METHODS OF GLUING HYPERGRAPHS WITH NICE TOPOLOGICAL AND COMBINATORIAL PROPERTIES

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ABSTRACT. Given an arbitrary hypergraph $\mathcal{H}$, we may glue to $\mathcal{H}$ a family of hypergraphs to get a new hypergraph $\mathcal{H}'$ having $\mathcal{H}$ as an induced subhypergraph. In this paper, we introduce three gluing techniques for which the topological and combinatorial properties (such as Cohen-Macaulayness, shellability, vertex-decomposability etc.) of the resulting hypergraph $\mathcal{H}'$ is under control in terms of the glued components. This enables us to construct broad classes of simplicial complexes containing a given simplicial complex as induced subcomplex satisfying nice topological and combinatorial properties.

INTRODUCTION

A *simplicial complex* Δ on a vertex set V is a collection of subsets of V such that $\cup\Delta = V$ and Δ is closed under the operation of taking subsets. The elements of Δ are called *faces* and the maximal faces of Δ, under inclusion, are called the *facets* of Δ. A simplicial complex with facets $F_1, \ldots, F_m$ is often denoted by $\langle F_1, \ldots, F_m \rangle$. A *simplex* is a simplicial complex with only one facet.

A simplicial complex Δ is called *shellable* if there is a total order on facets of Δ, say $F_1, \ldots, F_m$, such that $\langle F_1, \ldots, F_{i-1} \rangle \cap \langle F_i \rangle$ is generated by a non-empty set of maximal proper subsets of F_i for $2 \leq i \leq m$. The notion of shellability is used to give (an inductive) proof for the Euler-Poincaré formula in any dimension. If f_i denotes the number of i-faces of a d-dimensional polytope (with $f_{-1} = f_d = 1$), then the Euler-Poincaré formula states that $\sum_{i=-1}^{d}(-1)^i f_i = 1$. Shellable complexes are themselves an intermediate family among two other important families of simplicial complexes, namely vertex-decomposable and sequentially Cohen-Macaulay simplicial complexes. Indeed, we have the following implications

$$\text{vertex-decomposable} \Longrightarrow \text{shellable} \Longrightarrow \text{sequentially Cohen-Macaulay},$$

and both of these inclusions are known to be strict.

A vertex-decomposable simplicial complex Δ is defined recursively in terms of link and deletion of vertices of Δ. In a more general setting, the *link* and the *deletion* of a face F of Δ are defined as follows:

$$\begin{aligned} \operatorname{link}_\Delta(F) &= \{G \in \Delta\colon\ G \cap F = \varnothing \text{ and } G \cup F \in \Delta\}, \\ \Delta \setminus F &= \{G \in \Delta\colon\ G \cap F = \varnothing\}. \end{aligned}$$

In view of the above settings, Δ is *vertex-decomposable* if either it is a simplex or else there exists a vertex $v \in V$ such that

(i) any facet of $\Delta \setminus v$ is a facet of Δ;
(ii) both complexes $\operatorname{link}_\Delta(v)$ and $\Delta \setminus v$ are vertex-decomposable.

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Sequentially Cohen-Macaulay complexes are defined slightly different. Let Δ be a simplicial complex on $[n]$, where $[n] = \{1, \dots, n\}$. The *pure i-skeleton* of Δ is the simplicial complex $\Delta^{[i]} = \langle F \in \Delta\colon\ |F| = i+1 \rangle$. A simplicial complex Δ is *Cohen-Macaulay* over $\mathbb{K}$ if the *Stanley-Reisner ring* $\mathbb{K}[\Delta] := S/I_\Delta$ is a Cohen-Macaulay ring, where $S = \mathbb{K}[x_1, \dots, x_n]$ is the polynomial ring with coefficients in $\mathbb{K}$ and $I_\Delta = \langle \prod_{i \in F} x_i\colon\ F \notin \Delta \rangle$. It turns out that Δ is Cohen-Macaulay if and only if $\tilde{H}_i(\mathrm{link}_\Delta(F), \mathbb{K}) = 0$, for all $F \in \Delta$ and $i < \dim \mathrm{link}_\Delta(F)$ (Reisner's Theorem, see e.g. [15, Corollary 4.2]). Consequently, as stated in [15, Proposition 4.3], Cohen-Macaulayness is a topological property in the sense that Δ is Cohen-Macaulay if and only if the relative singular homologies $H_i(\|\Delta\|, \|\Delta\| - p, \mathbb{K})$ of the geometric realization $\|\Delta\|$ of Δ vanish for all $i < \dim \|\Delta\|$ and $p \in \|\Delta\|$. Note that Cohen-Macaulay complexes are *pure* in the sense that all of their facets have the same cardinality (see [2, Corollary 5.1.5]). Accordingly, Δ is *sequentially Cohen-Macaulay* if every pure i-skeleton of Δ is Cohen-Macaulay, which is equivalent to say that $\mathbb{K}[\Delta]$ is a sequentially Cohen-Macaulay ring (see [7, Theorem 3.3]). Recall that a (graded) S-module M is *sequentially Cohen-Macaulay* if there exists a filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M \tag{1}$$

of (graded) submodules of M such that each quotient M_i/M_{i-1} is Cohen-Macaulay and

$$\dim M_1/M_0 < \dim M_2/M_1 < \cdots < \dim M_r/M_{r-1},$$

where $\dim N$ denotes the Krull dimension of S-module N.

