Interpolative Fusions

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Introduction

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Theme: How do the model-theoretic properties of the theories of the individual expansions determine the properties of the interpolative fusion?

Outline

Questions:

1. What are interpolative fusions?
2. What are some examples?
3. When do they exist, and how can we axiomatize them?
4. How much quantifier elimination do they have?
5. What about other model-theoretic properties?
(stability, NIP, simplicity, NSOP_1, etc.)

This is joint work with Erik Walsberg and Minh Chieu Tran.

- *Interpolative Fusions*, `https://arxiv.org/abs/1811.06108`.
- *Interpolative Fusions II: Preservation Results*, in preparation.

Interpolative structures

Suppose we have:

- A language $L_\cap$.
- A family of languages $(L_i)_{i \in I}$ with $L_i \cap L_j = L_\cap$ for $i \neq j$.
- $L_\cup = \bigcup_{i \in I} L_i$.
- $\mathcal{M}_\cup$ an $L_\cup$-structure with reducts $\mathcal{M}_i$ to L_i and $\mathcal{M}_\cap$ to $L_\cap$.

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We say $\mathcal{M}_\cup$ is an **interpolative structure** if for all families $(X_i)_{i \in J}$ such that $J \subseteq I$ is finite and each X_i is an $\mathcal{M}_i$-definable set, either:

1. $\bigcap_{i \in J} X_i$ is non-empty, or
2. There is a family $(Y^i)_{i \in J}$ of $\mathcal{M}_\cap$-definable sets such that

$$X_i \subseteq Y^i \text{ for all } i \in J, \text{and } \bigcap_{i \in J} Y^i \text{ is empty.}$$

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In the case $|I| = 2$: Disjoint $\mathcal{M}_1/\mathcal{M}_2$-definable sets are separated by an $\mathcal{M}_\cap$-definable set. Analogous to the Craig interpolation theorem.

Interpolative fusions

Now suppose we have:

- An $L_\cap$-theory $T_\cap$.
- An L_i-theory T_i for each $i \in I$, such that $T_\cap$ is the set of $L_\cap$-consequences of T_i.
- $T_\cup = \bigcup_{i\in I} T_i$.

If the class of interpolative models of $T_\cup$ is elementary, we call the theory $T_\cup^*$ of this class the **interpolative fusion** of $(T_i)_{i\in I}$ over $T_\cap$.
Important note: The interpolative fusion may not exist!

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Proposition

If each T_i is model-complete, then:

1. *$\mathcal{M}_\cup \models T_\cup$ is interpolative if and only if it is existentially closed among models of $T_\cup$.*
2. *$T^*_\cup$ is the model companion of $T_\cup$ (if it exists).*

For the purposes of this talk, we can assume the T_i are model-complete.

Examples: Winkler's thesis

Theorem (Winkler)

Let T_1 and T_2 be theories in disjoint languages L_1 and L_2. If T_1 and T_2 are model complete and eliminate $\exists^\infty$, then $T_1 \cup T_2$ has a model companion.

The model companion is the interpolative fusion of T_1 and T_2 over $T_\cap$, where $T_\cap$ is the theory of an infinite set in the empty language $L_\cap$.

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Example: The **ordered random graph** is the Fraïssé limit of the class of finite ordered graphs.

Its theory is the interpolative fusion of

- T_1: the theory of the random graph (in the language $L_1 = \{R\}$).
- T_2: the theory of $\mathbb{Q}$ (in the language $L_2 = \{\leq\}$).
- Over $T_\cap$: the theory of infinite sets (in the empty language).

Examples: Fields with multiple structures

Example:
ACF = the theory of algebraically closed fields.
ACVF = the theory of algebraically closed valued fields.

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ACF $=$ the theory of algebraically closed fields.
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The interpolative fusion of n copies of ACFV over ACF is the generic theory of fields with n independent valuations.

More generally, you can put together copies of your favorite structures on fields (valuations, derivations, automorphisms, etc.) over ACF – when the interpolative fusion exists.

