Dynamic Topological Logic
Day 1

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A point $x \in X$ 'moves' along its orbit

$$x, S(x), S^2(x), \ldots, S^n(x), \ldots$$

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Example
The real line $\mathbb{R}$ is equipped with its standard topology where $U \subseteq \mathbb{R}$ is open iff

$$\forall x \in U \exists \varepsilon > 0 \forall y \in \mathbb{R} \left(|x - y| < \varepsilon \Rightarrow y \in U \right)$$

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- If $(W, \preccurlyeq)$ is a partially ordered set, then W can be endowed with the down-set topology by letting $U \subset W$ be open if
$$\forall w \succcurlyeq v \; \big(w \in U \Rightarrow v \in U\big).$$
The up-set topology is defined dually.

Continuous functions

$S\colon \mathbb{R} \to \mathbb{R}$ is continuous if

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Examples

- $S\colon \mathbb{R} \to \mathbb{R}$ is continuous whenever S is a polynomial.
- If $(W, \preccurlyeq)$ is a preorder then $S\colon W \to W$ is continuous iff increasing:

$$\forall v, w \; \big(w \preccurlyeq v \Rightarrow S(w) \preccurlyeq S(v) \big)$$

Topological recurrence

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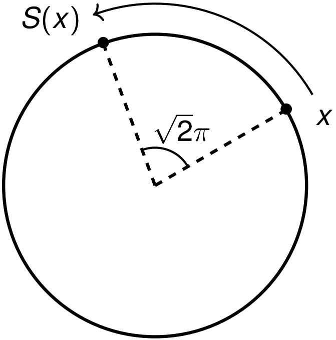
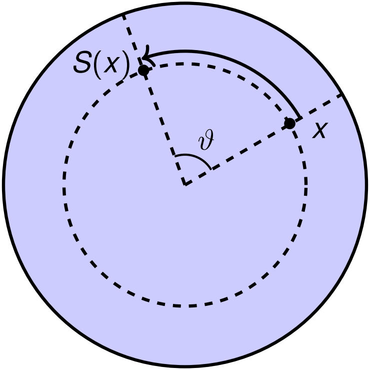
A Minimal System



A Probability-Preserving System



S4C

Language ($\mathcal{L}_{\blacksquare\circ}$):

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Rules:		
	MP	Modus ponens
	Subs	Substitution
	$\mathrm{N}_\odot$	Necessitation for $\odot = \blacksquare, \circ$

Positive results on S4C

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Kremer and Mints axioms

Kremer and Mints proposed the axiomatization KM of DTL given by

$$\mathrm{KM} = \mathrm{S4C} + \mathrm{Fix}_\Box + \mathrm{Ind}_\Box + \mathrm{N}_\Box$$

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- Kremer and Mints left the question of completeness open.

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  $$\exists\blacksquare\varphi \to \forall\Diamond\varphi$$

Here, $\forall$ and $\exists$ are the universal modalities.

Negative results

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2014 DFD showed that DTL is not finitely axiomatizable, hence KM is incomplete.

The Post Correspondence Problem

Fix a sequence of pairs

$$P = \big((\boldsymbol{v}_0, \boldsymbol{u}_0), \ldots, (\boldsymbol{v}_k, \boldsymbol{u}_k)\big)$$

with

$$\begin{aligned} \boldsymbol{v}_i &= b^i_0 \ldots b^i_{\ell_i} \\ \boldsymbol{u}_i &= c^i_0 \ldots c^i_{r_i} \end{aligned}$$

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Problem (PCP): Does there exist a sequence $i_1, \dots, i_N$ with

$$\boldsymbol{v}_{i_1} * \dots * \boldsymbol{v}_{i_N} = \boldsymbol{u}_{i_1} * \dots * \boldsymbol{u}_{i_N} ?$$

A Solvable PCP

$$\begin{aligned} \boldsymbol{v}_0 &= ab & \boldsymbol{u}_0 &= a \\ \boldsymbol{v}_1 &= d & \boldsymbol{u}_1 &= cd \\ \boldsymbol{v}_2 &= c & \boldsymbol{u}_2 &= b \end{aligned}$$

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Theorem (Post)

The set of PCP instances without a solution is not computably enumerable.

