Cardinalities of definable sets in finite structures

Dugald Macpherson

Joint work with Anscombe, Steinhorn, Wolf; IPM Tehran

July 29, 2020

Structure of Talk

1. Chatzidakis - van den Dries - Macintyre Theorem on definability in finite fields, asymptotic classes, measurable structures.

Structure of Talk

1. Chatzidakis - van den Dries - Macintyre Theorem on definability in finite fields, asymptotic classes, measurable structures.
2. Generalisation to multidimensional asymptotic classes and generalised measurable structures. Examples.

Structure of Talk

1. Chatzidakis - van den Dries - Macintyre Theorem on definability in finite fields, asymptotic classes, measurable structures.
2. Generalisation to multidimensional asymptotic classes and generalised measurable structures. Examples.
3. Multidimensional EXACT classes and approximations of homogeneous structures.

CDM Theorem

Theorem [Chatzidakis, van den Dries and Macintyre 1992] Let $\varphi(x_1, \ldots, x_n; y_1, \ldots, y_m)$ be a formula in the language $L_{\text{rings}} = (+, -, \times, 0, 1)$ of rings. Then

CDM Theorem

Theorem [Chatzidakis, van den Dries and Macintyre 1992] Let $\varphi(x_1,\ldots,x_n;y_1,\ldots,y_m)$ be a formula in the language $L_{\text{rings}} = (+,-,\times,0,1)$ of rings. Then

(i) there is a positive constant C and finitely many pairs (d_i,μ_i), with $d_i \in \{0,1,\ldots,n\}$ and $\mu_i \in \mathbb{Q}^{>0}$ such that for each finite field $\mathbb{F}_q$, and each $\bar{a} \in \mathbb{F}_q^m$, if the set

$$\varphi(\mathbb{F}_q^n,\bar{a}) := \{\bar{b} \in \mathbb{F}_q^n : \mathbb{F}_q \models \varphi(\bar{b},\bar{a})\}$$

is nonempty, then

$$\left||\varphi\left(\mathbb{F}_q^n,\bar{a}\right)| - \mu_i q^{d_i}\right| < Cq^{d_i-(1/2)}$$

for some i;

CDM Theorem

Theorem [Chatzidakis, van den Dries and Macintyre 1992] Let $\varphi(x_1, \ldots, x_n; y_1, \ldots, y_m)$ be a formula in the language $L_{\text{rings}} = (+, -, \times, 0, 1)$ of rings. Then

(i) there is a positive constant C and finitely many pairs (d_i, μ_i), with $d_i \in \{0, 1, \ldots, n\}$ and $\mu_i \in \mathbb{Q}^{>0}$ such that for each finite field $\mathbb{F}_q$, and each $\bar{a} \in \mathbb{F}_q^m$, if the set

$$\varphi(\mathbb{F}_q^n, \bar{a}) := \{\bar{b} \in \mathbb{F}_q^n : \mathbb{F}_q \models \varphi(\bar{b}, \bar{a})\}$$

is nonempty, then

$$\left| |\varphi\left(\mathbb{F}_q^n, \bar{a}\right)| - \mu_i q^{d_i} \right| < C q^{d_i - (1/2)}$$

for some i;

(ii) for each pair (d_i, μ_i), there is a formula $\psi_i(y_1, \ldots, y_m)$ in the language of rings such that $\psi_i\left(\mathbb{F}_q^m\right)$ consists of those $\bar{a} \in \mathbb{F}_q^m$ for which the corresponding inequality holds.

CDM Theorem

Theorem [Chatzidakis, van den Dries and Macintyre 1992] Let $\varphi(x_1, \ldots, x_n; y_1, \ldots, y_m)$ be a formula in the language $L_{\text{rings}} = (+, -, \times, 0, 1)$ of rings. Then

(i) there is a positive constant C and finitely many pairs (d_i, μ_i), with $d_i \in \{0, 1, \ldots, n\}$ and $\mu_i \in \mathbb{Q}^{>0}$ such that for each finite field $\mathbb{F}_q$, and each $\bar{a} \in \mathbb{F}_q^m$, if the set

$$\varphi(\mathbb{F}_q^n, \bar{a}) := \{\bar{b} \in \mathbb{F}_q^n : \mathbb{F}_q \models \varphi(\bar{b}, \bar{a})\}$$

is nonempty, then

$$\left| |\varphi\left(\mathbb{F}_q^n, \bar{a}\right)| - \mu_i q^{d_i} \right| < C q^{d_i - (1/2)}$$

for some i;

(ii) for each pair (d_i, μ_i), there is a formula $\psi_i(y_1, \ldots, y_m)$ in the language of rings such that $\psi_i\left(\mathbb{F}_q^m\right)$ consists of those $\bar{a} \in \mathbb{F}_q^m$ for which the corresponding inequality holds.

Corollary: There is no L_{rings}-formula (even with parameters) which uniformly in all finite fields $\mathbb{F}_{p^2}$ defines the prime subfield $\mathbb{F}_p$.

The CDM theorem was turned into a definition of an abstract model-theoretic framework in work of M + Steinhorn (2008), Elwes, Ryten,...

The CDM theorem was turned into a definition of an abstract model-theoretic framework in work of M + Steinhorn (2008), Elwes, Ryten,...

A **1-dimensional asymptotic class** is essentially a class of finite structures (in any first order language) satisfying the conclusion of the theorem.

The CDM theorem was turned into a definition of an abstract model-theoretic framework in work of M + Steinhorn (2008), Elwes, Ryten,...

A **1-dimensional asymptotic class** is essentially a class of finite structures (in any first order language) satisfying the conclusion of the theorem.

Elwes (2007): notion of N-dimensional asymptotic class (e.g. class of groups $\mathrm{SL}_2(q)$ is a 3-dimensional asymptotic class; likewise the class of groups $E_6(q)$ is a 78-dimensional asymptotic class).

The CDM theorem was turned into a definition of an abstract model-theoretic framework in work of M + Steinhorn (2008), Elwes, Ryten,...

A **1-dimensional asymptotic class** is essentially a class of finite structures (in any first order language) satisfying the conclusion of the theorem.

Elwes (2007): notion of N-dimensional asymptotic class (e.g. class of groups $\mathrm{SL}_2(q)$ is a 3-dimensional asymptotic class; likewise the class of groups $E_6(q)$ is a 78-dimensional asymptotic class).

Ryten (PhD thesis 2007): For any fixed Lie type τ, possibly twisted, the class of all finite simple groups of type τ is an asymptotic class.

The CDM theorem was turned into a definition of an abstract model-theoretic framework in work of M + Steinhorn (2008), Elwes, Ryten,...

A **1-dimensional asymptotic class** is essentially a class of finite structures (in any first order language) satisfying the conclusion of the theorem.

