{Hom}_{\mathcal{T}}(C', V_n) \xrightarrow{q_n(C')} \mathrm{Hom}_{\mathcal{T}}(C', U_n) \xrightarrow{p_n(C')} F(C') \longrightarrow 0,$$

hence we find an element $\beta_n \in \mathrm{Hom}_{\mathcal{T}}(C', V_n)$ satisfying $\alpha_n = q_n(C')(\beta_n)$. Note from condition (c) of (Ⅲ) that there is an exact sequence

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathcal{T}}(C', V_n) & \xrightarrow{q_n(C')} & \mathrm{Hom}_{\mathcal{T}}(C', U_n) & \xrightarrow{\mathrm{Hom}(C', i_n)} & \mathrm{Hom}_{\mathcal{T}}(C', U_{n+1}) \\ \beta_n & \longmapsto & \alpha_n & & \end{array}$$

Therefore we have $i_n \circ \alpha_n = 0$. This forces that $\alpha = j_n \circ \alpha_n = j_{n+1} \circ i_n \circ \alpha_n = 0$, hence we have shown that $\varphi(C')$ is an injective map.

We now have completed the proof of Theorem 9.8. $\square$

10. Keller's Rickard Theorem

We start this section with the following exercise.

Exercise 10.1. Let R and S be commutative rings. Assume there is an equivalence of module categories; $\mathrm{Mod}\text{-}R \simeq \mathrm{Mod}\text{-}S$. Then prove that R is isomorphic to S as a ring.

The answer of this exercise is simply a consequence of the following general result:

Consider the identity functor $I_R = id_{\mathrm{Mod}\text{-}R}$ on the category $\mathrm{Mod}\text{-}R$, and take the endomorphism ring of I_R, that is, the set of natural transformations;

$$E = \{\varphi : I_R \to I_R \mid \text{natural transformation}\}$$

Then E is a commutative ring that is isomorphic to R.

However it cannot be generalized to non-commutative rings. In general, let A be a non-commutative ring and let I_A be the identity functor on $\mathrm{Mod}\text{-}A$ by which we denote the category of right A-modules and right A-module homomorphisms. Then we can show that the endomorphism ring of I_A is a commutative ring that is isomorphic to the center of A. Therefore in non-commutative case, equivalence of module categories does not imply ring isomorphism.

Proposition 10.2 (Morita equivalence). *Let A and B be (not necessarily commutative) rings that are right Noetherian. Then the following two conditions are equivalent:*

(1) *There is an equivalence of categories;* $\mathrm{Mod}\text{-}A \simeq \mathrm{Mod}\text{-}B$.
(2) *There exists a finitely generated projective A-module P that is a generator*[7] *of* $\mathrm{Mod}\text{-}A$ *and* $B \cong \mathrm{End}_A(P)$.

[7] An object P in an abelian category $\mathcal{A}$ is called a generator if it satisfies that $\mathrm{Hom}_{\mathcal{A}}(P, f) = 0$ implies $f = 0$ for a morphism f

Proof. We only prove (1) $\Rightarrow$ (2): Let $F : \text{Mod-}B \to \text{Mod-}A$ be an equivalence. Since the object $B \in \text{Mod-}B$ is a generator that is compact and projective, so is $P := F(B)$ in Mod-A. Then P is finitely generated as we have shown in 7.7. Now the following isomorphisms of rings hold:

$$B \cong \text{End}_B(B) \cong \text{End}_A(F(B)) = \text{End}_A(P).$$

$\square$

Exercise 10.3. Prove (2) $\Rightarrow$ (1) in the proposition.
Hint: The functor $\text{Hom}_A(P, -) : \text{Mod-}A \to \text{Mod-}B$ gives an equivalence.

In the dg algebra case, the situation is quite different. If A is a dg R-algebra, then the object A in $\mathcal{C}(A)$ is neither a projective object nor a generator. (In fact $\mathcal{C}(A)$ has no enough projectives in general.) For this reason we focus on the derived categories for equivalence problem for dg modules, so that in the following we are interested in when a category equivalence $\mathcal{D}(A) \simeq \mathcal{D}(B)$ happens for dg R-algebras A and B.

Exercise 10.4. If A is a dg R-algebra, then prove A is compact in $\mathcal{C}(A)$.

In the rest of the section A is a dg R-algebra. The following is one of Keller's theorems.