A *hypergraph* $\mathcal{H}$ is simply a pair (V, E) of vertices V and edges $E \subseteq 2^V$. The *independence complex* $\Delta_\mathcal{H}$ of $\mathcal{H}$ is the simplicial complex of all independent sets in $\mathcal{H}$. Clearly, every simplicial complex is the independence complex of a hypergraph. One say that $\mathcal{H}$ is (sequentially) Cohen-Macaulay/shellable/vertex-decomposable/pure if $\Delta_\mathcal{H}$ is so. In this paper, we consider a hypergraph $\mathcal{H}'$ obtained by gluing some hypergraphs to a central "arbitrary" hypergraph $\mathcal{H}$ and study the topological and combinatorial properties (such as Cohen-Macaulayness, shellability, vertex-decomposability etc.) of $\mathcal{H}'$. In this regard, Villarreal [17, Proposition 2.2] proves that the graph obtained from a graph G by adding a pendant (also known as whisker) to each vertex is Cohen-Macaulay. Next Villarreal [18, Proposition 5.4.10] improves his result by showing that such graphs are pure and shellable. Later Dochtermann and Engström [6, Theorem 4.4] prove that such graphs are indeed pure and vertex-decomposable. Replacing pendants with complete graphs in the Villarreal's construction, Hibi et al. [11, Theorem 1.1] show that the resulting graph is still pure and vertex-decomposable (see also [3]). The idea of making small modifications to a graph in order to obtain a (sequentially) Cohen-Macaulay/shellable/vertex-decomposable graph is further explored in other papers too (see [1, 6, 9, 13, 14]). Our results unify all these results not only in the case of graphs but also hypergraphs. In algebraic setting our results turns into constructive and generic approaches to the following problem:

> Let $J \subseteq S = \mathbb{K}[x_1, \dots, x_n]$ be a square-free monomial ideal and $I \subseteq S' = \mathbb{K}[x_1, \dots, x_n, y_1, \dots, y_m]$ be a square-free monomial ideal such that $I \cap S = J$. Under which conditions on J and I, the ring S'/I is (sequentially) Cohen-Macaulay?

This paper is organized as follows: In the first section, we quickly review some algebraic and combinatorial backgrounds, which will be used in the sequel. Hybrid hypergraphs are introduced in Section 2. These hypergraphs are constructed by gluing a family of hypergraphs to a central one via a family of triples, which are assumed to satisfy the proper independence property (see the definition of PIP-triples). The main theorem of this section establishes some combinatorial/topological properties of hybrid hypergraphs and determines under which conditions a hybrid hypergraph is sequentially Cohen-Macaulay/shellable/vertex-decomposable (Theorem 2.1). It is shown that all of the results mentioned above are consequences of our main theorem of Section 2 (see Example 2). The idea behind the proof of Theorem 2.1 is very flexible and can be applied to other suitably constructed families of hypergraphs. In Section 3, we present two such families of hypergraph constructions and show that the results of Theorem 2.1 can be extended to these families too (Theorems 3.1 and 3.3). While the gluing methods here are less general than the hybrid case, the glued components need not to satisfy the proper independence property.

1. Preliminaries

In this section, we recall basic notions of simplicial complexes, hypergraphs, and their associated ideals, which we meet in this paper.

1.1. **Hypergraphs, clutters, and their associated ideals.** Let $\mathcal{H}$ be a hypergraph with vertex set $V = V(\mathcal{H})$ and edge set $E = E(\mathcal{H})$. Following [4], there are two ways to remove a vertex v from $\mathcal{H}$. The *strong vertex deletion* $\mathcal{H} \setminus v$ is the hypergraph with vertex set $V(\mathcal{H}) \setminus \{v\}$ and edge set $\{e \in E(\mathcal{H})\colon\ v \notin e\}$. The *weak vertex deletion* $\mathcal{H}/v$ has the same vertex set as $\mathcal{H} \setminus v$ but the edge set is $\{e \setminus \{v\}\colon\ e \in E(\mathcal{H})\}$. One observe that $\mathcal{H} \setminus v$ deletes all edges containing v, while $\mathcal{H}/v$ removes v from each edge containing it. It is straightforward to see that, if $v \neq w$ are vertices of $\mathcal{H}$, then:

$$(\mathcal{H} \setminus v) \setminus w = (\mathcal{H} \setminus w) \setminus v, \quad (\mathcal{H}/v)/w = (\mathcal{H}/w)/v, \quad (\mathcal{H} \setminus v)/w = (\mathcal{H}/w) \setminus v.$$

Let W be a set of vertices of $\mathcal{H}$ and $f \in \{0,1\}^W$ be a binary function on W. A (W, f)-*deletion* of $\mathcal{H}$ is a hypergraph obtained from $\mathcal{H}$ by repeatedly strongly deleting all vertices w of W with $f(w) = 0$ and weakly deleting all vertices w of W with $f(w) = 1$. The number of weak vertex deletions in a (W, f)-deletion $\mathcal{H}'$ of $\mathcal{H}$ is denoted by $\mathrm{wd}_f(\mathcal{H}')$ that is $\mathrm{wd}_f(\mathcal{H}') = \sum_{w\in W} f(w)$.