Examples: Fields with multiple structures

Generic theories of fields with several independent valuations were first studied by Lou van den Dries in his thesis. Here is a quote from that thesis:

> "P. Winkler treats in [Wi] some general constructions on model complete theories giving, under certain conditions, new model complete theories. For instance, he proves that the disjoint union of two theories each having an algebraically bounded model companion has a model companion. Now in our case not a disjoint union of theories is considered, but what might call, an amalgamated union, with the theory of domains as common part. It seems to me that something like algebraic boundedness is really behind the proof of (1.6). All this suggests a common generalization of Winkler's an my results."

(1.6) is the existence of the model companion for theories of fields with several orderings and valuations.

Examples: Minh's theory

Let $\chi\colon \overline{\mathbb{F}_p}^\times \to \mathbb{C}^\times$ be an injective multiplicative character.

The image of χ is contained in the unit circle in $\mathbb{C}$, so it induces a circular order C_χ on $\overline{\mathbb{F}_p}^\times$.

Theorem (Tran)

$\mathrm{Th}(\overline{\mathbb{F}_p}, 0, 1, +, -, \times, C_\chi)$ *is (a completion of) the interpolative fusion of* $\mathrm{Th}(\overline{\mathbb{F}_p}, 0, 1, +, -, \times)$ *and* $\mathrm{Th}(\overline{\mathbb{F}_p}, 0, 1, \times, C_\chi)$ *over* $\mathrm{Th}(\overline{\mathbb{F}_p}, 0, 1, \times)$*.*

The proof uses the Lang-Weil estimates (and other black boxes).

$+$ and C_χ interact "randomly" / "generically" over $\times$.

A case study: ACFA

Consider the following example:

1. $T_\cap$ is the two-sorted theory of two algebraically closed fields K and K' of the same characteristic (with no connection between them).

2. T_1 is the expansion of $T_\cap$ by an isomorphism $\sigma_1\colon K \to K'$.
   - T_1 is bi-interpretable with ACF.

3. T_2 is the expansion of $T_\cap$ by an isomorphism $\sigma_2\colon K \to K'$.
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4. $T_\cup$ is the theory of two algebraically closed fields K and K' with two isomorphisms $\sigma_1, \sigma_2\colon K \to K'$.

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4. $T_\cup$ is the theory of two algebraically closed fields K and K' with two isomorphisms $\sigma_1, \sigma_2 \colon K \to K'$.

$T_\cup$ is bi-interpretable with the theory ACF_σ of an algebraically closed field equipped with an automorphism σ.

- If $(K;\sigma) \models \mathsf{ACF}_\sigma$, then $(K, K; \sigma, \mathrm{id}_K) \models T_\cup$.
- If $(K, K'; \sigma_1, \sigma_2) \models T_\cup$, then $(K; (\sigma_2)^{-1} \circ \sigma_1) \models \mathsf{ACF}_\sigma$.

A case study: ACFA

This interpretation involves only quantifier-free formulas, so it restricts to a bi-interpretation between the interpolative fusion $T_\cup^*$ and the model companion of ACF_σ.

This model companion is called ACFA: the theory of a Algebraically Closed Fields equipped with a (generic) Automorphism.

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More generally:

For arbitrary T, the theory T_A of a model of T with a generic automorphism is bi-interpretable with an interpolative fusion of two theories, each bi-interpretable with T (when it exists).

More examples via bi-interpretations

- DCF_0 (the theory of differentially closed fields of characteristic 0) is bi-interpretable with an interpolative fusion of two theories, each bi-interpretable with ACF_0.
    - More generally: Any theory $\mathcal{D}{-}\mathrm{CF}_0$ of generic $\mathcal{D}$-fields in the sense of Moosa and Scanlon is bi-interpretable with an interpolative fusion of two theories, each bi-interpretable with ACF_0.

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- The theory of the random graph is bi-interpretable with an interpolative fusion of two theories, each of which is interpretable in the pure set.
  - More generally: Theories of generic L-structures (in an arbitrary language L, with no axioms), generic hypergraphs, generic tournaments, generic simplicial complexes, etc. are all bi-interpretable with an interpolative fusion of two theories, each interpretable in the pure set.