Elwes (2007): notion of N-dimensional asymptotic class (e.g. class of groups $\mathrm{SL}_2(q)$ is a 3-dimensional asymptotic class; likewise the class of groups $E_6(q)$ is a 78-dimensional asymptotic class).

Ryten (PhD thesis 2007): For any fixed Lie type τ, possibly twisted, the class of all finite simple groups of type τ is an asymptotic class.

Corollary. Fix a Lie type τ, giving an N-dimensional asymptotic class of finite simple groups.

The CDM theorem was turned into a definition of an abstract model-theoretic framework in work of M + Steinhorn (2008), Elwes, Ryten,...

A **1-dimensional asymptotic class** is essentially a class of finite structures (in any first order language) satisfying the conclusion of the theorem.

Elwes (2007): notion of N-dimensional asymptotic class (e.g. class of groups $\mathrm{SL}_2(q)$ is a 3-dimensional asymptotic class; likewise the class of groups $E_6(q)$ is a 78-dimensional asymptotic class).

Ryten (PhD thesis 2007): For any fixed Lie type τ, possibly twisted, the class of all finite simple groups of type τ is an asymptotic class.

Corollary. Fix a Lie type τ, giving an N-dimensional asymptotic class of finite simple groups.
There is a finite set E of pairs $(d_i, \mu_i) \in \{0, \ldots, N\} \times \mathbb{Q}^{\geq 0}$ such that for any finite simple group G of Lie type τ, each conjugacy class of G has cardinality roughly $\mu|G|^{d/N}$ for some $(d, \mu) \in E$.

The CDM theorem was turned into a definition of an abstract model-theoretic framework in work of M + Steinhorn (2008), Elwes, Ryten,...

A **1-dimensional asymptotic class** is essentially a class of finite structures (in any first order language) satisfying the conclusion of the theorem.

Elwes (2007): notion of N-dimensional asymptotic class (e.g. class of groups $\mathrm{SL}_2(q)$ is a 3-dimensional asymptotic class; likewise the class of groups $E_6(q)$ is a 78-dimensional asymptotic class).

Ryten (PhD thesis 2007): For any fixed Lie type τ, possibly twisted, the class of all finite simple groups of type τ is an asymptotic class.

Corollary. Fix a Lie type τ, giving an N-dimensional asymptotic class of finite simple groups.
There is a finite set E of pairs $(d_i, \mu_i) \in \{0, \ldots, N\} \times \mathbb{Q}^{\geq 0}$ such that for any finite simple group G of Lie type τ, each conjugacy class of G has cardinality roughly $\mu |G|^{d/N}$ for some $(d, \mu) \in E$.

Proof: Apply Ryten result to formula $\phi(x, y)$ of form $\exists z(z^{-1}xz = y)$.

Measurable structures

Corresponding notion of **measurable structure** (M + Steinhorn), for *infinite* structures. This means we can assign pairs (d, μ) to definable sets, with similar counting properties as for asymptotic classes.

Measurable structures

Corresponding notion of **measurable structure** (M + Steinhorn), for *infinite* structures. This means we can assign pairs (d, μ) to definable sets, with similar counting properties as for asymptotic classes.

Fact:

(i) Any ultraproduct of an asymptotic class is measurable. So every pseudofinite field is measurable. (The idea: an ultraproduct of sets of size roughly μq^d is assigned the pair (d, μ)).

Measurable structures

Corresponding notion of **measurable structure** (M + Steinhorn), for *infinite* structures. This means we can assign pairs (d, μ) to definable sets, with similar counting properties as for asymptotic classes.

Fact:

(i) Any ultraproduct of an asymptotic class is measurable. So every pseudofinite field is measurable. (The idea: an ultraproduct of sets of size roughly μq^d is assigned the pair (d, μ)).

(ii) Every measurable structure is supersimple of finite SU-rank. (Recall that the class of simple theories contains the stable theories, that forking gives a nice notion of independence in simple theories, and that supersimple + stable = superstable.)

Measurable structures

Corresponding notion of **measurable structure** (M + Steinhorn), for *infinite* structures. This means we can assign pairs (d, μ) to definable sets, with similar counting properties as for asymptotic classes.

Fact:

(i) Any ultraproduct of an asymptotic class is measurable. So every pseudofinite field is measurable. (The idea: an ultraproduct of sets of size roughly μq^d is assigned the pair (d, μ)).

(ii) Every measurable structure is supersimple of finite SU-rank. (Recall that the class of simple theories contains the stable theories, that forking gives a nice notion of independence in simple theories, and that supersimple + stable = superstable.)

(iii) $(\mathbb{C}, +, \times)$ is not measurable, due to the 2-1 surjection $x \mapsto x^2$ $\mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$.

We aim to broaden this framework, e.g.

(a) allow parts of the structure (sorts?) to vary independently,

We aim to broaden this framework, e.g.

(a) allow parts of the structure (sorts?) to vary independently,

(b) not require that ultraproducts have finite rank, or even have simple theory,

We aim to broaden this framework, e.g.

(a) allow parts of the structure (sorts?) to vary independently,

(b) not require that ultraproducts have finite rank, or even have simple theory,

(c) not worry about the form of the functions giving approximate cardinalities (no longer just $q \mapsto \mu q^d$ as with finite fields).

We aim to broaden this framework, e.g.

(a) allow parts of the structure (sorts?) to vary independently,
(b) not require that ultraproducts have finite rank, or even have simple theory,
(c) not worry about the form of the functions giving approximate cardinalities (no longer just $q \mapsto \mu q^d$ as with finite fields).

We continue to work with a class of finite structures, and retain a key finiteness property:

We aim to broaden this framework, e.g.

(a) allow parts of the structure (sorts?) to vary independently,

(b) not require that ultraproducts have finite rank, or even have simple theory,

(c) not worry about the form of the functions giving approximate cardinalities (no longer just $q \mapsto \mu q^d$ as with finite fields).

We continue to work with a class of finite structures, and retain a key finiteness property:
for any formula $\phi(\bar{x}, \bar{y})$ determining in each structure a family of definable sets, there is a number n_ϕ such that the sets have one of n_ϕ possible approximate sizes (in each finite structure).

We aim to broaden this framework, e.g.

(a) allow parts of the structure (sorts?) to vary independently,
(b) not require that ultraproducts have finite rank, or even have simple theory,
(c) not worry about the form of the functions giving approximate cardinalities (no longer just $q \mapsto \mu q^d$ as with finite fields).

We continue to work with a class of finite structures, and retain a key finiteness property:
for any formula $\phi(\bar{x}, \bar{y})$ determining in each structure a family of definable sets, there is a number n_ϕ such that the sets have one of n_ϕ possible approximate sizes (in each finite structure).

Note: Our framework will NOT include the class of total orders, due to the formula $x < y$.

