The Herwig–Lascar property of groups, ultraextensive structures and Vershik's conjecture

Mahmood Etedadi Aliabadi

University of North Texas

mahmood.etedadialiabadi@unt.edu

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Results in this talk are joint work with Su Gao, Franois Le Matre and Julien Melleray.

Overview

Herwig–Lascar extension theorem

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The main problem: extending the semigroup of partial automorphisms (partial symmetries) of a "nice" structure X to a group of automorphisms (full symmetries) of a "nice" extension of X, Y.

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One of the first results of this sort was proven by Hrushovski for finite (simple) graphs.

Theorem (Hrushovski, 1992)

Every finite (simple) graph Γ has an extension Γ' such that every partial automorphism of Γ can be extended to an automorphism of Γ'.

Herwig–Lascar extension theorem

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Definition

Let $\mathcal{C}$ be a class of $\mathcal{L}-$structures (containing both finite and infinite structures). $\mathcal{C}$ is said to have **the extension property for partial automorphisms (EPPA)** if whenever C_1 and C_2 are structures in $\mathcal{C}$, C_1 is finite, $C_1 \subseteq C_2$, and every partial automorphism of C_1 extends to an automorphism of C_2, then there exist a finite structure C_3 in $\mathcal{C}$ such that every partial automorphism of C_1 extends to an automorphism of C_3.

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If M is an $\mathcal{L}$-structure and $\mathcal{T}$ a set of $\mathcal{L}$-structures, we say that M is $\mathcal{T}$-**free** if there is no structure $T \in \mathcal{T}$ and homomorphism $h : T \to M$.

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Theorem (Herwig–Lascar, 1999)

Let $\mathcal{L}$ be a finite relational language and $\mathcal{T}$ a finite set of finite $\mathcal{L}$-structures. Then the class of all finite $\mathcal{T}$-free $\mathcal{L}$-structures has the EPPA.

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Theorem (Solecki, 2005)

Let X be a finite metric space. There exists a finite metric space Y such that X isometrically embeds into Y and every partial isometry of X extends to an (full) isometry of Y.

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That is, the class of metric spaces has the EPPA.

Alternative proofs presented by Pestov (2008) and Vershik (2007), independently. Also, Jan Hubička, Matěj Konečný and Jaroslav Nešetřil recently presented combinatorial proofs of Solecki and HL theorem.

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Let C be an $\mathcal{L}$-structure. An **HL-extension** of C is a pair (D, ϕ), where D is an extension of C, and $\phi : \mathcal{P}_C \to \mathrm{Aut}(D)$ such that $\phi(p)$ extends p for all $p \in \mathcal{P}_C$.

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HL theorem can be restated as: For a finite relational structure $\mathcal{L}$, if a finite $\mathcal{L}$-structure C has an HL-extension then it has a finite HL-extension.

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definition

Let C be an $\mathcal{L}$-structure and (D,ϕ) be an HL-extension of C. We say that (D,ϕ) is **minimal** if for all $b \in D \setminus C$ there are $p_1,\ldots,p_n \in \mathcal{P}_C$ and $a \in C$ such that $b = \phi(p_1)\ldots\phi(p_n)(a)$.

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Let X be a finite metric space. An **S-extension** of X is a pair (Y, ϕ), where $Y \supseteq X$ is an extension of X, and $\phi : \mathcal{P}_X \to \mathrm{Iso}(Y)$ such that $\phi(p)$ extends p for all $p \in \mathcal{P}_X$. The map ϕ is called an S-map for X.

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Solecki's theorem can be restated as: Every finite metric space has a finite S-extension.

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Note that for every finite $\mathcal{L}$-structure C there is a unique partition of C into substructures $\{C_i \ : \ i = 1, \dots, n\}$ such that each C_i is a maximal subset of C satisfying that for every $a, b \in C_i$, the map that sends a to b (that is, the map $\{(a, b)\}$) is a partial isomorphism of C.