If W is a set of vertices of $\mathcal{H}$, then the subhypergraph $\mathcal{H}[W]$ of $\mathcal{H}$ *induced* on W is the subhypergraph of $\mathcal{H}$ with vertex set W and edge set $\{e \in E(\mathcal{H})\colon\ e \subseteq W\}$.

For a non-empty hypergraph $\mathcal{H}$ on vertex set $[n]$, we define the ideal $I(\mathcal{H})$ to be

$$I(\mathcal{H}) = (\mathbf{x}_T\colon \quad T \in E(\mathcal{H}))\,,$$

and we set $I(\varnothing) = 0$. The ideal $I(\mathcal{H})$ is called the *edge ideal* of $\mathcal{H}$. Let $\Delta_{\mathcal{H}}$ be the simplicial complex on the vertex set $[n]$ with $I_{\Delta_{\mathcal{H}}} = I(\mathcal{H})$. The simplicial complex $\Delta_{\mathcal{H}}$ is called the *independence complex* of $\mathcal{H}$. Notice that $F \subseteq [n]$ belongs to $\Delta_{\mathcal{H}}$ if and only if it is an *independent set* in $\mathcal{H}$, that is $e \nsubseteq F$ for every $e \in E(\mathcal{H})$. The *independence number* $\alpha(\mathcal{H})$ of $\mathcal{H}$ is the maximum size of independent sets of $\mathcal{H}$ or equivalently $\dim \Delta_{\mathcal{H}} + 1$.

A *clutter* $\mathcal{C}$ with vertex set X is an antichain of 2^X such that $X = \cup\mathcal{C}$. The elements of $\mathcal{C}$ are called *circuits* of $\mathcal{C}$. A clutter $\mathcal{C}$ is d-*uniform* if every circuit of $\mathcal{C}$ has d vertices. As a hypergraph, to every clutter $\mathcal{C}$ one corresponds its ideal $I(\mathcal{C})$. This correspondence is clearly bijective, the fact that is not valid for hypergraphs in general.

If $\mathcal{C}$ is a d-uniform clutter on $[n]$, then we define the *complement* $\bar{\mathcal{C}}$ of $\mathcal{C}$ as

$$\bar{\mathcal{C}} = \{F \subseteq [n]\colon \quad |F| = d,\ F \notin \mathcal{C}\}.$$

In this case, the simplicial complex $\Delta(\mathcal{C})$ on the vertex set $[n]$ with $I_{\Delta(\mathcal{C})} = I\left(\bar{\mathcal{C}}\right)$ is called the *clique complex* of $\mathcal{C}$. A face $F \in \Delta(\mathcal{C})$ is called a *clique* in $\mathcal{C}$. It is easily seen that $F \subseteq [n]$ is a clique in $\mathcal{C}$ if and only if either $|F| < d$ or else all d-subsets of F belong to $\mathcal{C}$.

1.2. **Criteria for (sequentially) Cohen-Macaulayness and shellability.** The recursive definition of vertex-decomposability states that a simplicial complex Δ is vertex-decomposable if Δ admits a shedding vertex v such that $\operatorname{link}_\Delta(v)$ and $\Delta \setminus v$ are both vertex-decomposable provided that Δ is not a simplex. In the following, among other results, we show that analogous arguments work for (sequentially) Cohen-Macaulayness and shellability as well. Indeed, in the proof of our main theorems, we do not use the formal definitions of shellable or (sequentially) Cohen-Macaulay complexes as it is introduced in the introduction. Instead, the following theorem plays a crucial role in our arguments.

Theorem 1.1. *Let Δ and Δ' be simplicial complexes.*
- (i) *If Δ is (sequentially) Cohen-Macaulay/shellable/vertex-decomposable, then so is $\operatorname{link}_\Delta(F)$, for every face F of Δ.*
- (ii) *If Δ has a shedding vertex v such that both $\operatorname{link}_\Delta(v)$ and $\Delta \setminus v$ are (sequentially) Cohen-Macaulay/shellable, then so is Δ.*
- (iii) *$\Delta \star \Delta'$ is (sequentially) Cohen-Macaulay/shellable/vertex-decomposable if and only if both Δ and Δ' are so.*

In the rest of paper, we introduce three hypergraph constructions by gluing a family of hypergraphs to a given central hypergraph and examine when the resulting hypergraphs satisfy our desired properties, namely (sequentially) Cohen-Macaulayness, shellability, and vertex-decomposability.

2. First construction

In this section we introduce our first (and main) hypergraph gluing. Under mild assumptions, i.e. the PIP-condition, we may control the topological and combinatorial properties of the resulting hypergraphs in terms of the glued components.