Possible examples

Possible examples to keep in mind:

Possible examples

Possible examples to keep in mind:

1. Pairs $(V, \mathbb{F}_q)$, V a finite-dim vector space over $\mathbb{F}_q$.

Possible examples

Possible examples to keep in mind:

1. Pairs $(V, \mathbb{F}_q)$, V a finite-dim vector space over $\mathbb{F}_q$.
2. Triples $(V, W, \mathbb{F}_q)$ with $(V, \mathbb{F}_q)$ as above and W a subspace of V.

Possible examples

Possible examples to keep in mind:

1. Pairs $(V, \mathbb{F}_q)$, V a finite-dim vector space over $\mathbb{F}_q$.
2. Triples $(V, W, \mathbb{F}_q)$ with $(V, \mathbb{F}_q)$ as above and W a subspace of V.
3. Disjoint unions of complete graphs all of same size (n copies of K_m, so 2 parameters varying freely).

Possible examples

Possible examples to keep in mind:

1. Pairs $(V, \mathbb{F}_q)$, V a finite-dim vector space over $\mathbb{F}_q$.
2. Triples $(V, W, \mathbb{F}_q)$ with $(V, \mathbb{F}_q)$ as above and W a subspace of V.
3. Disjoint unions of complete graphs all of same size (n copies of K_m, so 2 parameters varying freely).
4. Finite abelian groups.

Possible examples

Possible examples to keep in mind:

1. Pairs $(V, \mathbb{F}_q)$, V a finite-dim vector space over $\mathbb{F}_q$.
2. Triples $(V, W, \mathbb{F}_q)$ with $(V, \mathbb{F}_q)$ as above and W a subspace of V.
3. Disjoint unions of complete graphs all of same size (n copies of K_m, so 2 parameters varying freely).
4. Finite abelian groups.
5. Finite graphs of bounded degree.

$\emptyset$-definable finite partitions

For a class $\mathcal{C}$ of finite $\mathcal{L}$-structures and a tuple $\bar{y}$ of variables, we denote by $(\mathcal{C}, \bar{y})$ the set $\{(M, \bar{a}) \mid M \in \mathcal{C}, \bar{a} \in M^{|\bar{y}|}\}$ of pairs ('pointed structures') consisting of a structure in $\mathcal{C}$ and a $\bar{y}$-tuple from that structure.

$\emptyset$-definable finite partitions

For a class $\mathcal{C}$ of finite $\mathcal{L}$-structures and a tuple $\bar{y}$ of variables, we denote by $(\mathcal{C}, \bar{y})$ the set $\{(M, \bar{a}) \mid M \in \mathcal{C}, \bar{a} \in M^{|\bar{y}|}\}$ of pairs ('pointed structures') consisting of a structure in $\mathcal{C}$ and a $\bar{y}$-tuple from that structure.

A finite partition Φ of $(\mathcal{C}, \bar{y})$ (i.e. finitely many parts) is $\emptyset$**-definable** if for each $P \in \Phi$ there exists an $\mathcal{L}$-formula $\phi_P(\bar{y})$ without parameters such that

$$\phi_P(M) = \{\bar{b} \in M^{|\bar{y}|} \mid (M, \bar{b}) \in P\},$$

for each $M \in \mathcal{C}$.

$\emptyset$-definable finite partitions

For a class $\mathcal{C}$ of finite $\mathcal{L}$-structures and a tuple $\bar{y}$ of variables, we denote by $(\mathcal{C}, \bar{y})$ the set $\{(M, \bar{a}) \mid M \in \mathcal{C}, \bar{a} \in M^{|\bar{y}|}\}$ of pairs ('pointed structures') consisting of a structure in $\mathcal{C}$ and a $\bar{y}$-tuple from that structure.

A finite partition Φ of $(\mathcal{C}, \bar{y})$ (i.e. finitely many parts) is **$\emptyset$-definable** if for each $P \in \Phi$ there exists an $\mathcal{L}$-formula $\phi_P(\bar{y})$ without parameters such that

$$\phi_P(M) = \{\bar{b} \in M^{|\bar{y}|} \mid (M, \bar{b}) \in P\},$$

for each $M \in \mathcal{C}$.

The idea: we partition $\bar{y}$-space uniformly (across $\mathcal{C}$) into a fixed finite number of parts, each part (uniformly) $\emptyset$-definable in each structure in $\mathcal{C}$.

Definition of R-m.a.c.

Let R be ANY set of functions $\mathcal{C} \longrightarrow \mathbb{R}^{\geq 0}$. A class $\mathcal{C}$ of finite $\mathcal{L}$-structures is an **R-multidimensional asymptotic class** (or an **R-m.a.c.** for short) if for every formula $\phi(\bar{x}, \bar{y})$ there is a finite $\emptyset$-definable partition Φ of $(\mathcal{C}, \bar{y})$ and a set $H_\Phi := \{h_P \in R \mid P \in \Phi\}$ of functions such that:

$$\big| |\phi(M^{|\bar{x}|}; \bar{b})| - h_P(M) \big| = o(h_P(M)) \tag{1}$$

for $(M, \bar{b}) \in P$ as $|M| \longrightarrow \infty$.

Definition of R-m.a.c.

Let R be ANY set of functions $\mathcal{C} \longrightarrow \mathbb{R}^{\geq 0}$. A class $\mathcal{C}$ of finite $\mathcal{L}$-structures is an R**-multidimensional asymptotic class** (or an R**-m.a.c.** for short) if for every formula $\phi(\bar{x}, \bar{y})$ there is a finite $\emptyset$-definable partition Φ of $(\mathcal{C}, \bar{y})$ and a set $H_\Phi := \{h_P \in R \mid P \in \Phi\}$ of functions such that:

$$\big||\phi(M^{|\bar{x}|}; \bar{b})| - h_P(M)\big| = o(h_P(M)) \tag{1}$$

for $(M, \bar{b}) \in P$ as $|M| \longrightarrow \infty$.

$\mathcal{C}$ is an R-**m.e.c** (multidimensional **exact** class) if above we have

$$|\phi(M^{|\bar{x}|}, \bar{b})| = h_P(M).$$

Definition of R-m.a.c.

Let R be ANY set of functions $\mathcal{C} \longrightarrow \mathbb{R}^{\geq 0}$. A class $\mathcal{C}$ of finite $\mathcal{L}$-structures is an ***R*-multidimensional asymptotic class** (or an ***R*-m.a.c.** for short) if for every formula $\phi(\bar{x}, \bar{y})$ there is a finite $\emptyset$-definable partition Φ of $(\mathcal{C}, \bar{y})$ and a set $H_\Phi := \{h_P \in R \mid P \in \Phi\}$ of functions such that:

$$\Big| |\phi(M^{|\bar{x}|}; \bar{b})| - h_P(M) \Big| = o(h_P(M)) \qquad (1)$$

for $(M, \bar{b}) \in P$ as $|M| \longrightarrow \infty$.

