Model Theory of Adeles II

Jamshid Derakhshan
St Hildas College, University of Oxford
Joint work with
Angus Macintyre
University of London, University of Edinburgh

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Model Theory of Restricted Products
(Feferman-Vaught & beyond)

This started with the work of Feferman-Vaught in 1959. They called these structures weak products But most of their analysis was done for direct products. We revisited this topic. Extended their work to restricted products and in most generality for many-sorted languages with relation and function symbols. (DM, APAL 2014, 2020).

This was needed for model theory of adeles. Later we proved an analogue for rings which gave a Converse to the Feferman-Vaught theorems. This gave general axioms for restricted products (DM 2020).

Here I present the basic case.

Let L be a language and $(\mathcal{M}_i)_{i\in I}$ an indexed family of L-structures Given an L-formula $\Phi(x_1,\dots,x_n)$ and elements $a_1,\dots,a_n \in \prod_{i\in I}\mathcal{M}_i$, define the Boolean value:

$$[\![\Phi(a_1,\dots,a_n)]\!] = \{ i\in I \mid \mathcal{M}_i \models \Phi(\underbrace{a_1(i)}_{(a_1(i))},\dots,a_n(i))\}$$

Let $\mathcal{L}_{Boolean} = \{\wedge,\vee,\neg,0,1\}$ be the language of Boolean algebras $\underbrace{\mathcal{L}^+_{Boolean}}$ denote an expansion of $\mathcal{L}_{Boolean}$.

Let $\mathbb{P}(I)$ [illegible]

algebras $\mathcal{L}'_{Boolean}$ denote an expansion of $\mathcal{L}_{Boolean}$.

Let $\mathbb{P}(I)$ denote the powerset of I. Consider it as an enriched Boolean algebra and an $\mathcal{L}^+_{Boolean}$-structure $\mathbb{P}(I)^+$

For any $\mathcal{L}^+_{Boolean}$-formula $\Theta(z_1,\dots,z_m)$ and any L-formulas $\varphi_1(x_1,\dots,x_n), \dots, \varphi_m(x_1,\dots,x_n)$ let $\Theta \circ \langle \varphi_1,\dots,\varphi_m \rangle$ denote the relation on $\prod_{i\in I} \mathcal{M}_i$ defined by

$$\prod_{i\in I} \mathcal{M}_i \models \Theta \circ \langle \varphi_1,\dots,\varphi_m\rangle (a_1,\dots,a_n) \iff$$

$$\mathbb{P}(I)^+ \models \Theta([\![\varphi_1(a_1,\dots,a_n)]\!], \dots, [\![\varphi_m(a_1,\dots,a_n)]\!]).$$

Here $a_i \in \prod_{i\in I} \mathcal{M}_i$.

Extend L by adding a new relation symbol, of appropriate arity, for each $\langle \Theta, \varphi_1,\dots,\varphi_m\rangle$. $\prod \models R(a_1,\dots,a_n)$

$\mathbb{P}(I) \models [\![R(a_1,\dots,a_n)]\!] = 1$

We denote the resulting language by $\mathcal{L}^+_{Boolean}(L)$

$\prod_{i\in I} \mathcal{M}_i$ has thus been given an $\mathcal{L}^+_{Boolean}(L)$-structure

This was defined by Feferman-Vaught, who called it the language of generalized products.

Assume that we are given an L-formula $\Phi(x)$ and that for all $i\in I$, the set $\Phi(\mathcal{M}_i)$ is an L-substructure of $\mathcal{M}_i$.

Then we define the restricted product of $\mathcal{M}_i$ with respect to $\Phi(x)$ to be

$$\prod_{i\in I}{}^{\Phi} \mathcal{M}_i = \left\{ (a(i))_i \in \prod_{i\in I} \mathcal{M}_i \;\middle|\; \mathcal{M}_i \models \Phi(a(i)) \text{ for all but finitely many } i\in I \right\}$$

$\mathcal{L}^+_{Boolean}(L)$-substructure of $\prod_{i\in I} \mathcal{M}_i$

Theorem: Let L, $\mathcal{L}^+_{Boolean}$, and $\Phi(x)$ be given. Then for any

Theorem: Let L, $\mathcal{L}^+_{Boolean}$, and $\Phi(x)$ be given. Then for any $\mathcal{L}^+_{Boolean}(L)$-formula $\underbrace{\varphi(x_1, \ldots, x_n)}$ there are L-formulas $\psi_1, \ldots, \psi_m$ in the same variables as φ, and an $\mathcal{L}^+_{Boolean}$-formula $\theta(z_1, \ldots, z_m)$ such that for any indexed family of L-structures $(\mathcal{M}_i)_{i \in I}$ and any $a_1, \ldots, a_n \in \prod^{\Phi}_{i \in I} \mathcal{M}_i$ we have

$\cancel{\forall_w} \exists z$

$$\prod^{\Phi}_{i \in I} \mathcal{M}_i \models \varphi(a_1, \ldots, a_n) \iff$$

$$\mathbb{P}(I)^+ \models \theta([\![\psi_1(a_1, \ldots, a_n)]\!], \ldots, [\![\psi_m(a_1, \ldots, a_n)]\!])$$

This extends Feferman-Vaught for products.