Embedding the PCP in DTL

Goal: For an alphabet A and a PCP instance P, define $\varphi_{A,P}$ so that P has a solution iff $\varphi_{A,P}$ is satisfiable over the class of dynamical systems with an interior map.

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- $\mathit{lw}_i = \mathit{left}_{b^i_0} \wedge \blacklozenge_{\mathit{stripe}}\Big(\mathit{left}_{b^i_2} \wedge \cdots \wedge \blacklozenge_{\mathit{stripe}} \mathit{left}_{b^i_{\ell_i}}\Big)$

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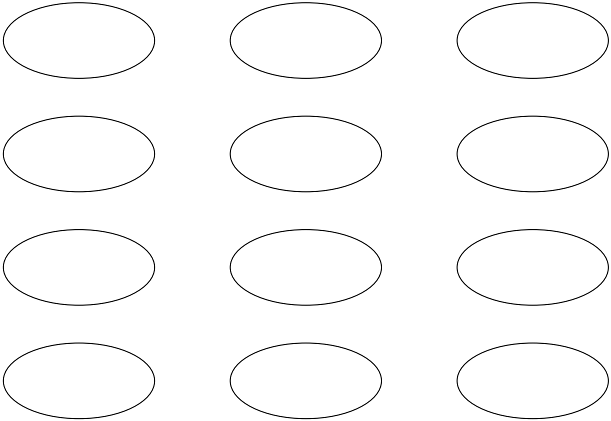
The formula φ_{right} is defined similarly, replacing *left* by *right*, etc.

Satisfying $\varphi_{A,P}$ on a dynamical system



The set of worlds W

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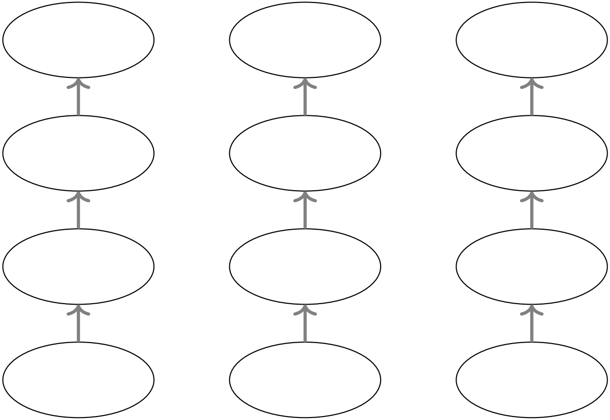

The relation $\preccurlyeq$

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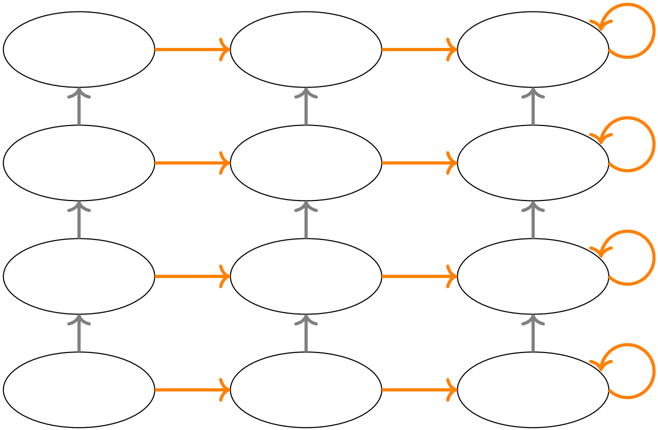