Theorem 10.5. *Let A be a dg R-algebra that is not necessarily non-negatively graded, and assume that $H(A) = H_0(A) =: S$, i.e.*

$$H_n(A) = \begin{cases} S & (n = 0) \\ 0 & (n \neq 0) \end{cases}.$$

Then there is a category equivalence $\mathcal{D}(A) \simeq \mathcal{D}(S)$.

Proof. Note that we have shown in Theorem 6.19 that a quasi-isomorphism $A \to B$ induces a category equivalence $\mathcal{D}(A) \simeq \mathcal{D}(B)$. However note that there is no direct dg algebra homomorphism between A and S.

Let us consider the subset $A_{\geqslant 0} := Z_0(A) \oplus \bigoplus_{i>0} A_i \subset A$ that is a non-negatively graded dg subalgebra of A. Then there are a pair of quasi-ismorphisms;

$$A \hookleftarrow A_{\geqslant 0} \to H_0(A) = S.$$

Thus it follows from Theorem 6.19 that $\mathcal{D}(A) \simeq \mathcal{D}(A_{\geqslant 0}) \simeq \mathcal{D}(S)$. $\square$

Definition 10.6. Let $T, X \in \mathcal{K}(A)$ be dg A-modules. Recalling from Remark 3.2 we define

$$\text{Hom}_A(T, X) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{\text{gr}A\text{-mod}}(T, X[n]),$$

where $\text{Hom}_{\text{gr}A\text{-mod}}(T, X[n])$ is the set of A-module homomorphisms $f : T \to X$ mapping T_i to X_{i+n} for all $i \in \mathbb{Z}$. The differential $\partial^{\text{Hom}_A(T,X)}$ is defined as

$$\partial^{\text{Hom}_A(T,X)}((f_i)_{i\in\mathbb{Z}}) = \left(f_{i-1} \circ d_i^T - (-1)^n d_{i+n}^X \circ f_i\right)_{i\in\mathbb{Z}}$$

for $(f_i)_{i\in\mathbb{Z}} \in \text{Hom}_{\text{gr}A\text{-mod}}(T, X[n])$. Then $\text{Hom}_A(T, X)$ is a dg R-module (=an R-complex) with the differential $\partial^{\text{Hom}_A(T,X)}$.

The endomorphism dg algebra of T is denoted by $\text{End}_A(T)$ which is defined as $\text{End}_A(T) = \text{Hom}_A(T, T)$ with differential $d^{\text{End}_A(T)} = \partial^{\text{Hom}_A(T,T)}$. Clearly $\text{End}_A(T)$ is a grade algebra by composition of homomorphisms as multiplication, and hence $\text{End}_A(T)$ is a dg R-algebra with differential $d^{\text{End}_A(T)}$. Compare with Example 1.11.

Note that $\mathrm{Hom}_A(T, X)$ is a graded $\mathrm{End}_A(T)$-module by defining the right action of $\mathrm{End}_A(T)$ as composition of mappings. Then it is not hard to see that $\mathrm{Hom}_A(T, X)$ is a dg $\mathrm{End}_A(T)$-module.

Exercise 10.7. Verify every statement in Definition 10.6.

Exercise 10.8. Let $T, N \in \mathcal{K}^{sf}(A)$ and assume that N is null. Then prove $\mathrm{Hom}_A(T, N)$ is also a null object in $\mathcal{C}(\mathrm{End}_A(T))$

Definition 10.9 (RHom)**.** Assume T is a semi-free dg A-module. By the previous exercise the functor $\mathrm{Hom}_A(T, -) : \mathcal{C}(A) \to \mathcal{C}(\mathrm{End}_A(T))$ preserves null objects, and hence it also preserves null-homotopic morphisms. (See Lemma 3.6.) Therefore the functor $\mathrm{Hom}_A(T, -) : \mathcal{K}(A) \to \mathcal{K}(\mathrm{End}_A(T))$ is induced. By Corollary 4.18 we can see that this functor preserves quasi-isomorphisms too. Hence it induces the functor between the derived categories, which we denote by $\mathbf{R}\mathrm{Hom}_A(T, -) : \mathcal{D}(A) \to \mathcal{D}(\mathrm{End}_A(T))$.