We prove also many-sorted version

Recall that there is an $\exists\forall$ $\mathcal{L}_{rings}$-formula that defines uniformly the valuation ring of all Henselian valued fields with finite or pseudo-finite residue field (Cluckers-D-Leenknegt-Macintyre) APAL 2013

Let us call this formula $\underbrace{\Phi_{val}(x)}$

Therefore $\mathbb{A}^{fin}_K$ is the restricted direct product of the completions K_v, $v \in V_K$, with respect to $\underbrace{\Phi_{val}(x)}$

where $\mathbb{A}^{fin}_K = \left\{ (a(v)) \in \prod_{v \notin Arch(K)} K_v \mid a(v) \in \mathcal{O}_v \text{ for all but finitely many } v \right\}$

and $\mathbb{A}_K \cong \prod_{v \in Arch(K)} K_v \times \mathbb{A}^{fin}_K$ (algebraically & topologically)

Corollary Let $\varphi(x_1, \ldots, x_n)$ be an $\mathcal{L}_{rings}$-formula. Then there are $\mathcal{L}_{rings}$-formulas $\psi_1(x_1, \ldots, x_n), \ldots, \psi_m(x_1, \ldots, x_n)$ and a $\mathcal{L}_{rings}$-formula $\theta(z_1, \ldots, z_m)$ which is an addition a Boolean combination of $Fin(x)$ and $C_j(x)$, such that

$$\mathbb{A}_K \models \forall x_1, \ldots, \forall x_n \left(\varphi(x_1, \ldots, x_n) \iff \theta(\underbrace{[\![\psi_1(x_1, \ldots, x_n)]\!], \ldots, [\![\psi_m(x_1, \ldots, x_n)]\!]}_{\text{systems}}) \right).$$

$\otimes E$ for $\mathbb{A}_K$ showed

Theorem: $Fin(x)$, $C_j(x)$ are $\mathcal{L}_{rings}$-definable

$$\underset{\sim}{x} \in Fin_K \iff \mathbb{A}_K \models x = x^2 \wedge \forall y \exists e \exists w \exists z (e = e^2 \wedge yx = ew \wedge e = yxz).$$

$(0, 1, 1, 0, 1 \ldots)$

This says: x is idempotent and for all y, $\underset{\sim}{yx}$ is "von Neumann regular"

Thus Fin_K is $\forall\exists$-definable.

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Thus Fin_K is $\forall\exists$ - definable.

Note: Only true in $\mathbb{A}_K$ not the product.

Let $\mathrm{Fin}(x)$ be a 1-place predicate interpreted in an atomic Boolean algebra as there are only finitely many atoms $\leq x$ In $\mathbb{P}(\mathbb{N})$ as "finite"

Axioms for $\mathrm{Fin}(x)$ state that Fin defines a Boolean ideal and

$$\forall x \quad \neg \mathrm{Fin}(x) \Rightarrow \exists y \quad (y < x \wedge \neg \mathrm{Fin}(y) \wedge \neg \mathrm{Fin}(x \wedge \neg y))$$

$C_j(x)$ states $\exists$ at least j many distinct atoms $\leq x$

Theorem (Tarski). The theory T_{Bool} of infinite atomic Boolean algebras is complete and decidable and has quantifier-elimination in $\mathcal{L}_{Boolean} \cup \{C_j : j \geq 1\}$

Let $\mathcal{L}^{fin}_{Boolean} = \mathcal{L}_{Boolean} \cup \{C_j(x) : j \geq 1\} \cup \{\mathrm{Fin}(x)\}$

Theorem (FV, DM Fund Math 2017). The theory T^{fin} of infinite atomic Boolean algebras in the language $\mathcal{L}^{fin}_{Boolean}$ is complete, decidable, and has quantifier-elimination