$\mathcal{C}$ is an R-**m.e.c** (multidimensional **exact** class) if above we have

$$|\phi(M^{|\bar{x}|}, \bar{b})| = h_P(M).$$

The idea: The size of $\phi(M^{|\bar{x}|}, \bar{b})$ is a function of M, the function depending just on the *part* of $\bar{b}$ (i.e. the *part* of $(M, \bar{b})$). The notion of R-m.e.c. is a strengthening of R-m.a.c..

Definition of R-m.a.c.

Let R be ANY set of functions $\mathcal{C} \longrightarrow \mathbb{R}^{\geq 0}$. A class $\mathcal{C}$ of finite $\mathcal{L}$-structures is an R**-multidimensional asymptotic class** (or an R**-m.a.c.** for short) if for every formula $\phi(\bar{x}, \bar{y})$ there is a finite $\emptyset$-definable partition Φ of $(\mathcal{C}, \bar{y})$ and a set $H_\Phi := \{h_P \in R \mid P \in \Phi\}$ of functions such that:

$$\left||\phi(M^{|\bar{x}|}; \bar{b})| - h_P(M)\right| = o(h_P(M)) \tag{1}$$

for $(M, \bar{b}) \in P$ as $|M| \longrightarrow \infty$.

$\mathcal{C}$ is an R-**m.e.c** (multidimensional **exact** class) if above we have

$$|\phi(M^{|\bar{x}|}, \bar{b})| = h_P(M).$$

The idea: The size of $\phi(M^{|\bar{x}|}, \bar{b})$ is a function of M, the function depending just on the *part* of $\bar{b}$ (i.e. the *part* of $(M, \bar{b})$). The notion of R-m.e.c. is a strengthening of R-m.a.c..

weak R-m.a.c. (or R-m.e.c.) – drop the definability clause on the partition Φ.

Observations

Basic facts about R-m.a.c.s and R-m.e.c.s.

1. To prove a class $\mathcal{C}$ is an R-m.a.c. or R-m.e.c it suffices to prove the condition for formulas $\phi(x, \bar{y})$ (with x a single variable), replacing R by the ring generated by R. (Fibering argument, using definability. Compare how o-minimality is a one-variable condition but implies cell decomposition.)

Observations

Basic facts about R-m.a.c.s and R-m.e.c.s.

1. To prove a class $\mathcal{C}$ is an R-m.a.c. or R-m.e.c it suffices to prove the condition for formulas $\phi(x, \bar{y})$ (with x a single variable), replacing R by the ring generated by R. (Fibering argument, using definability. Compare how o-minimality is a one-variable condition but implies cell decomposition.)

2. (Wolf) If $\mathcal{C}$ is a m.a.c. or m.e.c. then so is any class of finite structures uniformly bi-interpretable with $\mathcal{C}$. (Note: These conditions are not closed under interpretability or taking reducts, as the definability clause may be lost.)

Observations

Basic facts about R-m.a.c.s and R-m.e.c.s.

1. To prove a class $\mathcal{C}$ is an R-m.a.c. or R-m.e.c it suffices to prove the condition for formulas $\phi(x, \bar{y})$ (with x a single variable), replacing R by the ring generated by R. (Fibering argument, using definability. Compare how o-minimality is a one-variable condition but implies cell decomposition.)

2. (Wolf) If $\mathcal{C}$ is a m.a.c. or m.e.c. then so is any class of finite structures uniformly bi-interpretable with $\mathcal{C}$. (Note: These conditions are not closed under interpretability or taking reducts, as the definability clause may be lost.)

3. Any class uniformly interpretable in a m.a.c. is a weak m.a.c..

Examples

1. (Garcia, M, Steinhorn) Class of 2-sorted structures $(V, \mathbb{F}_q)$, with V a finite-dimensional vector space over $\mathbb{F}_q$. Here, given a formula $\phi(\bar{x}, \bar{y})$ there is a finite set E_ϕ of polynomials $g(\mathbb{V}, \mathbb{F})$ over $\mathbb{Q}$ such that if $M = (V, F)$ then each $h_P(M)$ has form $g(|V|, |F|)$ for some $g \in E_\phi$.

Examples

1. (Garcia, M, Steinhorn) Class of 2-sorted structures $(V, \mathbb{F}_q)$, with V a finite-dimensional vector space over $\mathbb{F}_q$. Here, given a formula $\phi(\bar{x}, \bar{y})$ there is a finite set E_ϕ of polynomials $g(\mathbb{V}, \mathbb{F})$ over $\mathbb{Q}$ such that if $M = (V, F)$ then each $h_P(M)$ has form $g(|V|, |F|)$ for some $g \in E_\phi$.

Note that ultraproducts are supersimple, but the V-sort may have rank ω.

Examples

1. (Garcia, M, Steinhorn) Class of 2-sorted structures $(V, \mathbb{F}_q)$, with V a finite-dimensional vector space over $\mathbb{F}_q$. Here, given a formula $\phi(\bar{x}, \bar{y})$ there is a finite set E_ϕ of polynomials $g(\mathbb{V}, \mathbb{F})$ over $\mathbb{Q}$ such that if $M = (V, F)$ then each $h_P(M)$ has form $g(|V|, |F|)$ for some $g \in E_\phi$.
Note that ultraproducts are supersimple, but the V-sort may have rank ω.

2. More generally, fix a quiver Q (digraph) of **finite representation type** (A_n, D_n, E_6, E_7, E_8). Over the field F, this has a finite-dimensional **path algebra** FQ, which has finitely many isomorphism types of indecomposable representations. Let

$$\mathcal{C}_Q := \{(V, FQ, F) : F \text{ finite field }, V \text{ finite module for } FQ\}$$

(3-sorted, with the natural language). Then $\mathcal{C}_Q$ is an R-mac with the functions h given by polynomials $g(F, W_1, \ldots, W_t)$, where the W_i variables correspond to the indecomposables for the quiver Q.

Examples

3. (Bello Aguirre) In the language of rings, for fixed $d \in \mathbb{N}$, let $\mathcal{C}_d$ be the collection of all finite residue rings $\mathbb{Z}/n\mathbb{Z}$, where n is a product of powers of at most d primes, each with exponent at most d.

Examples

3. (Bello Aguirre) In the language of rings, for fixed $d \in \mathbb{N}$, let $\mathcal{C}_d$ be the collection of all finite residue rings $\mathbb{Z}/n\mathbb{Z}$, where n is a product of powers of at most d primes, each with exponent at most d.