The main questions

So there are many interesting examples of interpolative fusions!

We seek to generalize results about these individual examples, placing them in the abstract framework of interpolative fusions.

Axiomatization results:
When does $T^*_\cup$ exist? i.e., when is the class of interpolative models of $T_\cup$ elementary?

Preservation results:
How can we understand properties of $T^*_\cup$ in terms of properties of the T_i and $T_\cap$?

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The pseudo-topological setting

Recall that $\mathcal{M}_\cup$ is an **interpolative structure** if for all families $(X_i)_{i \in J}$ such that $J \subseteq I$ is finite and each X_i is an $\mathcal{M}_i$-definable set, either:

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This quantification over $\mathcal{M}_\cap$-definable sets is not elementary in general.

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Idea: If all the X_i are "dense" in the same $\mathcal{M}_\cap$-definable set, they can't be separated by $\mathcal{M}_\cap$-definable sets.

The pseudo-topological setting

Let $\mathcal{M} \models T$, and let dim assign an ordinal or the formal symbol $-\infty$ to each $\mathcal{M}$-definable set, such that for all $\mathcal{M}$-definable X, X':

1. $\dim(X \cup X') = \max\{\dim X, \dim X'\}$,
2. $\dim X = -\infty$ if and only if $X = \emptyset$,
3. $\dim X = 0$ if and only if X is nonempty and finite,

We call such $\dim$ an **ordinal rank** on T.

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Let X be a definable set and A be an arbitrary set.

- A is **pseudo-dense** in X if A intersects every non-empty definable $X' \subseteq X$ such that $\dim X' = \dim X$.
- X is a **pseudo-closure** of A if $A \subseteq X$ and A is pseudo-dense in X. (Note the pseudo-closure is not unique, in general.)

The pseudo-topological setting

Let $\mathcal{M}'$ be an expansion of $\mathcal{M}$. Then $\mathcal{M}'$ is **approximable** if every $\mathcal{M}'$-definable set admits an $\mathcal{M}$-definable pseudo-closure.
We also say $T' = \mathrm{Th}(\mathcal{M}')$ is **approximable** over T.

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T' **defines pseudo-density** over T if for all L-formulas $\varphi(x, y)$ and L'-formulas $\psi(x, z)$ there is an L'-formula $\delta'(y, z)$ such that $\psi(\mathcal{M}', c)$ is pseudo-dense in $\varphi(\mathcal{M}', b)$ if and only if $\mathcal{M}' \models \delta(b, c)$.

Theorem

If $T_\cap$ admits an ordinal rank, and each T_i is approximable over $T_\cap$ and defines pseudo-density over $T_\cap$, then $T^*_\cup$ exists.

From the proof of the theorem, we obtain an axiomatization of $T^*_\cup$.

For each finite family $(X_i)_{i\in J}$, where each X_i is an $\mathcal{M}_i$-definable set, and each $M_\cap$-definable set Z, we add to $T_\cup$ the axiom:

If each X_i is pseudo-dense in Z, then $\bigcap_{i\in J} X_i$ is non-empty.

Consequences: ω-stable base

The content of the previous theorem can be elaborated in different ways in various contexts. Here are two sample applications.

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When T is ω-stable, the Morley rank is an ordinal rank on T.

Theorem

If T is ω-stable, then any expansion of T is approximable over T.

So if each T_i defines pseudo-density, then $T^*_\cup$ exists.

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So if each T_i defines pseudo-density, then $T^*_\cup$ exists.

If $T_\cap$ is additionally ω-categorical, then elimination of $\exists^\infty$ is sufficient.

Theorem

Suppose that $T_\cap$ is ω-stable and ω-categorical with weak elimination of imaginaries. If each T_i eliminates $\exists^\infty$, then $T^*_\cup$ exists.

Consequences: Tame topological base

When T is o-minimal, the o-minimal dimension is an ordinal rank on T (in fact it is ω-valued).