Definition 1. Let $\mathcal{H}$ be a hypergraph with vertex partition $U_1\dot\cup\cdots\dot\cup U_m\dot\cup V$, and $\mathcal{H}_1,\ldots,\mathcal{H}_m$ be hypergraphs such that $\mathcal{H},\mathcal{H}_1,\ldots,\mathcal{H}_m$ are pairwise disjoint. Let $D_1,\ldots,D_m$ be sets of non-negative integers. The hypergraph with vertex set $V(\mathcal{H})\cup V(\mathcal{H}_1)\cup\cdots\cup V(\mathcal{H}_m)$ and edge set

$$E(\mathcal{H}) \cup \bigcup_{i=1}^{m} \{e\cup e'\colon \quad e\subseteq U_i,\ \varnothing\neq e'\in E(\mathcal{H}_i),\ |e\cup e'|\in D_i\}, \tag{2}$$

denoted by $(\mathcal{H},(U_i,D_i,\mathcal{H}_i)_{i=1}^m)$, is called the *hybrid hypergraph* of $\mathcal{H}$ with respect to the gluing triples $(U_i,D_i,\mathcal{H}_i)_{i=1}^m$. The hypergraphs $\mathcal{H}_1,\ldots,\mathcal{H}_m$ are the *glued components* of $(\mathcal{H},(U_i,D_i,\mathcal{H}_i)_{i=1}^m)$.

Remark. From the definition of hybrid hypergraphs, it is evident that $\mathcal{H}$ is an induced subhypergraph of $(\mathcal{H},(U_i,D_i,\mathcal{H}_i)_{i=1}^m)$. As a result, every independent set in $\mathcal{H}$ is an independent set in $(\mathcal{H},(U_i,D_i,\mathcal{H}_i)_{i=1}^m)$ as well.

The independence complex of hybrid hypergraphs and their facets look wild if there is no constraints on the gluing triples $(U_i,D_i,\mathcal{H}_i)$. To resolve this, we apply the PIP-condition (see definition below) on gluing triples in order to describe the independence complex of hybrid hypergraphs. In what follows, we consider the *Minkowski difference* $X-Y$ of two sets X,Y of integers as the set $\{x-y\colon\ x\in X,\ y\in Y\}$. Also, for a hypergraph $\mathcal{H}$, let $\mathcal{H}^X$ stand for the spanning subhypergraph of $\mathcal{H}$ including all edges of sizes belonging to

X, and $\text{Size}(\mathcal{H})$ denote the set $\{|e|\colon\ e \in E(\mathcal{H})\}$. If a, b are integers with $a \leq b$, then the set of all integers x with $a \leq x \leq b$ is denoted by the interval $[a, b]$.

Definition 2. Let $\mathcal{H}$ be a hypergraph, D be a set of non-negative integers, and α be a positive integer. The triple $(\alpha, D, \mathcal{H})$ satisfies the *proper independence property* (PIP) if

(a) every G in $\mathcal{F}(\Delta_{\mathcal{H}^{D-[0,i+1]}})$ is contained properly in some G' in $\mathcal{F}(\Delta_{\mathcal{H}^{D-[0,i]}})$,
(b) every G' in $\mathcal{F}(\Delta_{\mathcal{H}^{D-[0,i]}})$ contains properly some G in $\mathcal{F}(\Delta_{\mathcal{H}^{D-[0,i+1]}})$,

for all $0 \leq i < \alpha$. It turns out that $[\alpha] \subseteq D - \text{Size}(\mathcal{H})$. More precisely, every $i \in [\alpha]$ belongs to $D - (\text{Size}(\mathcal{H}) \setminus (D - [0, i-1]))$.

If $(\alpha, D, \mathcal{H})$ is any triple, then in general we have the following series of simplicial complexes

$$\Delta_{\mathcal{H}^{D-[0,\alpha]}} \subseteq \Delta_{\mathcal{H}^{D-[0,\alpha-1]}} \subseteq \cdots \subseteq \Delta_{\mathcal{H}^{D-[0,1]}} \subseteq \Delta_{\mathcal{H}^{D-[0,0]}}$$

showing that every facet G of $\Delta_{\mathcal{H}^{D-[0,i+1]}}$ is contained in a facet G' of $\Delta_{\mathcal{H}^{D-[0,i]}}$ for all $0 \leq i < \alpha$. Being a PIP-triple indicates that not only every facet G of $\Delta_{\mathcal{H}^{D-[0,i+1]}}$ is contained "properly" in a facet G' of $\Delta_{\mathcal{H}^{D-[0,i]}}$ for all $0 \leq i < \alpha$ but also every facet G' of $\Delta_{\mathcal{H}^{D-[0,i]}}$ contains a facet G of $\Delta_{\mathcal{H}^{D-[0,i+1]}}$ properly for all $0 \leq i < \alpha$.

Example 1.