Then $\mathcal{C}_d$ is a weak m.a.c., and is a m.a.c. after appropriate expansion by unary predicates. If just one prime is involved, this is an asymptotic class (e.g. $\{\mathbb{Z}/p^2\mathbb{Z} : p \text{ prime}\}$ is a 2-dim asymptotic class). Ultraproducts are supersimple of finite SU-rank.

Examples

3. (Bello Aguirre) In the language of rings, for fixed $d \in \mathbb{N}$, let $\mathcal{C}_d$ be the collection of all finite residue rings $\mathbb{Z}/n\mathbb{Z}$, where n is a product of powers of at most d primes, each with exponent at most d.

Then $\mathcal{C}_d$ is a weak m.a.c., and is a m.a.c. after appropriate expansion by unary predicates. If just one prime is involved, this is an asymptotic class (e.g. $\{\mathbb{Z}/p^2\mathbb{Z} : p \text{ prime}\}$ is a 2-dim asymptotic class). Ultraproducts are supersimple of finite SU-rank.

(Idea: $\mathbb{Z}/p^d\mathbb{Z}$ is coordinatised uniformly by $\mathbb{Z}/p\mathbb{Z}$.)

Generalised measurable structures

Let $(S, +, \cdot, 0, 1, <)$ be a (commutative) ordered semiring (so $(S, +, 0)$, $(S, \cdot, 1)$ are commutative monoids, least element 0, etc.).

Generalised measurable structures

Let $(S, +, \cdot, 0, 1, <)$ be a (commutative) ordered semiring (so $(S, +, 0)$, $(S, \cdot, 1)$ are commutative monoids, least element 0, etc.).
Define $\sim$ on S with $a \sim b$ iff $a \leq b \leq na$ or $b \leq a \leq nb$ for some $n \in \mathbb{N}$.

Generalised measurable structures

Let $(S, +, \cdot, 0, 1, <)$ be a (commutative) ordered semiring (so $(S, +, 0)$, $(S, \cdot, 1)$ are commutative monoids, least element 0, etc.).
Define $\sim$ on S with $a \sim b$ iff $a \leq b \leq na$ or $b \leq a \leq nb$ for some $n \in \mathbb{N}$.
Put $D := S/\sim$, and $d_S : S \to D$ the natural 'dimension' map. Say S is a **measuring semiring** if

$$\forall x, y, z \in S((x < y \wedge d(y) = d(z)) \to x + z < y + z).$$

Generalised measurable structures

Let $(S, +, \cdot, 0, 1, <)$ be a (commutative) ordered semiring (so $(S, +, 0)$, $(S, \cdot, 1)$ are commutative monoids, least element 0, etc.).
Define $\sim$ on S with $a \sim b$ iff $a \le b \le na$ or $b \le a \le nb$ for some $n \in \mathbb{N}$.
Put $D := S/\sim$, and $d_S : S \to D$ the natural 'dimension' map. Say S is a **measuring semiring** if

$$\forall x, y, z \in S((x < y \wedge d(y) = d(z)) \to x + z < y + z).$$

Let S be a measuring semiring and let M be an L-structure. We say that M is **S-measurable** if there is a function $h : \mathrm{Def}(M) \longrightarrow S$ such that

1. *finite sets*: $h(X) = |X|$ for finite X;
2. *finite additivity*: if $X, Y \in \mathrm{Def}(M)$ are disjoint, $h(X \cup Y) = h(X) + h(Y)$;
3. *mac condition*: for each definable family $\mathcal{X}$ (given by a formula $\phi(\bar{x}, \bar{y})$) there exists a finite set $F \subseteq S$ such that $h(\mathcal{X}) = F$ and for each $f \in F$, $h^{-1}(f)$ is a $\emptyset$-definable family; and
4. *Fubini*: suppose $p : X \longrightarrow Y$ is a definable function and there exists $f \in S$ such that for all $\bar{a} \in Y$, $h(p^{-1}(\bar{a})) = f$; then we have $h(X) = f \cdot h(Y)$.

In finite fields, by CDM, definable sets had size roughly μq^d. Likewise, if M is S-measurable, and D is formed as above via $d_S : S \to D$, then $(D, \max, \oplus, -\infty, 0, <)$ has a semiring structure (with max and $\oplus$ induced from + and $\times$ via d_S), and we may form

$$E = \mathbb{R}^{\geq 0}X^D = \{\mu X^d : \mu \in \mathbb{R}^{\geq 0}, d \in D\}.$$

In finite fields, by CDM, definable sets had size roughly μq^d. Likewise, if M is S-measurable, and D is formed as above via $d_S : S \to D$, then $(D, \max, \oplus, -\infty, 0, <)$ has a semiring structure (with max and $\oplus$ induced from + and $\times$ via d_S), and we may form

$$E = \mathbb{R}^{\geq 0} X^D = \{\mu X^d : \mu \in \mathbb{R}^{\geq 0}, d \in D\}.$$

Here $\mu_1 X^{d_1} \times \mu_2 X^{d_2} = \mu_1 \mu_2 X^{d_1 \oplus d_2}$ and

$$\mu_1 X^{d_1} + \mu_2 X^{d_2} = \begin{cases} (\mu_1 + \mu_2) X^{d_1} & d_1 = d_2 \\ \max\{\mu_1 X^{d_1}, \mu_2 X^{d_2}\} & \text{otherwise} \end{cases}$$

In finite fields, by CDM, definable sets had size roughly μq^d. Likewise, if M is S-measurable, and D is formed as above via $d_S : S \to D$, then $(D, \max, \oplus, -\infty, 0, <)$ has a semiring structure (with max and $\oplus$ induced from + and $\times$ via d_S), and we may form

$$E = \mathbb{R}^{\geq 0} X^D = \{\mu X^d : \mu \in \mathbb{R}^{\geq 0}, d \in D\}.$$

Here $\mu_1 X^{d_1} \times \mu_2 X^{d_2} = \mu_1 \mu_2 X^{d_1 \oplus d_2}$ and

$$\mu_1 X^{d_1} + \mu_2 X^{d_2} = \begin{cases} (\mu_1 + \mu_2) X^{d_1} & d_1 = d_2 \\ \max\{\mu_1 X^{d_1}, \mu_2 X^{d_2}\} & \text{otherwise} \end{cases}$$

Now E is a measuring semiring, there is a dimension map $d_E : E \to D$ with $d_E(\mu X^d) = d$, and a semiring homomorphism $\phi : S \to E$ with $d_S = d_E \circ \phi$.