Theorem

If T is o-minimal, then any expansion of T defines pseudo-density over T.

So if each T_i is approximable over $T_\cap$, then $T^*_\cup$ exists.

The same holds in several other "tame topological" settings, e.g. when $T_\cap$ is C-minimal or P-minimal.

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The same holds in several other "tame topological" settings, e.g. when $T_\cap$ is C-minimal or P-minimal.

Theorem

Suppose $T_\cap$ is o-minimal. If $T_\cap$ is an open core of each T_i (the topological closure of every $\mathcal{M}_i$-definable set is $\mathcal{M}_\cap$-definable), then $T^*_\cup$ exists.

Time for a break

Let's take a 10 minute break.

I'm happy to answer any questions you have during the break.

The main questions

Axiomatization results:
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From now on, we assume $T^*_\cup$ exists.

Model completeness

Recall that a theory T is **model-complete** if for all models $\mathcal{M}, \mathcal{N} \models T$, every embedding $f\colon \mathcal{M} \to \mathcal{N} \models T$ is partial elementary.

Fact

A theory T is model-complete if and only if every formula is equivalent to an existential formula.

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Our earlier proposition can be viewed as a (trivial) preservation result:

Proposition

If each T_i is model-complete, then $T_\cup^*$ is the model companion of $T_\cup$. In particular, $T_\cup^*$ is model-complete.

Substructure completeness = quantifier elimination

A theory T is **substructure-complete** if for all $\mathcal{M} \models T$, and for every substructure $A = \langle A \rangle \subseteq \mathcal{M}$, every embedding $f\colon A \to \mathcal{N} \models T$ is partial elementary.

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A theory T is substructure-complete if and only if it has quantifier elimination (QE).

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Fact

A theory T is substructure-complete if and only if it has quantifier elimination (QE).

Too much to hope for!

$T^*_\cup$ typically does not have QE, even if each T_i does.

Example: Failure of QE

Example

- $T_\cap =$ the theory of an infinite set in the empty language.
- $T_1 =$ ACF in the language of rings.
- $T_2 =$ the theory in the language $\{P\}$ which says P is an infinite and cofinite unary predicate.
- Then $T^*_\cup$ is the theory of an algebraically closed field equipped with a generic predicate.

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is not equivalent to a quantifier-free formula.

Note, however, that this example only involves a quantifier over $\mathrm{acl}(x)$.

acl-completeness

Recall that b is **algebraic** over A if there is some formula $\varphi(y, x)$ and some tuple a from A such that there are only finitely many solutions to $\varphi(y, a)$ in $\mathcal{M}$, and b is among them.

$\mathrm{acl}(A)$ is the set of all $b \in \mathcal{M}$ which are algebraic over A.

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A theory T is **acl-complete** if for all $\mathcal{M} \models T$, and $A = \mathrm{acl}(A) \subseteq \mathcal{M}$, every embedding $f \colon A \to \mathcal{N} \models T$ is partial elementary.

Equivalently, if $A = \mathrm{acl}(A)$, then

$$T \cup \mathrm{qftp}(A) \models \mathrm{tp}(A).$$

acl-completeness

The **combined closure**, $\mathrm{ccl}(A)$, of a subset A of $\mathcal{M}_\cup \models T^*_\cup$ is the smallest set containing A which is acl_i-closed for each $i \in I$:

$$b \in \mathrm{ccl}(A) \iff b \in \mathrm{acl}_{i_n}(\ldots(\mathrm{acl}_{i_1}(A))\ldots) \text{ for some } i_1, \ldots, i_n \in I.$$

(Here acl_i is acl in the reduct to L_i.)

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Theorem

Suppose $T_\cap$ is stable with weak elimination of imaginaries (w.e.i), and each T_i is acl-complete. Then $\mathrm{acl}_\cup = \mathrm{ccl}$ and $T^*_\cup$ is acl-complete.