The function S

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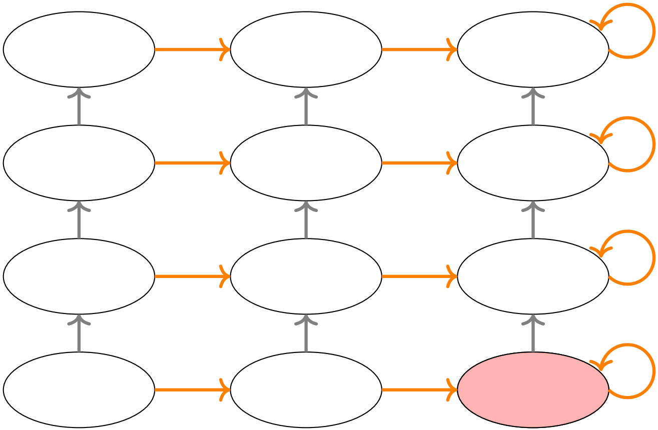

The valuation of *end* for $\Diamond end$

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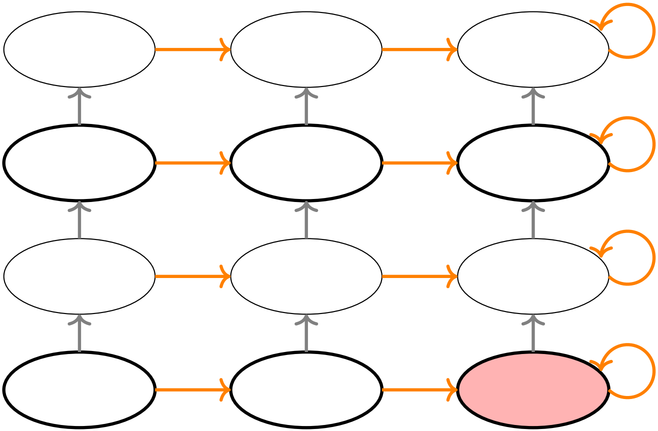

The valuation of *stripe* for $\Box^{<end}\blacksquare(stripe \leftrightarrow \circ stripe)$

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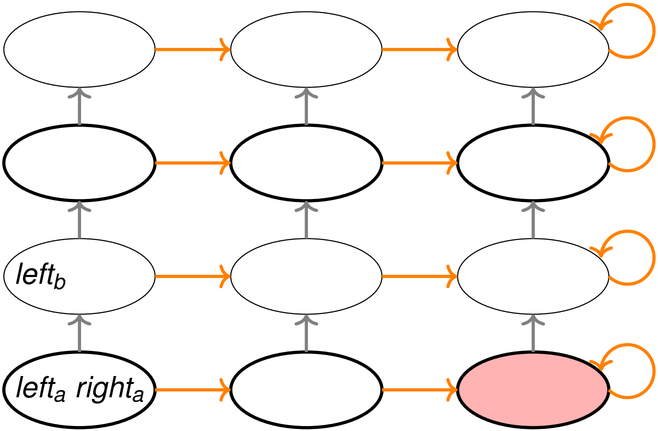

The coding of $\boldsymbol{v}_0$ and $\boldsymbol{u}_0$

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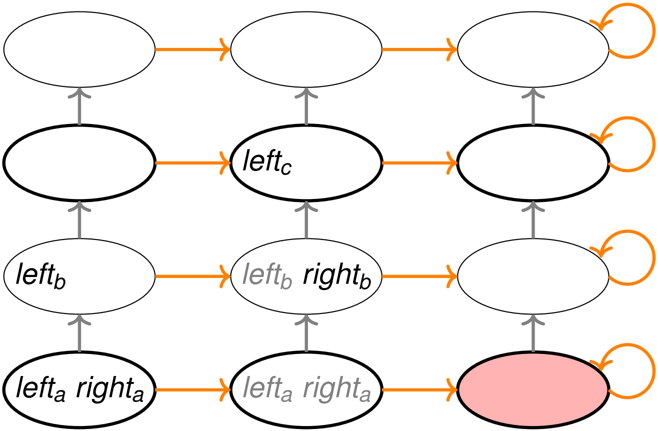