(i) Let $\mathcal{C}$ be a d-uniform clutter, $D = \{d\}$, and $0 < \alpha \leq d$. If $\mathcal{H}$ is the hypergraph induced by the edge-set $\mathcal{C} \cup (\langle V(\mathcal{C})\rangle^{[d-2]} \setminus X)$, where $X \subseteq \langle V(\mathcal{C})\rangle^{[d-\alpha-2]}$, then

$$\mathcal{C}'^{D-[0,i]} = \mathcal{C} \cup (\langle V(\mathcal{C})\rangle^{[d-2]} \setminus \langle V(\mathcal{C})\rangle^{[d-i-2]})$$

for all $0 \leq i \leq \alpha$. It follows that $\Delta_{\mathcal{C}'^{D-[0,i]}} = \langle V(\mathcal{C})\rangle^{[d-i-2]}$, for all $1 \leq i \leq \alpha$. Hence $(\alpha, D, \mathcal{C}')$ is a PIP-triple satisfying $\mathcal{C}'^{D} = \mathcal{C}^{D}$.

(ii) Let $\mathcal{S}$ be a simplicial complex of dimension $d-1$, $D = \{d\}$, and $0 < \alpha < d$. Then the triple $(\alpha, D, \mathcal{S})$ satisfies PIP(a) but not PIP(b) in general. Indeed, if $\mathcal{S}$ is a simplicial complex and G is any independent set in $\mathcal{S}^{D-[0,i+1]}$ with $i < \alpha$, then any $G' \supset G$ with $|G' \setminus G| = 1$ is an independent set in $\mathcal{S}^{D-[0,i]}$. On the other hand, the triple $(\alpha, \{3\}, \mathcal{S})$, where $\mathcal{S}$ is the simplicial complex $\langle 124, 134, 234, 235, 136, 127\rangle \cup \langle 45, 46, 47, 56, 57, 67\rangle$ and $\alpha > 0$ does not satisfy PIP(b). To see this, we observe that the facet 123 of $\Delta_{\mathcal{S}^{D-[0,0]}}$ does not contain any of the facets of $\Delta_{\mathcal{S}^{D-[0,1]}} = \langle 15, 26, 37\rangle$.

(iii) The triple $(\alpha, \{3\}, \mathcal{H})$, where $E(\mathcal{H}) = \{1, 2, 3, 4, 12, 24, 34, 123\}$ and $0 < \alpha < 3$ satisfies PIP while it is not a simplicial complex.

In order to state our main result of this section, we need some preparations and preliminary lemmas.

Definition 3. Let $\mathcal{H}$ be a hypergraph. A set D of vertices of $\mathcal{H}$ is a *strong dominating set* in $\mathcal{H}$ if $\alpha(\mathcal{H}/D) = 0$ or equivalently every singleton subset of $V(\mathcal{H}/D)$ is an edge of $\mathcal{H}/D$. Here by $\mathcal{H}/D$ we mean the hypergraph whose edge set is

$$\{e \setminus D\colon\ e \in E(\mathcal{H})\}.$$

According to the above settings, we are in the position to state and prove our results on the structure and combinatorial/topological properties of hybrid hypergraphs.

Theorem 2.1. *Let $\mathcal{H}' = (\mathcal{H}, (U_i, D_i, \mathcal{H}_i)_{i=1}^{m})$ be a hybrid hypergraph of $\mathcal{H}$, where $\mathcal{H}$ is a hypergraph with vertex partition $U_1 \dot\cup \cdots \dot\cup U_m \dot\cup V$. If $(\alpha(\mathcal{H}[U_i]), D_i, \mathcal{H}_i)_{i=1}^{m}$ is a family of PIP-triples, then*

(i) $\dim \Delta_{\mathcal{H}'} = \sum_{i=1}^m \dim \Delta_{\mathcal{H}_i^{D_i}} + \dim \Delta_{\mathcal{H}[V]} + m$,

(ii) $\Delta_{\mathcal{H}'}$ *is pure if and only if* $\Delta_{\mathcal{H}[V]}$ *is pure, and* $\Delta_{\mathcal{H}_i^{D_i-[0,s]}}$ *is pure and*

$$\dim \Delta_{\mathcal{H}_i^{D_i-[0,s]}} - \dim \Delta_{\mathcal{H}_i^{D_i-[0,t]}} = t - s,$$

for all $1 \leq i \leq m$ *and* $0 \leq s \leq t \leq \alpha(\mathcal{H}[U_i])$,

(iii) *Let* W_i *be a strong dominating independent set in* $\mathcal{H}[U_i]$, *for every* $1 \leq i \leq m$. *Then* $\mathcal{H}'$ *is sequentially Cohen-Macaulay/shellable/vertex-decomposable if and only if*

$$\mathcal{H}[V], \quad \mathcal{H}_i^{D_i-[0,\alpha(\mathcal{U}_i)+\mathrm{wd}_{f_i}(\mathcal{U}_i)]}$$

are so for all $1 \leq i \leq m$ *and* (W_i, f_i)*-deletions* $\mathcal{U}_i$ *of* $\mathcal{H}[U_i]$.