So if $A = \mathrm{ccl}(A)$, then

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$$T^*_\cup \cup \bigcup_{i \in I} \mathrm{qftp}_{L_i}(A) \models \mathrm{tp}_{L_\cup}(A).$$

acl-completeness

The **combined closure**, $\mathrm{ccl}(A)$, of a subset A of $\mathcal{M}_\cup \models T_\cup^*$ is the smallest set containing A which is acl_i-closed for each $i \in I$:

$$b \in \mathrm{ccl}(A) \iff b \in \mathrm{acl}_{i_n}(\dots(\mathrm{acl}_{i_1}(A))\dots) \text{ for some } i_1, \dots, i_n \in I.$$

(Here acl_i is acl in the reduct to L_i.)

Theorem

Suppose $T_\cap$ is stable with weak elimination of imaginaries (w.e.i), and each T_i is acl-complete. Then $\mathrm{acl}_\cup = \mathrm{ccl}$ and $T_\cup^*$ is acl-complete.

So if $A = \mathrm{ccl}(A)$, then

$$T_\cup^* \cup \bigcup_{i \in I} \mathrm{qftp}_{L_i}(A) \models \mathrm{tp}_{L_\cup}(A).$$

Crucial: Understand $L_\cup$-types in terms of L_i-types.

Quantifier elimination, stability, NIP

From now on: $T_\cap$ is stable with w.e.i. and T_i has QE for all i.

Theorem (QE)

If $\mathrm{acl}_i(A) = \langle A \rangle_i$ for all sets A and all $i \in I$, then $T^*_\cup$ has QE.

Theorem (Better QE)

If $\mathrm{acl}_i(A) = \mathrm{dcl}_\cap(A)$ for all sets A and all $i \in I$, then every $L_\cup$-formula is $T^*_\cup$-equivalent to a Boolean combination of quantifier-free L_i-formulas.

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Corollary

If $\mathrm{acl}_i(A) = \mathrm{dcl}_\cap(A)$ for all sets A and T_i is stable/NIP for all $i \in I$, then $T^*_\cup$ is stable/NIP.

Proof: Preservation of stability/NIP under Boolean combinations.

Slightly weaker (but more technical) hypotheses suffice. But we can't hope to get QE or stability/NIP except under tight control on acl.

Example: QE and TP_2

Example:

- $T_\cap =$ the theory of an infinite set in the empty language.
- $T_1 =$ the theory of $\mathbb{Q}$-vector spaces in the language $\{0, +, -, (c)_{c\in\mathbb{Q}}\}$, where c is a unary function symbol for scalar multiplication by c.
- T_2 is the theory of an equivalence relation with infinitely many infinite classes in the language $\{E\}$.
- Then $T_\cup^*$ is the theory of a $\mathbb{Q}$-vector space equipped with a generic equivalence relation.

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$T_\cup^*$ has QE by our first test: $\mathrm{acl}_i(A) = \langle A\rangle_i$ for $i = 1$ and 2.

But it does not have "better QE": $\mathrm{acl}_1 \neq \mathrm{dcl}_\cap$, and the formula

$$(x + y)Ez$$

is not equivalent to a Boolean combination of quantifier-free L_i-formulas.

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T_1 and T_2 are stable (even ω-stable), but $\varphi(x; y, z)$:

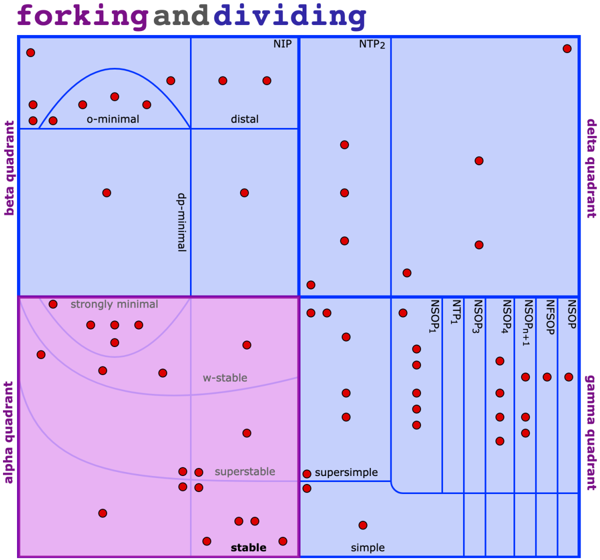
$$(x + y)Ez$$

has TP_2 in $T_\cup^*$, so $T_\cup^*$ is not even a simple theory.