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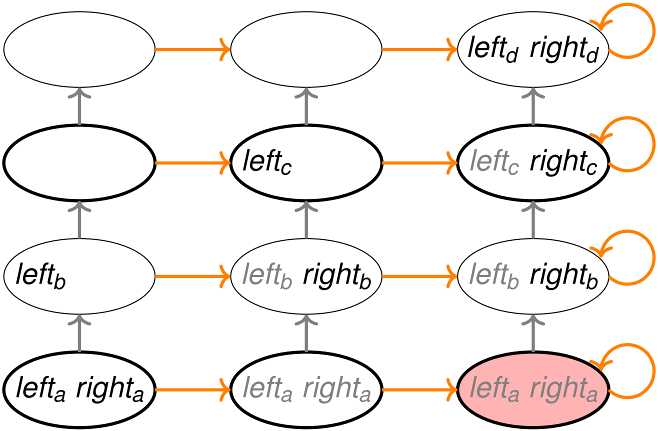

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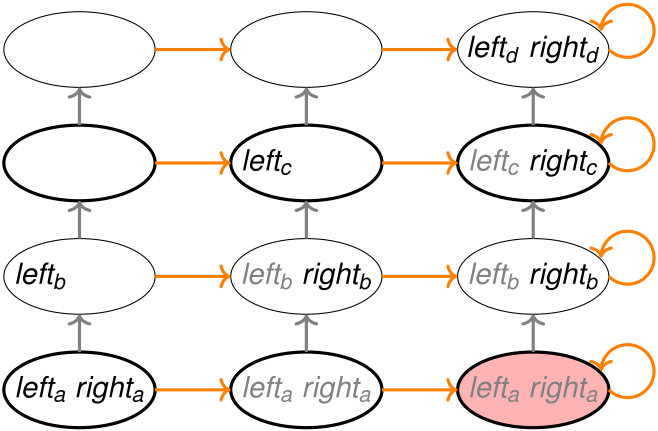

The formula $\varphi_{eq} := \Diamond\big(end \wedge \bigwedge_{a \in A} \blacksquare(left_a \leftrightarrow right_a)\big)$ is satisfied!

The PCP and interior maps

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The following are equivalent:

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Beware: Topologically satisfiable formulas are not always Kripke-satisfiable.

Non-axiomatizability and undecidability

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The proof proceeds by a similar (but more involved) reduction of a reachability problem for lossy channel systems.

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($\Rightarrow$) If X is Alexandroff, define $x \preccurlyeq y$ if

$$y \in \bigcap \{U : U \text{ is open and } x \in U\}$$

$\square$

Kripke incompleteness

Proposition

The formula

$$\Box\blacksquare p \to \blacksquare\Box p$$

is Kripke-valid but not topologically valid

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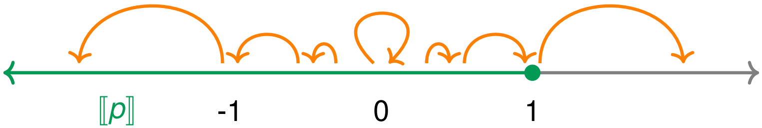
Falsifying $\Box\blacksquare p \to \blacksquare\Box p$ topologically

- $X = \mathbb{R}$
- $[\![p]\!] = (-\infty, 1]$
- $S(x) = 2x$



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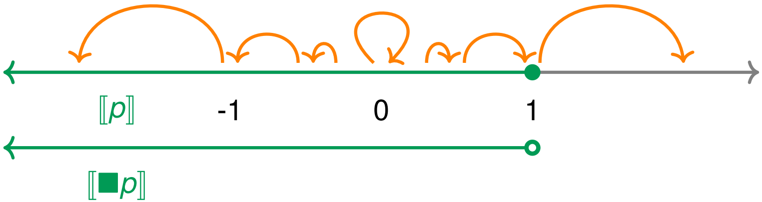