In finite fields, by CDM, definable sets had size roughly μq^d. Likewise, if M is S-measurable, and D is formed as above via $d_S : S \to D$, then $(D, \max, \oplus, -\infty, 0, <)$ has a semiring structure (with max and $\oplus$ induced from + and $\times$ via d_S), and we may form

$$E = \mathbb{R}^{\geq 0} X^D = \{\mu X^d : \mu \in \mathbb{R}^{\geq 0}, d \in D\}.$$

Here $\mu_1 X^{d_1} \times \mu_2 X^{d_2} = \mu_1 \mu_2 X^{d_1 \oplus d_2}$ and

$$\mu_1 X^{d_1} + \mu_2 X^{d_2} = \begin{cases} (\mu_1 + \mu_2) X^{d_1} & d_1 = d_2 \\ \max\{\mu_1 X^{d_1}, \mu_2 X^{d_2}\} & \text{otherwise} \end{cases}$$

Now E is a measuring semiring, there is a dimension map $d_E : E \to D$ with $d_E(\mu X^d) = d$, and a semiring homomorphism $\phi : S \to E$ with $d_S = d_E \circ \phi$.

In particular, M is E-measurable.

In finite fields, by CDM, definable sets had size roughly μq^d. Likewise, if M is S-measurable, and D is formed as above via $d_S : S \to D$, then $(D, \max, \oplus, -\infty, 0, <)$ has a semiring structure (with max and $\oplus$ induced from + and $\times$ via d_S), and we may form

$$E = \mathbb{R}^{\geq 0} X^D = \{\mu X^d : \mu \in \mathbb{R}^{\geq 0}, d \in D\}.$$

Here $\mu_1 X^{d_1} \times \mu_2 X^{d_2} = \mu_1 \mu_2 X^{d_1 \oplus d_2}$ and

$$\mu_1 X^{d_1} + \mu_2 X^{d_2} = \begin{cases} (\mu_1 + \mu_2) X^{d_1} & d_1 = d_2 \\ \max\{\mu_1 X^{d_1}, \mu_2 X^{d_2}\} & \text{otherwise} \end{cases}$$

Now E is a measuring semiring, there is a dimension map $d_E : E \to D$ with $d_E(\mu X^d) = d$, and a semiring homomorphism $\phi : S \to E$ with $d_S = d_E \circ \phi$.

In particular, M is E-measurable.

Proposition. If M is S-measurable, and the associated set of dimensions $(d_S \circ h)(\mathrm{Def}(M))$ is well-ordered, then M is supersimple. (Idea: Forking ensures drop in dimension.)

Generalised measurable structures

Proposition. Let M be (weakly) generalised measurable. Then

(i) M does not have the strict order property (i.e. there is no definable partial order on any M^n with an infinite totally ordered subset);

Generalised measurable structures

Proposition. Let M be (weakly) generalised measurable. Then

(i) M does not have the strict order property (i.e. there is no definable partial order on any M^n with an infinite totally ordered subset);

(ii) M is **functionally unimodular**, that is, if $f_i : A \to B$ (for $i = 1, 2$) are definable surjections with f_i k_i-to-1, then $k_1 = k_2$.

Generalised measurable structures

Proposition. Let M be (weakly) generalised measurable. Then

(i) M does not have the strict order property (i.e. there is no definable partial order on any M^n with an infinite totally ordered subset);

(ii) M is **functionally unimodular**, that is, if $f_i : A \to B$ (for $i = 1, 2$) are definable surjections with f_i k_i-to-1, then $k_1 = k_2$.

Note: It follows from (ii) that $(\mathbb{C}, +, \times)$ is not generalised measurable. In fact, by an argument of Scanlon, if K is a generalised measurable field then $\mathrm{Aut}(K^{\mathrm{alg}}/K) \cong (\mathbb{Z}, +)$.

Generalised measurable structures

Proposition. Let M be (weakly) generalised measurable. Then

(i) M does not have the strict order property (i.e. there is no definable partial order on any M^n with an infinite totally ordered subset);

(ii) M is **functionally unimodular**, that is, if $f_i : A \to B$ (for $i = 1, 2$) are definable surjections with f_i k_i-to-1, then $k_1 = k_2$.

Note: It follows from (ii) that $(\mathbb{C}, +, \times)$ is not generalised measurable. In fact, by an argument of Scanlon, if K is a generalised measurable field then $\mathrm{Aut}(K^{\mathrm{alg}}/K) \cong (\mathbb{Z}, +)$.

Example (Anscombe). If M is a Fraïssé limit of a free amalgamation class then M is generalised measurable (note for example the generic triangle-free graph is such a Fraïssé limit and has TP1 and TP2 theory).

Generalised measurable structures

Proposition.
1. If $\mathcal{C}$ is a m.a.c. then any ultraproduct is generalised measurable (so NSOP, functionally unimodular, etc.)

Generalised measurable structures

Proposition.
1. If $\mathcal{C}$ is a m.a.c. then any ultraproduct is generalised measurable (so NSOP, functionally unimodular, etc.)

2. If $\mathcal{C}$ is a m.e.c. (rather than just m.a.c.) then any ultraproduct is T-measurable for some ordered **ring** T (rather than just semiring).

Generalised measurable structures

Proposition.
1. If $\mathcal{C}$ is a m.a.c. then any ultraproduct is generalised measurable (so NSOP, functionally unimodular, etc.)

2. If $\mathcal{C}$ is a m.e.c. (rather than just m.a.c.) then any ultraproduct is T-measurable for some ordered **ring** T (rather than just semiring).

Note. The above supersimplicity result applies to ultraproducts of examples like

$$\{(V, \mathbb{F}_q) : q \text{ prime power }, V \text{ finite dim. over } \mathbb{F}_q\}$$

and the quiver example, where the defining functions are given by polynomials in several variables, so the corresponding set of dimensions is well-ordered (they are given by the polynomial degrees, which are ordered like $\mathbb{N}^d$).

Examples of m.e.c.s

1. (Essentially by Pillay) Let M be any pseudofinite strongly minimal set. Then there is a m.e.c. whose infinite ultraproducts are all elementarily equivalent to M, with the functions determining cardinalities given as polynomials (over $\mathbb{Z}$) in the cardinalities of the finite structures.

Examples of m.e.c.s

1. (Essentially by Pillay) Let M be any pseudofinite strongly minimal set. Then there is a m.e.c. whose infinite ultraproducts are all elementarily equivalent to M, with the functions determining cardinalities given as polynomials (over $\mathbb{Z}$) in the cardinalities of the finite structures.

2. (Wolf, based on Cherlin-Hrushovski) For a fixed language L and $d \in \mathbb{N}$, let $\mathcal{C}_{L,d}$ be the collection of all finite L-structures M with at most d 4-types (equivalently, $\mathrm{Aut}(M)$ has at most d orbits on M^4). Then $\mathcal{C}_{L,d}$ is a m.e.c (functions given by polynomials in the coordinatising Lie geometries).

Examples of m.e.c.s

1. (Essentially by Pillay) Let M be any pseudofinite strongly minimal set. Then there is a m.e.c. whose infinite ultraproducts are all elementarily equivalent to M, with the functions determining cardinalities given as polynomials (over $\mathbb{Z}$) in the cardinalities of the finite structures.