Proof: Let $(v_n)_{n \in \omega}$ be distinct, let $(e_i)_{i \in \omega}$ be representatives of distinct equivalence classes, and consider the array $(v_n, e_i)_{n,i \in \omega}$.

- $\{(x + v_n)Ee_{\sigma(n)} \mid n < \omega\}$ is consistent for all $\sigma \colon \omega \to \omega$.
- $\{(x + v_n)Ee_i, (x + v_n)Ee_j\}$ is inconsistent for all $i \neq j$.

Fusions may not be simple. What about $NSOP_1$?



Map by Gabe Conant, `forkinganddividing.com`.

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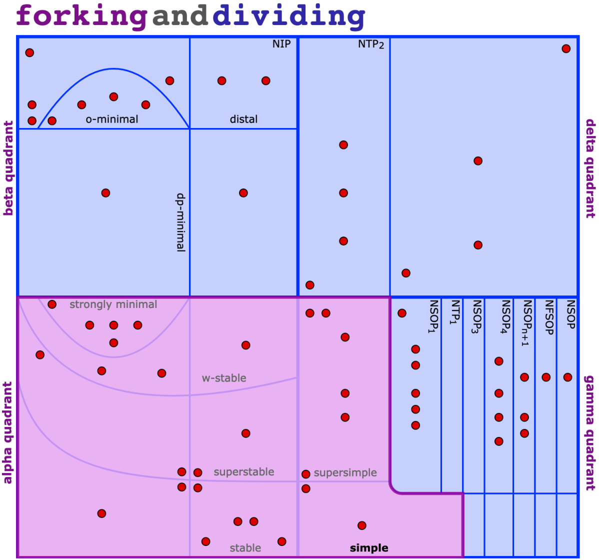

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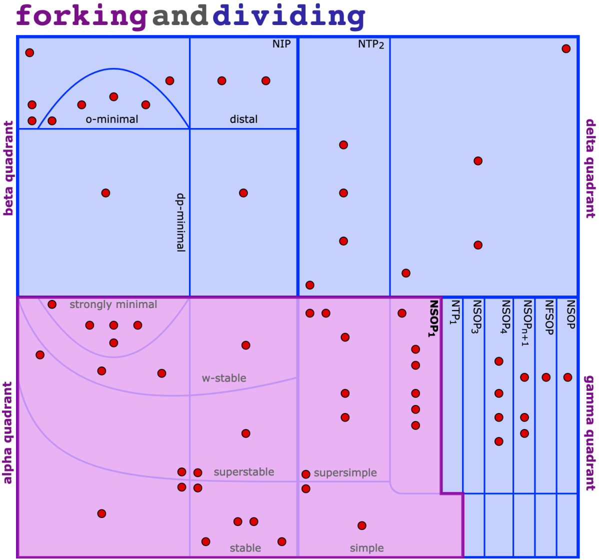

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What is NSOP$_1$?

Theorem (Chernikov–Ramsey, Kaplan–Ramsey)

T is NSOP$_1$ if and only if there is a relation $\mathop{\smile\!\!\!\!\!\!|}^K_M$ (Kim independence) defined on subsets of the monster model $\mathbb{M}$ for all $M \prec \mathbb{M}$ such that:

1. *Invariance: If $A \mathop{\smile\!\!\!\!\!\!|}^K_M B$ and $A'B'M' \equiv ABM$, then $A' \mathop{\smile\!\!\!\!\!\!|}^K_{M'} B'$.*
2. *Symmetry: If $A \mathop{\smile\!\!\!\!\!\!|}^K_M B$, then $B \mathop{\smile\!\!\!\!\!\!|}^K_M A$.*
3. *Monotonicity: If $A' \subseteq A$, $B' \subseteq B$, and $A \mathop{\smile\!\!\!\!\!\!|}^K_M B$, then $A' \mathop{\smile\!\!\!\!\!\!|}^K_M B'$.*
4. *Existence: $A \mathop{\smile\!\!\!\!\!\!|}^K_M M$.*
5. *Strong finite character: if $A \not\mathop{\smile\!\!\!\!\!\!|}^K_M B$, then there is a formula $\varphi(x;b) \in \mathrm{tp}(A/MB)$ such that for any $a' \models \varphi(x;b)$, $a' \not\mathop{\smile\!\!\!\!\!\!|}^K_M b$.*
6. **The independence theorem:** *If $a \mathop{\smile\!\!\!\!\!\!|}^K_M B$, $a' \mathop{\smile\!\!\!\!\!\!|}^K_M C$, $B \mathop{\smile\!\!\!\!\!\!|}^K_M C$, and $a \equiv_M a'$, then there exists a'' such that $a'' \equiv_{MB} a$, $a'' \equiv_{MC} a$, and $a'' \mathop{\smile\!\!\!\!\!\!|}^K_M BC$.*

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Fact: If T is simple, then $\mathop{\smile\!\!\!\!\!|}^K = \mathop{\smile\!\!\!\!\!|}^f$.

Deficiencies of $\mathop{\smile\hskip-0.9em\vert}^K$

- Currently, we only know how to define $A \mathop{\smile\hskip-0.9em\vert}^K_M B$ over a model M.
  - But in every known NSOP_1 theory, $\mathop{\smile\hskip-0.9em\vert}^K$ has a natural extension to an independence relation defined over arbitrary sets.
  - In recent work, Dobrowolski, Kim, and Ramsey have shown how to extend $\mathop{\smile\hskip-0.9em\vert}^K$ to a relation over arbitrary sets, assuming that every set is an extension base for $\mathop{\smile\hskip-0.9em\vert}^f$.

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- $\mathop{\smile\!\!\!\!\!\!|}^K$ does not satisfy base monotonicity in general.

Fact: An NSOP$_1$ is simple if and only if $\mathop{\smile\!\!\!\!\!\!|}^K$ satisfies base monotonicity.

If $M \subseteq N \subseteq B$ and $A \mathop{\smile\!\!\!\!\!\!|}^K_M B$, then $A \mathop{\smile\!\!\!\!\!\!|}^K_N B$.

Fusions of stable theories are NSOP$_1$

Theorem

If each T_i is stable, then $T^*_\cup$ is NSOP$_1$. Further:

$$A \mathop{\smile\!\!\!\!\!\!\;|}^K_M B \text{ in } T^*_\cup \iff \mathrm{acl}_\cup(MA) \mathop{\smile\!\!\!\!\!\!\;|}^f_M \mathrm{acl}_\cup(MB) \text{ in } T_i \text{ for all } i \in I.$$

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We also get preservation of NSOP$_1$ over stable bases:

If $T_\cap$ is stable and each T_i is NSOP$_1$ and satisfies a technical strengthening of the independence theorem, then $T_\cup^*$ is NSOP$_1$.

Why is ACFA simple?

Slogan: NSOP_1 is an upper bound for the complexity of generic constructions in the unordered realm.

... but sometimes we get simplicity.

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Theorem (Chatzidakis-Hrushovski)

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... but sometimes we get simplicity.

Theorem (Chatzidakis-Hrushovski)

ACFA is simple.

Theorem (Chatzidakis-Pillay)

Suppose T is stable and T_A exists. Then T_A is simple.

Relatively disintegrated acl in ACFA

Recall:

- $T_\cap$ is the two-sorted theory of two algebraically closed fields K and K' of the same characteristic (with no connection between them).
- T_i is the expansion of $T_\cap$ by an isomorphism $\sigma_i \colon K \to K'$.