Corollary 2.2. *Let* $\mathcal{H}$ *be a hypergraph with vertex partition* $U_1\dot\cup\cdots\dot\cup U_m$ *and* $d_i = \alpha(\mathcal{H}[U_i])$, *for* $i = 1, \ldots, m$. *Let* $\mathcal{C}_i$ *be a* d_i*-uniform clutter and* $\mathcal{H}_i$ *be the hypergraph induced by the edge-set* $\mathcal{C}_i \cup \langle V(\mathcal{C}_i)\rangle^{[d_i-2]}$ *for* $i = 1, \ldots, m$. *Let* $\mathcal{H}'$ *be the hybrid hypergraph* $(\mathcal{H}, (U_i, \{d_i\}, \mathcal{H}_i)_{i=1}^m)$. *Then*

(i) $\dim \Delta_{\mathcal{H}'} = \sum_{i=1}^m \dim \Delta_{\mathcal{C}_i} + m - 1$,

(ii) $\Delta_{\mathcal{H}'}$ *is pure if and only if* $\mathcal{C}_i = \binom{V(\mathcal{C}_i)}{d_i}$ *is complete* d_i*-clutter, for all* $i = 1, \ldots, m$,

(iii) $\mathcal{H}'$ *is vertex-decomposable if and only if* $\mathcal{C}_i$ *is vertex-decomposable for all* $1 \leq i \leq m$ *such that* $\mathcal{H}[U_i]$ *has a unique maximal independent set,*

Corollary 2.3. *Let* $\mathcal{C}$ *be a* d*-uniform clutter,* $U_1, \ldots, U_m$ *be a clique partition of* $\mathcal{C} \setminus V$ *for some subset* V *of* $V(\mathcal{C})$, *and let* $\mathcal{H}_1, \ldots, \mathcal{H}_m$ *be hypergraphs such that the triples* $(\alpha(\mathcal{C}[U_i]), \{d\}, \mathcal{H}_i)$ *satisfy the PIP-conditions, for* $i = 1, \ldots, m$. *Let* $\mathcal{C}'$ *be the* d*-uniform clutter defined as*

$$\mathcal{C}' = \mathcal{C} \cup \bigcup_{i=1}^m \{e \subseteq U_i \cup V(\mathcal{H}_i)\colon \quad |e| = d \text{ and } e \cap V(\mathcal{H}_i) \in E(\mathcal{H}_i)\}.$$

Then

(i) $\dim \Delta_{\mathcal{C}'} = \sum_{i=1}^m \dim \Delta_{\mathcal{H}_i^{\{d\}}} + \dim \Delta_{\mathcal{C}[V]} + m$,

(ii) $\Delta_{\mathcal{C}'}$ *is pure if and only if* $\Delta_{\mathcal{C}[V]}$ *is pure, and* $\Delta_{\mathcal{H}_i^{[s,d]}}$ *is pure and*

$$\dim \Delta_{\mathcal{H}_i^{[t,d]}} - \dim \Delta_{\mathcal{H}_i^{[s,d]}} = t - s,$$

for all $i = 1, \ldots, m$ *and* $\max\{d - |U_i|, 1\} \leq s \leq t \leq d$,

(iii) $\mathcal{C}'$ *is sequentially Cohen-Macaulay/shellable/vertex-decomposable if and only if* $\mathcal{C}[V]$ *and* $\mathcal{H}_i^{[d-j,d]}$ *are so for all* $j \in J$, *where* $J = \{\beta, \ldots, \alpha(\mathcal{C}[U_i])\}$ *with*

$$\beta = \begin{cases} 0, & |U_i| < d, \\ \min\{|U_i| - (d-1), d-1\}, & |U_i| \geq d. \end{cases}$$

Corollary 2.4. *Let* $\mathcal{C}$ *be a* d*-uniform clutter,* $U_1, \ldots, U_m$ *be a clique partition of* $\mathcal{C}$, *and let* $\mathcal{H}_1, \ldots, \mathcal{H}_m$ *be disjoint simplexes of dimensions at least* $d-2$. *If*

$$\mathcal{C}' = \mathcal{C} \cup \bigcup_{i=1}^m \binom{U_i \cup V(\mathcal{H}_i)}{d},$$

then

(i) $\Delta_{\mathcal{C}'}$ *is pure vertex-decomposable of dimension* $(d-1)m - 1$, *and*

(ii) *the ring* $\mathbb{K}[V(\mathcal{C}')]/I(\mathcal{C}')$ *is Cohen-Macaulay of dimension* $(d-1)m$.

Theorem 2.1 and its related corollaries establish alternate proofs for previously known results we address here.

Example 2.