2. (Wolf, based on Cherlin-Hrushovski) For a fixed language L and $d \in \mathbb{N}$, let $\mathcal{C}_{L,d}$ be the collection of all finite L-structures M with at most d 4-types (equivalently, $\mathrm{Aut}(M)$ has at most d orbits on M^4). Then $\mathcal{C}_{L,d}$ is a m.e.c (functions given by polynomials in the coordinatising Lie geometries).

3. For any fixed d, the class $\mathcal{C}_d$ of finite graphs of degree at most d is a m.e.c.

Examples of m.e.c.s

1. (Essentially by Pillay) Let M be any pseudofinite strongly minimal set. Then there is a m.e.c. whose infinite ultraproducts are all elementarily equivalent to M, with the functions determining cardinalities given as polynomials (over $\mathbb{Z}$) in the cardinalities of the finite structures.

2. (Wolf, based on Cherlin-Hrushovski) For a fixed language L and $d \in \mathbb{N}$, let $\mathcal{C}_{L,d}$ be the collection of all finite L-structures M with at most d 4-types (equivalently, $\mathrm{Aut}(M)$ has at most d orbits on M^4). Then $\mathcal{C}_{L,d}$ is a m.e.c (functions given by polynomials in the coordinatising Lie geometries).

3. For any fixed d, the class $\mathcal{C}_d$ of finite graphs of degree at most d is a m.e.c.

Note. The class of finite fields is not even a **weak** m.e.c..

Examples of m.e.c.s

1. (Essentially by Pillay) Let M be any pseudofinite strongly minimal set. Then there is a m.e.c. whose infinite ultraproducts are all elementarily equivalent to M, with the functions determining cardinalities given as polynomials (over $\mathbb{Z}$) in the cardinalities of the finite structures.

2. (Wolf, based on Cherlin-Hrushovski) For a fixed language L and $d \in \mathbb{N}$, let $\mathcal{C}_{L,d}$ be the collection of all finite L-structures M with at most d 4-types (equivalently, $\mathrm{Aut}(M)$ has at most d orbits on M^4). Then $\mathcal{C}_{L,d}$ is a m.e.c (functions given by polynomials in the coordinatising Lie geometries).

3. For any fixed d, the class $\mathcal{C}_d$ of finite graphs of degree at most d is a m.e.c.

Note. The class of finite fields is not even a **weak** m.e.c..

(If $(m, q) = 1$ and $|m| \leq 2\sqrt{q}$ there is an elliptic curve E over $\mathbb{F}_q$ with $q + 1 - m$ $\mathbb{F}_q$-rational points.)

Examples of m.e.c.s

1. (Help from Kestner) The class of all finite abelian groups is a m.e.c. (in fact, for any fixed finite ring R, this holds for the class of all finite R-modules).

Examples of m.e.c.s

1. (Help from Kestner) The class of all finite abelian groups is a m.e.c. (in fact, for any fixed finite ring R, this holds for the class of all finite R-modules).

2. If $\mathcal{C}$ is a m.e.c. of groups, then there is $d \in \mathbb{N}$ such that the groups in $\mathcal{C}$ have (uniformly definable) soluble radical $R(G)$ of index at most d, and $R(G)/F(G)$ has derived length at most d, where $F(G)$ is the largest nilpotent normal subgroup of G.

Examples of m.e.c.s

1. (Help from Kestner) The class of all finite abelian groups is a m.e.c. (in fact, for any fixed finite ring R, this holds for the class of all finite R-modules).

2. If $\mathcal{C}$ is a m.e.c. of groups, then there is $d \in \mathbb{N}$ such that the groups in $\mathcal{C}$ have (uniformly definable) soluble radical $R(G)$ of index at most d, and $R(G)/F(G)$ has derived length at most d, where $F(G)$ is the largest nilpotent normal subgroup of G.

Question. Does every m.e.c. of finite groups consist of nilpotent-by-bounded groups?

Examples of m.e.c.s

1. (Help from Kestner) The class of all finite abelian groups is a m.e.c. (in fact, for any fixed finite ring R, this holds for the class of all finite R-modules).

2. If $\mathcal{C}$ is a m.e.c. of groups, then there is $d \in \mathbb{N}$ such that the groups in $\mathcal{C}$ have (uniformly definable) soluble radical $R(G)$ of index at most d, and $R(G)/F(G)$ has derived length at most d, where $F(G)$ is the largest nilpotent normal subgroup of G.

Question. Does every m.e.c. of finite groups consist of nilpotent-by-bounded groups?

Problem. Find a m.e.c. with ultraproduct having non-simple theory.

Homogeneous structures as limits of m.e.c.s

Conjecture. If M is a homogeneous structure over a finite relational language, then the following are equivalent.
(1) There is a m.e.c with ultraproduct elementarily equivalent to M.
(2) M is stable.

Homogeneous structures as limits of m.e.c.s

Conjecture. If M is a homogeneous structure over a finite relational language, then the following are equivalent.
(1) There is a m.e.c with ultraproduct elementarily equivalent to M.
(2) M is stable.

Remarks. 1. The direction (2) $\Rightarrow$ (1) follows from Lachlan + Wolf.

Homogeneous structures as limits of m.e.c.s

Conjecture. If M is a homogeneous structure over a finite relational language, then the following are equivalent.
(1) There is a m.e.c with ultraproduct elementarily equivalent to M.
(2) M is stable.

Remarks. 1. The direction (2) $\Rightarrow$ (1) follows from Lachlan + Wolf.

2. The Paley graphs form a m.a.c (but not m.e.c) with limit the random graph, which is unstable. (Paley graph P_q has vertex set $\mathbb{F}_q$ where $q \equiv 1 \pmod 4$, with x, y adjacent iff $x - y$ is a square.)

Homogeneous structures as limits of m.e.c.s

Conjecture. If M is a homogeneous structure over a finite relational language, then the following are equivalent.
(1) There is a m.e.c with ultraproduct elementarily equivalent to M.
(2) M is stable.

Remarks. 1. The direction (2) $\Rightarrow$ (1) follows from Lachlan + Wolf.

2. The Paley graphs form a m.a.c (but not m.e.c) with limit the random graph, which is unstable. (Paley graph P_q has vertex set $\mathbb{F}_q$ where $q \equiv 1 \pmod 4$, with x, y adjacent iff $x - y$ is a square.)

3. The Conjecture holds for graphs, by the Lachlan-Woodrow classification of homogeneous graphs and...

Theorem. Let M be any of the following homogeneous structures. Then there is no m.e.c. with an ultraproduct elementarily equivalent to M.

Theorem. Let M be any of the following homogeneous structures. Then there is no m.e.c. with an ultraproduct elementarily equivalent to M.