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definition

A **natural factorization** of C is of the form $\{(C_i, a_i) : i = 1, \ldots, n\}$, where $\{C_i \ : \ i = 1, \ldots, n\}$ is as above and $a_i \in C_i$ for every i.

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By $g(a_i) = a_i$ we mean that if $g = p_1 \cdots p_m$ with $p_1, \ldots, p_m \in \mathcal{P}_C$, then $p_1(\cdots(p_m(a_i))\cdots)$ is defined and

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Given any $\gamma \in \mathbb{F}(\mathcal{P}_C)$, the map Φ_γ defined by $\Phi_\gamma(gH_i) = \gamma g H_i$ is an automorphism of Γ.

Lemma

(Γ, Φ) is an HL-extension of C with $\Phi : \mathcal{P}_C \to \mathrm{Aut}(\Gamma)$ defined as $\Phi(p) = \Phi_p$.

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Lemma

(Γ, Φ) is an HL-extension of C with $\Phi : \mathcal{P}_C \to \mathrm{Aut}(\Gamma)$ defined as $\Phi(p) = \Phi_p$. Note that by definition, (Γ, Φ) is a minimal HL-extension of C.

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Note that $\Gamma = \mathbb{F}(\mathcal{P}_X)/H$ is an S-extension of X.

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Theorem (E., Gao)

Let (Y, ϕ) be a finite minimal S-extension of X. Let $N = \ker(\phi)$ and $G = \Phi_N(\mathbb{F}(\mathcal{P}_X))$. Then there is a G-invariant pseudometric ρ on Γ_N which is consistent with w_N such that (Y, ϕ) is isomorphic to $(\overline{\Gamma_N}, \overline{\Phi_N})$.

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Let G be a group. We say that G has the **HL-property**if for every finitely generated $H_1, \ldots, H_n \leq G$ and left system of equations on $H_1, \ldots, H_n$ that does not have a solution,there exist normal subgroups of finite index $N_1, \ldots, N_n \trianglelefteq G$ such that the same left system of equations on $N_1 H_1, \ldots, N_n H_n$ does not have a solution.

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All groups with RZ-property are residually finite, that is, the profinite topology is Hausdorff.

HL-property

Definition

Let G be a group acting by isometries on a metric space (X, d_X). We say that the action is **finitely approximable** if for any finite $A \subseteq X$ and finite $F \subseteq G$ there is a finite metric space (Y, d_Y), on which G acts by isometries, and an isometry $\pi : A \to Y$ such that whenever $g \in F$ and $x, gx \in A$, then $\pi(gx) = g\pi(x)$.

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Theorem (Rosendal, 2011)

The following are equivalent for a countable discrete group G:

1. *G has the RZ-property;*
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An $\mathcal{L}$-structure C is called a **Gaifman clique** if for every $a, b \in C$ there is a relation symbol $R \in \mathcal{L}$ with arity $m \geq 2$ and $c_1, \ldots, c_m \in C$ with $a, b \in \{c_1, \ldots, c_m\}$ and $R^C(c_1, \ldots, c_m)$.

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(iii) *Clause* (ii) *with the additional assumption that every structure $T \in \mathcal{T}$ is a Gaifman clique.*

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Theorem (EGLMM)

Let Γ_1, Γ_2 be two finite groups, and Λ be a common subgroup of Γ_1, Γ_2. Then $\Gamma_1 *_\Lambda \Gamma_2$, the amalgamated free product over Λ, has the HL-property

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Coherent HL-extensions and ultraextensive structures

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Definition (Solecki)

Let X be a metric space. An S-extension (Y, ϕ) of X is **strongly coherent** if for every triple (p, q, r) of partial isometries of X such that $p \circ q = r$, we have $\phi(p) \circ \phi(q) = \phi(r)$.

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We introduce a weaker notion of coherence.