(i) Let G be a graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$. The corona graph G' of G is a graph obtained from G by attaching a pendant to each vertex of G, that is G' is the graph with vertex set $V(G) \cup \{w_1, \ldots, w_n\}$ and edge set $E(G) \cup \{v_i w_i \colon 1 \leq i \leq n\}$. It is shown by Villarreal [17, Proposition 2.2] that the graph G' is Cohen-Macaulay. Villarreal [18, Proposition 5.4.10] improves his result by showing that G' is pure and shellable. Later Dochtermann and Engström [6, Theorem 4.4] prove that G' is indeed pure and vertex-decomposable. Hibi, Higashitani, Kimura, and O'Keefe [11] and Cook II and Nagel [3] give two generalizations of Villarreal's construction. In [11, Theorem 1.1], Hibi et. al. show that the graph obtained from identification of every vertex v of G with a vertex of a complete graph G_v is still pure and vertex-decomposable. Cook II and Nagel [3, Theorem 3.3 and Corollary 3.5] apply a different generalization and show that Δ_{G^π} is vertex-decomposable and Cohen-Macaulay if G is a graph and G^π is the graph obtained from G with a clique partition $\pi = \{W_1, \ldots, W_t\}$ as follows: $V(G^\pi) = V(G) \cup \{w_1, \ldots, w_t\}$ for some distinct vertices $w_1, \ldots, w_t$ not in G, and $E(G^\pi) = E(G) \cup \{v w_i \colon v \in W_i\}$. All of these results and consequences thereafter are special cases of Corollary 2.4.

(ii) In [6, Proposition 4.3] the authors show that if G_r is the graph obtained from an r-cycle with attaching a new vertex to two adjacent vertices of the cycle, then $I(G_r)$ is vertex-decomposable and hence sequentially Cohen-Macaulay. This follows simply from Corollary 2.3.

(iii) Let G be a chordal graph that is G has no induced cycles of length greater than 3. In [10, Theorem 3.2], the authors show that Δ_G is sequentially Cohen-Macaulay. Later, in [16, Theorem 2.13] it is shown that Δ_G is indeed shellable. This result is also strengthened by Woodroofe [20, Corollary 7(2)] (and independently by Dochtermann and Engström [6, Theorem 4.1]) by showing that Δ_G is vertex-decomposable. We use our method to obtain the mentioned results. First observe that the chordal graph G has a vertex v with complete neighborhood U (see [5]). If $V := V(G) \backslash (U \cup \{v\})$, then $(G \backslash v, (U, \{2\}, \langle v \rangle)) = G$. Hence, an inductive argument in conjunction with Corollary 2.3 shows that Δ_G is vertex-decomposable.

(iv) Let Δ be a simplicial complex on the vertex set $V = \{v_1, \ldots, v_n\}$. Following [1], an m-coloring χ of Δ is a partition $V = V_1 \cup \cdots \cup V_m$ of the vertices (where the sets V_i are allowed to be empty) such that $|F \cap V_i| \leq 1$ for all $F \in \Delta$ and $1 \leq i \leq m$. For such a coloring χ of Δ, define the simplicial complex Δ_χ on the vertex set $\{v_1, \ldots, v_n, w_1, \ldots, w_m\}$ with faces $\sigma \cup \tau$ where $\sigma \in \Delta$ and τ is any subset of $\{w_i \colon \sigma \cap V_i = \varnothing\}$. In [1, Theorem 7], it is shown that Δ_χ is pure and vertex-decomposable. In the following we show that this result is an immediate consequence of Theorem 2.1. Let $\mathcal{C}$ and $\mathcal{C}'$ be the clutters with $\Delta = \Delta_{\mathcal{C}}$ and $\Delta_\chi = \Delta_{\mathcal{C}'}$. It is easy to see that $\mathcal{C}' = (\mathcal{C}, (V_i, \{2\}, \langle w_i \rangle)_{i=1}^m)$. Suppose without loss of generality that $V_1, \ldots, V_m$ are non-empty. It follows from the definition of m-coloring that $\alpha(\mathcal{C}[V_i]) = 1$, hence $(\alpha(\mathcal{C}[V_i]), \{2\}, \langle w_i \rangle)$ is a PIP-triple for all $1 \leq i \leq m$. Now from Theorem 2.1 we conclude that $\Delta_\chi = \Delta_{\mathcal{C}'}$ is pure and vertex-decomposable.

3. Second and third constructions

The aim of this section is to give yet two more gluing techniques leading us to similar results as in Theorem 2.1. Though our constructions here coincide with the first one in very special cases, they are independent from the first construction in general. In what follows, our gluing processes are applied to clutters instead of hypergraphs and that every component is glued via a single vertex to the central clutter. The main idea of our proofs obey from that in Theorem 2.1. The following definition is used in order to describe the purity of the resulting clutters.

Definition 4. Let $\mathcal{C}$ be a clutter and $\mathcal{U}$ be an induced subclutter of $\mathcal{C}$. Then $\mathcal{U}$ is *independently embedded* in $\mathcal{C}$ if $F \cup X \in \mathcal{F}(\Delta_{\mathcal{C}})$ with $F \subseteq V(\mathcal{U})$ and $X \subseteq V(\mathcal{C}) \setminus V(\mathcal{U})$ implies $F \in \mathcal{F}(\Delta_{\mathcal{U}})$.