Theorem 10.10. *Assume $T \in \mathcal{D}(A)$ is a compact generator of $\mathcal{D}(A)$. Then there is a triangle equivalence of triangulated categories; $\mathcal{D}(A) \simeq \mathcal{D}(\mathrm{End}_A(T))$.*

Proof. Replacing T by its semi-free resolution, we may assume that the compact generator T is semi-free. Set $E = \mathrm{End}_A(T)$ to save notations, and we prove $F := \mathbf{R}\mathrm{Hom}_A(T, -) : \mathcal{D}(A) \to \mathcal{D}(E)$ gives an equivalence.

Consider the functor $G : \mathcal{D}(E) \to \mathcal{D}(A)$ in the reverse direction which is defined as $G(-) = - \otimes^{\mathbf{L}}_E T$.

Since there is a natural mapping $\mathrm{Hom}_A(T, X) \otimes_E T \to X$ given by $f \otimes t \mapsto f(t)$ for any $X \in \mathcal{C}(A)$, it induces the morphism between functors $\mu : G \circ F = \mathbf{R}\mathrm{Hom}_A(T, -) \otimes^{\mathbf{L}}_E T \to id_{\mathcal{D}(A)}$. Note by definition that $\mu(T) : \mathbf{R}\mathrm{Hom}_A(T, T) \otimes^{\mathbf{L}}_E T \to T$ is an isomorphism. Now consider the full subcategory $\mathcal{U} := \{X \in \mathcal{D}(A) \mid \mu(X) \text{ is an isomorphism}\}$ of $\mathcal{D}(A)$. One can easily see that $\mathcal{U} \subset \mathcal{D}(A)$ is a triangulated subcategory. Further, if $\{X_\lambda\}_\lambda \subset \mathcal{U}$, then $\coprod_\lambda X_\lambda \in \mathcal{U}$. In fact,

$$\mathrm{Hom}_A(T, \coprod_\lambda X_\lambda) \otimes_E T \cong \coprod_\lambda (\mathrm{Hom}_A(T, X_\lambda) \otimes_E T)$$

holds, since T is compact in $\mathcal{D}(A)$. Therefore $\mathcal{U}$ is a localizing subcategory of $\mathcal{D}(A)$. Since $T \in \mathcal{U}$, and since T is a generator, we have $\mathcal{U} = \mathcal{D}(A)$, hence μ is an isomorphism on whole $\mathcal{D}(A)$, i.e. $G \circ F \cong id_{\mathcal{D}(A)}$.

We can prove $F \circ G \cong id_{\mathcal{D}(E)}$ in a similar manner. Actually, there is a natural morphism $\nu : id_{\mathcal{D}(E)} \to \mathbf{R}\mathrm{Hom}_A(T, - \otimes^{\mathbf{L}}_E T)$. Now $\mathcal{V} := \{Y \in \mathcal{D}(E) \mid \nu(Y) \text{ is an isomorphism}\}$ is a localizing subcategory of $\mathcal{D}(E)$, since $- \otimes^{\mathbf{L}}_E T$ preserves coproducts and T is compact in $\mathcal{D}(A)$. Since $E \in \mathcal{V}$ and since E is a generator in $\mathcal{D}(E)$, we finally have $\mathcal{V} = \mathcal{D}(E)$, hence ν is an isomorphism. □

Theorem 10.11 (Keller)**.** *Let A, B be dg R-algebras. Assume that $H(B) = H_0(B)$, that is, the homology of B is concentrated in degree 0. Then the following conditions are equivalent:*

(1) *There is a triangle equivalence; $\mathcal{D}(A) \simeq \mathcal{D}(B)$.*
(2) *There is a compact generator T in $\mathcal{D}(A)$ satisfying;*
   (*i*) $\mathrm{Hom}_{\mathcal{D}(A)}(T, T[n]) = 0$ *if* $n \neq 0$.
   (*ii*) $\mathrm{Hom}_{\mathcal{D}(A)}(T, T) \cong H_0(B)$.

Proof. (2) $\Rightarrow$ (1): Under the assumption of (2), set $E := \mathrm{End}_A(T)$. Then we have $H(E) = H_0(E) \cong H_0(B)$ by (i) and (ii). It follows from Theorem 10.10 that $\mathcal{D}(E) \simeq \mathcal{D}(A)$. Now combine this with the equivalence in Theorem 10.5 to get $\mathcal{D}(A) \simeq \mathcal{D}(B)$.