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Definition

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(iii) letting $K_i = \phi_i(\mathcal{P}_{X_i}) \subseteq \mathrm{Iso}(Y_i)$ for $i = 1, 2$, and letting $\kappa : K_1 \to K_2$ be the map $\kappa(\phi_1(p)) = \phi_2(p)$ for all $p \in \mathcal{P}_{X_1}$, then κ extends uniquely to a group embedding from $\langle K_1 \rangle$ into $\langle K_2 \rangle$.

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Although the existence of coherent S-extensions follows from existence of strongly coherent S-extensions, we present a direct construction of coherent S-extensions using the algebraic method.

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Theorem (E., Gao)

Suppose $X_1 \subseteq X_2$ are finite metric spaces and (Y_1, ϕ_1) is a finite minimal S-extension of X_1. Then there is a finite minimal S-extension (Y_2, ϕ_2) of X_2 so that (Y_2, ϕ_2) is coherent with (Y_1, ϕ_1).

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The fact that RZ-property is closed under free products is essential in the proof of the above theorem.

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The following are some examples of ultraextensive spaces: the Urysohn space $\mathbb{U}$,

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For every separable ultraextensive metric space U, $\mathrm{Iso}(U)$ contains a dense locally finite subgroup.

Coherent HL-extensions and ultraextensive structures

The notions of strongly coherent, coherent and ultraextensive can be generalize to the context of HL-extensions.

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Definition

Let $C_1 \subseteq C_2$ be $\mathcal{L}$-structures and (D_i, ϕ_i) be an HL-extension of C_i for $i = 1, 2$. We say that (D_1, ϕ_1) and (D_2, ϕ_2) are **coherent** if

(i) D_2 extends D_1,

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Coherent HL-extensions and ultraextensive structures

Theorem (E., Gao)

Let $\mathcal{C}$ be a Fraïssé class and U be the Fraïssé limit of $\mathcal{C}$. Then, U is ultraextensive iff $\mathcal{C}$ has the coherent extension property.

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Corollary

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Theorem (E., Gao)

If U is a countable ultraextensive $\mathcal{L}$-structure then $\mathrm{Aut}(U)$ has a dense locally finite subgroup.

Vershik's conjecture and omnigenous groups

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There exists a countable locally finite group $\mathbb{H}$ that is determined up to isomorphism by the following properties:

(A) *any finite group can be embedded in $\mathbb{H}$, and*

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$\mathbb{H}$ can be viewed as the direct limit of $S_3 \to S_{S_3} \to \cdots$, where $S_n \to S_{S_n}$ is the map that sends $g \in S_n$ to $\phi_g \in S_{S_n}$ defined by $\phi_g(h) = gh$.

Vershik's conjecture and omnigenous groups

Theorem (Hall)

For every triple (G_1, G_2, Ψ), *where* G_1, G_2 *are finite subgroups of* $\mathbb{H}$ *and* $\Psi : G_1 \to G_2$ *is a group isomorphism, there exists* $h \in \mathbb{H}$ *such that for every* $g \in G_1$ *we have* $\Psi(g) = hgh^{-1}$.

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Proposition

Let H be a countable locally finite group with the following property:

(E) for every triple (G_1, G_2, Ψ_1), where $G_1 \leq G_2$ are finite groups and $\Psi_1 : G_1 \to H$ is a group embedding, there exists a group embedding $\Psi_2 : G_2 \to H$ such that $\Psi_2 \restriction G_1 = \Psi_1$.

Then H is isomorphic to $\mathbb{H}$.

Vershik's conjecture and omnigenous groups

Vershik's conjecture states that the isometry group of the universal Urysohn space ($\mathrm{Iso}(\mathbb{U})$) and the automorphism group of the countable random graph ($\mathrm{Aut}(\mathcal{R})$) each contain a dense subgroup that is isomorphic to $\mathbb{H}$.