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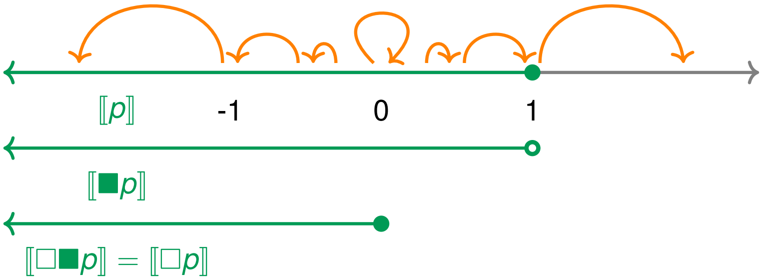

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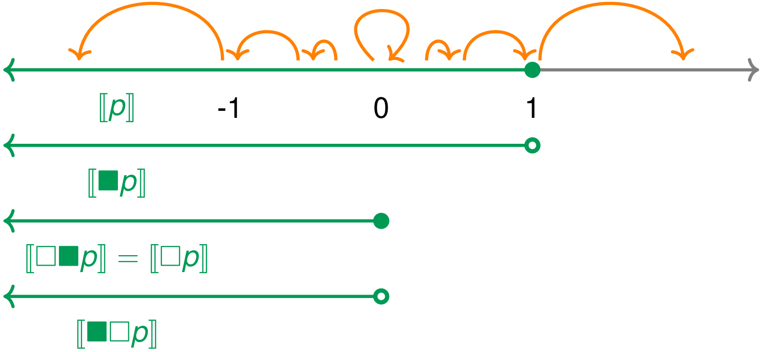

Rescuing Kripke semantics

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In the sequel we discuss non-deterministic quasimodels and their applications to DTL.

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A pair of sets of formulas $\Phi = (\Phi^+, \Phi^-)$ satisfying natural coherence conditions

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Definition: Labelled preorder

Triple $(W, \preccurlyeq, \ell)$ where ℓ assigns a type to each $w \in W$ according to the Kripke semantics

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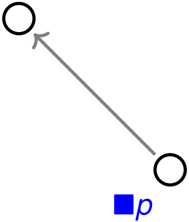

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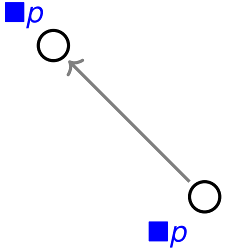

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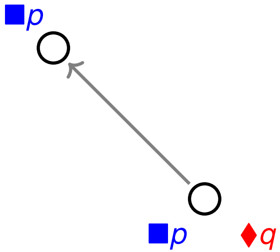

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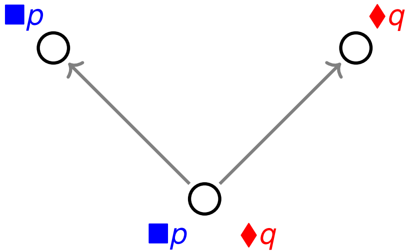

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Tuple $(W, \preccurlyeq, S, \ell)$ consisting of a locally finite labelled preorder with a forward-confluent relation S satisfying semantic conditions of the successor relation

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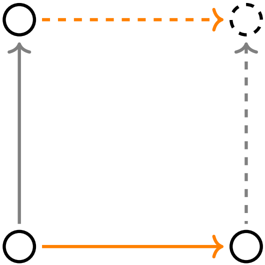

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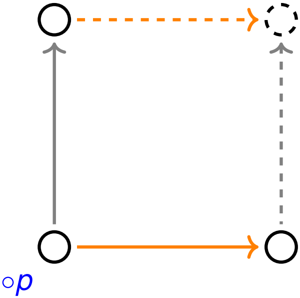

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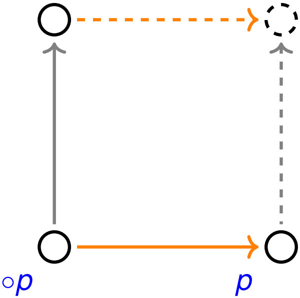

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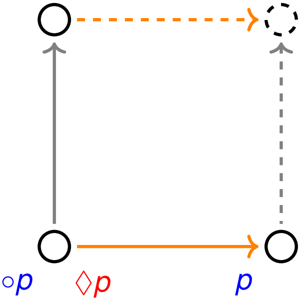