Remark. Let $\mathcal{C}$ be a clutter and $v \in V(\mathcal{C})$. Viewing $\mathcal{C}$ as a hypergraph, the weakly deletion $\mathcal{C}/v$ is a hypergraph but not a clutter in general. However, the set $\min(\mathcal{C}/v)$ of all minimal elements of $\mathcal{C}/v$ under inclusion is a clutter whose edge ideal is the same as that of $\mathcal{C}/v$. This is usually referred to hypergraph reduction of $\mathcal{C}/v$.

Theorem 3.1. *Let $\mathcal{C}$ be a clutter with vertex set $U\dot{\cup}V$ and $\{\mathcal{C}_u\}_{u\in U}$ be a family of non-empty clutters such that $\mathcal{C}$ and $\{\mathcal{C}_u\}_{u\in U}$ are pairwise disjoint. Let $(\mathcal{C}, \{\mathcal{C}_u\}_{u\in U})$ be the clutter obtained from $\mathcal{C}$ as follows:*

$$(\mathcal{C}, \{\mathcal{C}_u\}_{u\in U}) = \mathcal{C} \cup \bigcup_{u\in U} \{e \cup \{u\} \colon \quad e \in \mathcal{C}_u\}.$$

If $\mathcal{C}' := (\mathcal{C}, \{\mathcal{C}_u\}_{u\in U})$ and $\Delta' := \Delta_{\mathcal{C}'}$, then

(i) $\dim \Delta' = \sum_{u\in U} |V(\mathcal{C}_u)| + \dim \Delta_{\mathcal{C}[V]}$,
(ii) *Δ' is pure if and only if $|\mathcal{C}_u| = 1$ for all $u \in U$, $\Delta_{\mathcal{C}[V]}$ is pure, and $\mathcal{C}[V]$ is independently embedded in $\mathcal{C}$,*
(iii) *$\mathcal{C}'$ is sequentially Cohen-Macaulay/shellable/vertex-decomposable if and only if $\mathcal{C}[V]$ and $\mathcal{C}_u$ are so for all $u \in U$,*
(iv) *$\mathcal{C}'$ is Cohen-Macaulay if and only if it is pure and $\mathcal{C}[V]$ is Cohen-Macaulay.*

Corollary 3.2 (Compare with [8, Theorem 8.2])**.** *Let $\mathcal{C}$ be a clutter on $[n]$ and $C_1, \ldots, C_n$ be non-empty sets such that $V(\mathcal{C}), C_1, \ldots, C_n$ are pairwise disjoint. Let*

$$\mathcal{C}' := \mathcal{C} \cup \{C_i \cup \{i\} \colon \ i \in [n]\}.$$

Then $\Delta_{\mathcal{C}'}$ is a pure and vertex-decomposable simplicial complex, hence Cohen-Macaulay.

One observe that the above corollary covers the results of Villarreal and Dochtermann-Engström in Example 2(i).

Theorem 3.3. *Let $\mathcal{C}$ be a clutter with vertex set $U\dot{\cup}V$ and $\{\mathcal{C}_u\}_{u\in U}$ be a family of non-empty clutters such that $\mathcal{C}$ and $\{\mathcal{C}_u\}_{u\in U}$ are pairwise disjoint. Let $(\mathcal{C}, \{\mathcal{C}_u\}_{u\in U})^*$ be the clutter obtained from $\mathcal{C}$ as follows:*

$$(\mathcal{C}, \{\mathcal{C}_u\}_{u\in U})^* = \mathcal{C} \cup \bigcup_{u\in U} \mathcal{C}_u \cup \bigcup_{u\in U} \{\{u\}\} \star \mathcal{C}_u^*,$$

where $\mathcal{C}_u^*$ is the set of minimal elements of the set $\{e \setminus x \colon \ x \in e, \ e \in \mathcal{C}_u\}$ with respect to inclusion, for all $u \in U$. If $\mathcal{C}' := (\mathcal{C}, \{\mathcal{C}_u\}_{u\in U})^*$ and $\Delta' := \Delta_{\mathcal{C}'}$, then

(i) $\dim \Delta' = |U| + \sum_{u\in U} |\Delta_{\mathcal{C}_u}| + \dim \Delta_{\mathcal{C}[V]}$,

(ii) *Δ' is pure if and only if $\Delta_{\mathcal{C}[V]}$ is pure, $\mathcal{C}[V]$ is independently embedded in $\mathcal{C}$, $\Delta_{\mathcal{C}_u}$ and $\Delta_{\mathcal{C}_u^*}$ are pure and $\dim \Delta_{\mathcal{C}_u^*} = \dim \Delta_{\mathcal{C}_u} - 1$ for all $u \in U$,*

(iii) *$\mathcal{C}'$ is sequentially Cohen-Macaulay/shellable/vertex-decomposable if and only if $\mathcal{C}[V]$, and $\mathcal{C}_u$ and $\mathcal{C}_u^*$ are so for all $u \in U$.*

Notice that the independence complex of a complete graph is vertex-decomposable. Now, in view of Theorem 3.3, we conclude that the graph G' obtained form a graph G by attaching a complete graph to each vertex of G is vertex-decomposable. This covers the results of Hibi et. al. in Example 2(i).

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