(i) Any unstable homogeneous graph (using Cameron 1980 + Cameron-Goethals-Seidel 1978).

Theorem. Let M be any of the following homogeneous structures. Then there is no m.e.c. with an ultraproduct elementarily equivalent to M.

(i) Any unstable homogeneous graph (using Cameron 1980 + Cameron-Goethals-Seidel 1978).

(ii) Any homogeneous tournament (digraph such that for $x \neq y$, exactly one of $x \to y$ or $y \to x$ holds).

Theorem. Let M be any of the following homogeneous structures. Then there is no m.e.c. with an ultraproduct elementarily equivalent to M.

(i) Any unstable homogeneous graph (using Cameron 1980 + Cameron-Goethals-Seidel 1978).

(ii) Any homogeneous tournament (digraph such that for $x \neq y$, exactly one of $x \to y$ or $y \to x$ holds).

(iii) The digraph D_n for each $n \geq 3$ (universal subject to omitting an independent set I_n)

Theorem. Let M be any of the following homogeneous structures. Then there is no m.e.c. with an ultraproduct elementarily equivalent to M.

(i) Any unstable homogeneous graph (using Cameron 1980 + Cameron-Goethals-Seidel 1978).

(ii) Any homogeneous tournament (digraph such that for $x \neq y$, exactly one of $x \to y$ or $y \to x$ holds).

(iii) The digraph D_n for each $n \geq 3$ (universal subject to omitting an independent set I_n)

(iv) The generic bipartite graph.

Theorem. Let M be any of the following homogeneous structures. Then there is no m.e.c. with an ultraproduct elementarily equivalent to M.

(i) Any unstable homogeneous graph (using Cameron 1980 + Cameron-Goethals-Seidel 1978).

(ii) Any homogeneous tournament (digraph such that for $x \neq y$, exactly one of $x \to y$ or $y \to x$ holds).

(iii) The digraph D_n for each $n \geq 3$ (universal subject to omitting an independent set I_n)

(iv) The generic bipartite graph.

(v) (Ainslie) The universal homogeneous two-graph (a 3-uniform hypergraph reduct of the random graph, where the 3-edges consist of 3-sets containing an odd number of random graph edges.)

Proof of (ii) above, that the universal homogeneous tournament is not elementary equivalent to an ultraproduct of a m.e.c..

Proof of (ii) above, that the universal homogeneous tournament is not elementary equivalent to an ultraproduct of a m.e.c..

For a contradiction, consider a m.e.c. $\mathcal{C}$ of finite tournaments with all non-principal ultraproducts $\equiv$ the random tournament.

Proof of (ii) above, that the universal homogeneous tournament is not elementary equivalent to an ultraproduct of a m.e.c..

For a contradiction, consider a m.e.c. $\mathcal{C}$ of finite tournaments with all non-principal ultraproducts $\equiv$ the random tournament.

1. Any finite regular tournament has indegree equal to outdegree, so has an odd number of vertices (count in 2 ways the pairs (x, y) with $x \to y$).

Proof of (ii) above, that the universal homogeneous tournament is not elementary equivalent to an ultraproduct of a m.e.c..

For a contradiction, consider a m.e.c. $\mathcal{C}$ of finite tournaments with all non-principal ultraproducts $\equiv$ the random tournament.

1. Any finite regular tournament has indegree equal to outdegree, so has an odd number of vertices (count in 2 ways the pairs (x, y) with $x \to y$).

2. For any formula $\phi(\bar{x}, \bar{y})$, in a large enough finite tournament $M \in \mathcal{C}$ the cardinality $|\phi(M, \bar{a})|$ depends just on the isomorphism type of $\bar{a}$ (uses QE, + definability clause of m.e.c.).

Proof of (ii) above, that the universal homogeneous tournament is not elementary equivalent to an ultraproduct of a m.e.c..

For a contradiction, consider a m.e.c. $\mathcal{C}$ of finite tournaments with all non-principal ultraproducts $\equiv$ the random tournament.

1. Any finite regular tournament has indegree equal to outdegree, so has an odd number of vertices (count in 2 ways the pairs (x, y) with $x \to y$).

2. For any formula $\phi(\bar{x}, \bar{y})$, in a large enough finite tournament $M \in \mathcal{C}$ the cardinality $|\phi(M, \bar{a})|$ depends just on the isomorphism type of $\bar{a}$ (uses QE, + definability clause of m.e.c.).

3. If $M \in \mathcal{C}$ is large enough then M is regular, so $|M|$ is odd, by (1).

Proof of (ii) above, that the universal homogeneous tournament is not elementary equivalent to an ultraproduct of a m.e.c..

For a contradiction, consider a m.e.c. $\mathcal{C}$ of finite tournaments with all non-principal ultraproducts $\equiv$ the random tournament.

1. Any finite regular tournament has indegree equal to outdegree, so has an odd number of vertices (count in 2 ways the pairs (x, y) with $x \to y$).

2. For any formula $\phi(\bar{x}, \bar{y})$, in a large enough finite tournament $M \in \mathcal{C}$ the cardinality $|\phi(M, \bar{a})|$ depends just on the isomorphism type of $\bar{a}$ (uses QE, + definability clause of m.e.c.).

3. If $M \in \mathcal{C}$ is large enough then M is regular, so $|M|$ is odd, by (1).

4. If M is large enough finite and a, b are distinct vertices, then the tournaments M and on the sets $\{x : a, b \to x\}$, $\{x : x \to a, b\}$, $\{x : a \to x \to b\}$ and $\{x : b \to x \to a\}$ are all regular, so all of odd size.

Proof of (ii) above, that the universal homogeneous tournament is not elementary equivalent to an ultraproduct of a m.e.c..

For a contradiction, consider a m.e.c. $\mathcal{C}$ of finite tournaments with all non-principal ultraproducts $\equiv$ the random tournament.

1. Any finite regular tournament has indegree equal to outdegree, so has an odd number of vertices (count in 2 ways the pairs (x, y) with $x \to y$).

2. For any formula $\phi(\bar{x}, \bar{y})$, in a large enough finite tournament $M \in \mathcal{C}$ the cardinality $|\phi(M, \bar{a})|$ depends just on the isomorphism type of $\bar{a}$ (uses QE, + definability clause of m.e.c.).

3. If $M \in \mathcal{C}$ is large enough then M is regular, so $|M|$ is odd, by (1).

4. If M is large enough finite and a, b are distinct vertices, then the tournaments M and on the sets $\{x : a, b \to x\}$, $\{x : x \to a, b\}$, $\{x : a \to x \to b\}$ and $\{x : b \to x \to a\}$ are all regular, so all of odd size.

5. $|M| =$ the sum of four odd numbers +2, which is even – contradicting (3)!