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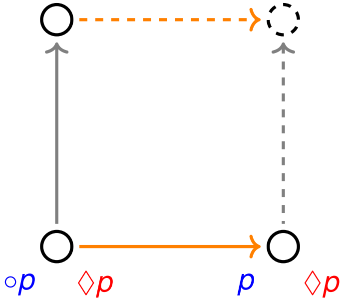

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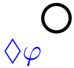

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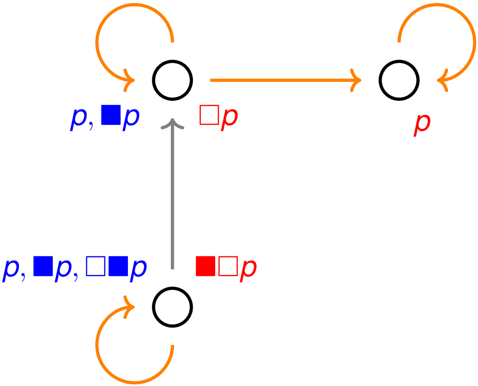
A quasimodel falsifying $\Box\blacksquare p \to \blacksquare\Box p$



From dynamical systems to quasimodels

Theorem

A formula $\varphi \in \mathcal{L}_{\blacksquare\circ\Box}$ *is valid over the class of dynamical systems iff it is valid over the class of quasimodels*

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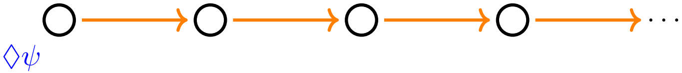

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$$\begin{tikzcd} \underset{\Diamond\psi}{\bigcirc} \arrow[r] & \bigcirc \arrow[r] & \bigcirc \arrow[r] & \underset{\psi}{\bigcirc} \arrow[r] & \cdots \end{tikzcd}$$

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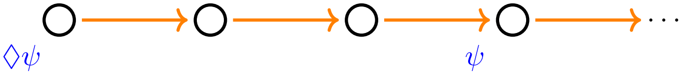

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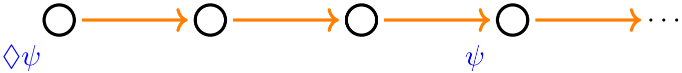

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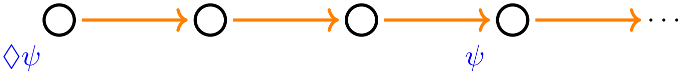

($\Leftarrow$) Fix a finite set of formulas Σ closed under subformulas

- Construct an initial, weak quasimodel $\mathcal{I}_\Sigma$
- Prove that if $\varphi \in \Sigma$ is topologically falsifiable, then it is falsifiable on some quasimodel $\mathcal{Q} \leq \mathcal{I}_\Sigma$

The initial weak quasimodel

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Fact: $\mathcal{I}_\Sigma$ is a weak quasimodel, but not necessarily a quasimodel.

Quasimodels by simulation

A simulation E between a weak quasimodel $\mathcal{Q} = (W, \preccurlyeq, R, \ell)$ and a dynamic topological model $\mathcal{M} = (X, S, [\![\cdot]\!])$ is a binary relation

$$E \subset W \times X$$

such that

1. E preserves types
2. E is continuous (preimages of opens are open)
3. E is dynamic if the following diagram can always be completed

$$\begin{tikzcd} w \arrow[r, dashed, "R"] \arrow[d, "E"'] & v \arrow[d, dashed, "E"] \\ x \arrow[r, "S"'] & y \end{tikzcd}$$

The maximal simulation

Let $\mathcal{Q} = (W, \preccurlyeq, R, \ell)$ be a weak quasimodel, $\mathcal{M} = (X, S, \llbracket\cdot\rrbracket)$ a dynamic topological model.

Lemma

If $E \subseteq W \times X$ is a dynamic simulation, then the domain of E is a quasimodel.

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So, any topologically satisfiable formula is satisfiable on a quasimodel.

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DTL *is computably enumerable.*

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Thank you for your attention!

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