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Theorem (Solecki, 2009; alternative proof by Rosendal, 2011)

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Vershik's conjecture and omnigenous groups

Theorem (EGLMM)

The following groups contain Hall's universal countable locally finite group $\mathbb{H}$ as a dense subgroup:

1. *Iso*$(\mathbb{U})$*, the isometry group of the Urysohn space;*
2. *Iso*$(\mathbb{QU})$*, the isometry group of the rational Urysohn space;*
3. *Iso*$(\mathbb{U}_\Delta)$*, the isometry group of the universal* Δ*-metric space, for any distance value set* Δ*;*
4. *Isometry groups of ultrametric Urysohn spaces;*
5. *Aut*$(\mathcal{R})$*, the automorphism group of the random graph; and*
6. *Aut*(H_n)*, the automorphism group of the universal* K_n*-free graph, for any* $n \geq 3$*.*

Definition

The structure M is said to be **ultrahomogeneous** if finite partial automorphism of M extends to an automorphism of M. (Since $\mathcal{L}$ is a relational language)

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Definition

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The age of any countable $\mathcal{L}$-structure contains only countably many members up to isomorphism; also, any two members of the age embed in a third one (the **joint embedding property**) and whenever $A \in \mathrm{Age}(M)$ and B is a substructure of A then also $B \in \mathrm{Age}(M)$ (the **hereditary property**). Ages of ultrahomogeneous structures are characterized by an additional condition.

Definition

Let $\mathcal{K}$ be a class of $\mathcal{L}$-structures. We say that $\mathcal{K}$ has the **amalgamation property** if, for any $A, B, C \in \mathcal{K}$ and any embedding $\beta\colon A \to B$, $\gamma\colon A \to C$, there exists $D \in \mathcal{K}$ and embeddings $\beta'\colon B \to D$ and $\gamma'\colon C \to D$ such that $\beta' \circ \beta(a) = \gamma' \circ \gamma(a)$ for all $a \in A$. $\mathcal{K}$ has the **strong amalgamation property** if in the above definition we have in addition $\beta'(B) \cap \gamma'(C) = \beta' \circ \beta(A)$.

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Theorem (Fraïssé)

The age of any ultrahomogeneous $\mathcal{L}$-structure satisfies the amalgamation property. Conversely, if $\mathcal{K}$ is a countable (up to isomorphism) class of finite $\mathcal{L}$-structures which has the joint embedding, hereditary and amalgamation properties then there exists a unique (up to isomorphism), ultrahomogeneous countable $\mathcal{L}$-structure M such that $Age(M) = \mathcal{K}$.

Definition

Let $\mathcal{T}$ be a finite set of finite $\mathcal{L}$-structures each of which is a Gaifman clique. Let $\mathcal{K}$ be the class of all pairs (M, G) such that

- M is a finite $\mathcal{T}$-free $\mathcal{L}$-structure,
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Theorem (EGLMM)

$\mathcal{K}$ is a Fraïssé class. Furthermore, if (N_∞, H_∞) is the Fraïssé limit of $\mathcal{K}$, then $H_\infty \cong \mathbb{H}$, H_∞ acts faithfully on N_∞, and H_∞ is dense in $Aut(N_\infty)$.

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Theorem (EGLMM)

There are continuum many pairwise nonisomorphic countable universal locally finite omnigenous groups.

Definition

A **distance value set** is a nonempty subset Δ of the open interval $(0, +\infty)$, such that

$$\forall x, y \in \Delta \quad \min(x + y, \sup(\Delta)) \in \Delta \ .$$

A **Δ-metric space** is a metric space whose nonzero distances belong to Δ.

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Let H be a countable omnigenous locally finite group. Then for any countable distance value set Δ, $Iso(\mathbb{U}_\Delta)$ contains H as a dense subgroup.

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There are continuum many pairwise nonisomorphic countable universal locally finite groups each of which can be embedded into $Iso(\mathbb{U}_\Delta)$ as a dense subgroup.

Open problems

Note that RZ-property and HL-property are both closed under free product and $\mathbb{Z}$ satisfies both properties.

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For $|\Delta| \geq 2$, are all dense locally finite subgroups of $\mathrm{Iso}(\mathbb{U}_\Delta)$ omnigenous?

Thank you!