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So T_i has **relatively disintegrated** acl over its reduct $T_\cap$:

$$\mathrm{acl}_i(A) = \mathrm{acl}_\cap\left(\bigcup_{a\in A} \mathrm{acl}_i(a)\right)$$

Preservation of simplicity

T_i has **relatively disintegrated** acl if

$$\mathrm{acl}_i(A) = \mathrm{acl}_\cap\left(\bigcup_{a\in A}\mathrm{acl}_i(a)\right)$$

(No (≥ 2)-ary algebraic dependencies that weren't already present in $T_\cap$.)

Theorem

If $T_\cap$ is stable and each T_i is simple with relatively disintegrated acl, then $T_\cup^*$ is simple.

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If $T_\cap$ is stable and each T_i is simple with relatively disintegrated acl, then $T_\cup^*$ is simple.

Idea of proof: We know $T_\cup^*$ is NSOP$_1$ and

$$A \mathop{\smile\!\!\!\!\!\!|}^{K}_{M} B \text{ in } T_\cup^* \iff \mathrm{acl}_\cup(MA) \mathop{\smile\!\!\!\!\!\!|}^{f}_{M} \mathrm{acl}_\cup(MB) \text{ in } T_i \text{ for all } i \in I.$$

Using relatively disintegrated acl, we show that $\mathop{\smile\!\!\!\!\!\!|}^{K}$ satisfies base monotonicity in $T_\cup^*$.

Simplicity and generic (≥ 2)-ary dependences

The result on preservation of simplicity resonates with (and generalizes) the following theorem:

Theorem (K.-Ramsey, Jeřábek)

For any language L, the empty L-theory has a model companion T_L^*.

- *T_L^* is always NSOP$_1$.*
- *If L has no function symbols of arity ≥ 2, then T_L^* is simple.*

The Henson graph in a fusion of simple theories

Example:

- $T_\cap$ = the theory of a generic 3-uniform hypergraph in the language $\{R\}$ (the Fraïssé limit of the class of finite 3-uniform hypergraphs).
- T_1 = the generic expansion of $T_\cap$ by a graph relation E_1 such that:

$$\forall x, y, z, (E_1(x,y) \wedge E_1(x,z) \wedge E_1(y,z) \to R(x,y,z)).$$

- T_2 = the generic expansion of $T_\cap$ by a graph relation E_2 such that:

$$\forall x, y, z, (E_2(x,y) \wedge E_2(x,z) \wedge E_2(y,z) \to \neg R(x,y,z)).$$

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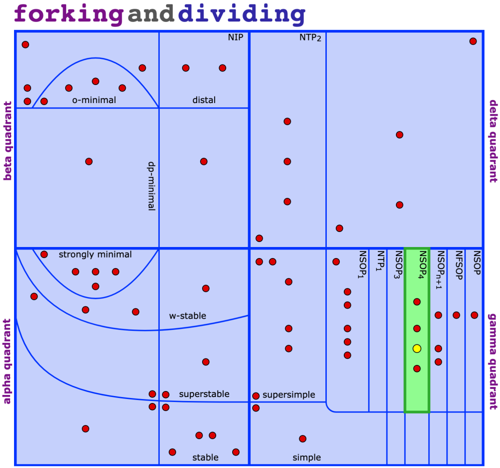
$T^*_\cup$ exists and is $\aleph_0$-categorical. Let $(M; R, E_1, E_2)$ be the countable model. Define a graph relation E on M by:

$$E(x,y) \iff E_1(x,y) \wedge E_2(x,y).$$

Then $(M; E)$ is isomorphic to the Henson graph (the Fraïssé limit of the class of finite $\triangle$-free graphs).

The Henson graph in a fusion of simple theories

In this example, $T_\cap$, T_1, and T_2 are all simple theories, but $T^*_\cup$ has SOP_3, since it interprets the theory of the Henson graph.



Moral: Without stability of $T_\cap$, interpolative fusions may not be NSOP_1.

Thank you!

For more information, see our papers:

- *Interpolative Fusions*, `https://arxiv.org/abs/1811.06108`.
- *Interpolative Fusions II: Preservation Results*, on arXiv soon!

I'll be happy to take questions.