(1) $\Rightarrow$ (2): Let $F : \mathcal{D}(B) \to \mathcal{D}(A)$ be a functor that gives a triangle equivalence of triangulated categories. Set $T = F(B)$. Since $B \in \mathcal{D}(B)$ is a compact generator, so is $T \in \mathcal{D}(A)$. Since F is an equivalence, we have an isomorphism of R-modules; $\mathrm{Hom}_{\mathcal{D}(A)}(T, T[n]) \cong \mathrm{Hom}_{\mathcal{D}(B)}(B, B[n]) = H_n(B)$, which vanishes if $n \neq 0$ by the assumption. $\square$

The following beautiful result is a direct consequence of Theorem 10.5.

Corollary 10.12 (Rickard)**.** *Let S, S' be R-algebras that we consider dg R-algebras concentrated in degree* 0*. Then the following conditions are equivalent:*

(1) *There is a category equivalence; $\mathcal{D}(S) \simeq \mathcal{D}(S')$.*
(2) *There is a compact object T in $\mathcal{D}(S)$ satisfying;*
  (i) $\mathrm{Ext}^i_S(T, T) = 0$ *if* $i \neq 0$.
  (ii) $\mathrm{End}_S(T) \cong S'$.

Recall from Theorem 7.15 that a compact object T of $\mathcal{D}(S)$ is a direct summand of perfect dg S-module in $\mathcal{D}(S)$, and a perfect dg module is a finite free complex over S of finite length. Therefore T is a complex of finite length consisting of finite projective S-modules. (In other words T has finite projective dimension.) Such a complex T satisfying conditions (i) and (ii) in Corollary 10.12 is called a *tilting* complex.

References

1. L. L. Avramov, *Infinite free resolutions*, Six lectures on commutative algebra (Bellaterra, 1996), Progr. Math., vol. 166, Birkhäuser, Basel, 1998, pp. 1–118.
2. L. L. Avramov, H.-B. Foxby, and S. Halperin, *Differential graded homological algebra*, in preparation.
3. L. L. Avramov and S. Halperin, *Through the looking glass: a dictionary between rational homotopy theory and local algebra*, Algebra, algebraic topology and their interactions (Stockholm, 1983), 1–27, Lecture Notes in Math., 1183, Springer, Berlin, 1986.
4. M. Bökstedt and A. Neeman, *Homotopy limits in triangulated categories*, Compositio Math. **86** (1993), no. 2, 209–234.
5. Y. Félix, S. Halperin, and J.-C. Thomas, *Rational homotopy theory*, Graduate Texts in Mathematics, vol. 205, Springer-Verlag, New York, 2001.
6. Tor H. Gulliksen and G. Levin, *Homology of local rings*, Queen's Paper in Pure and Applied Mathematics, No. 20 (1969), Queen's University, Kingston, Ontario, Canada.
7. B. Keller, *Deriving DG categories*, Ann. Sci. École Norm. Sup. (4) **27** (1994), no. 1, 63–102.
8. ———, *Derived categories and tilting*, Handbook of tilting theory, 49–104, London Math. Soc. Lecture Note Ser., 332, Cambridge Univ. Press, Cambridge, 2007.
9. M. Kontsevich and Y. Soibelman, Notes on A^∞-algebras, A^∞-categories and non-commutative geometry. Homological mirror symmetry, 153–219, Lecture Notes in Phys., 757, Springer, Berlin, 2009.
10. J. Rickard, *Morita theory for derived categories*, J. London Math. Soc. (2) **39** (1989), no. 3, 436–456.
11. ———, *Derived equivalences as derived functors*, J. London Math. Soc. (2) **43** (1991), no. 1, 37–48.
12. The Stacks Project, Part 1, Chapter 22, *Differential Graded Algebras*, https://stacks.math.columbia.edu/browse https://stacks.math.columbia.edu/download/dga.pdf

13. J. Tate, *Homology of Noetherian rings and local rings*, Illinois J. Math. **1** (1957), 14–27.
14. B. Toën and G. Vezzosi, Homotopical algebraic geometry. II. Geometric stacks and applications. Mem. Amer. Math. Soc. 193 (2008), no. 902, x+224 pp.
15. A. Yekutieli, *Derived categories*, Cambridge Studies in Advanced Mathematics, Series Number **183**, Cambridge University Press (2019).