New enrichments. Hilbert reciprocity Gauss QR

$\mathrm{Res}(n,m)(x)$ $\quad T^{fin, res}$

Theorem Definable subsets of $\mathbb{A}_K^m$ are Boolean combinations of sets defined by $\mathrm{Fin}([\![\psi(x_1,\dots,x_m)]\!])$ and $C_j([\![\phi(x_1,\dots,x_m)]\!])$

Theorem 0-Definable subsets of the set of minimal idempotents of $\mathbb{A}_K$ (primes of K) are (after removing finitely many min. idempotents corresponding to Arch valuations) finite unions of sets of the form

$\{v \in V_K^{fin} \mid k_v \models \sigma\}$, σ an $\mathcal{L}_{rings}$ - Sentence

By Ax If $K = \mathbb{Q}$, these sets Boolean combinations of sets of the form

$\{p$: the reduction of $f(x)$ (mod p) has a root in $\mathbb{F}_p\}$, $f(x) \in \mathbb{Z}[x]$.

Proof of decidability of $\mathbb{A}_\mathbb{Q}$ using Ax 1968 on $\{\mathbb{F}_p \; p \text{ prime}\}$ and Our quantifier elimination theorem

19.1. **A proof of decidability of $\mathbb{A}_\mathbb{Q}$.**

We give a short proof that $\mathbb{A}_\mathbb{Q}$ is decidable in the language of rings and in the other languages from Section 8 using Ax's work on finite fields [2] and Corollary 7.1.

The first proof of decidability of $\mathbb{A}_K$, K a number field, was given by Weispfenning in [52].

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We first give a proof of decidability for $\mathbb{A}_{\mathbb{Q}}$ in the language of rings, and then show it can be adapted to the case of the languages $\mathcal{L}^{+}_{Boolean}(L)$ where $\mathcal{L}^{+}_{Boolean}$ is $\mathcal{L}^{fin}_{Boolean}$ or $\mathcal{L}^{fin\_res}_{Boolean}$, and L is $\mathcal{L}_{rings}$, $\mathcal{L}_{Denef-Pas}$, $\mathcal{L}_{Pas}$, or $\mathcal{L}_{Basarab}$. The proof goes through when L is any 1-sorted or many-sorted language such that K_v have a.e. QE, which in the many-sorted case is for the field sort relative other decidable

sorts, and $\mathcal{L}^{+}_{Boolean}$ is any expansion of $\mathcal{L}_{Boolean}$ in which the theory of all infinite atomic Boolean algebras is complete and decidable. See Sections 8 and 6 for details on these languages.

Given an $\mathcal{L}_{rings}$-sentence φ, we want to decide if φ is true in $\mathbb{A}_{\mathbb{Q}}$. By Corollary 7.1, with an effective procedure, we are reduced to deciding the following statements:

(I) $Fin([[\phi]])$,

(II) $C_j([[\psi]])$,

where ϕ and ψ are $\mathcal{L}_{rings}$-sentences.

We shall use the following result of Ax [2] (cf. also [33], and Theorem 31.2.4 (a) in [32]).

Theorem 19.1. *Let ϕ be a sentence of $\mathcal{L}_{rings}$. It is possible to decide if ϕ is true in $\mathbb{F}_p$ for almost all p, and if so to list the exceptional primes.*

We now give the decision procedure for (I). Clearly it suffices to decide $Fin([[\phi]]^{na})$. This states that ϕ holds in finitely many $\mathbb{Q}_p$. Equivalently, it states that $\neg\phi$ holds in all but finitely many $\mathbb{Q}_p$.

By Theorem 8.1 (Ax-Kochen-Ershov) there is an $\mathcal{L}_{rings}$-sentence τ and an effectively computable $C > 0$ such that for any prime $p \geq C$ we have

$$\mathbb{Q}_p \models \neg\phi \Leftrightarrow \mathbb{F}_p \models \tau.$$

Thus it suffices to decide if τ holds in all but finitely many $\mathbb{F}_p$. This follows from 19.1.

Regarding (II), we need to decide if ϕ holds in at least j many K_v, where $j \geq 1$ is given. By (I) we can decide if $Fin([[\phi]])$ holds or not. If it does not hold, then $C_j([[\phi]])$ holds for all j. If $Fin([\phi]])$ holds, then we consider $\psi := \neg\phi$, which holds in all but finitely many $\mathbb{Q}_p$, and the exceptional primes are exactly the primes p where ϕ holds in $\mathbb{Q}_p$. By Theorem 19.1, we can list this finite set of primes and decide if this set has cardinality at least j or not, which is what we need. This completes the decision procedure for $\mathbb{A}_{\mathbb{Q}}$ in the language of rings.

Now we show how this proof can be adapted to the languages $\mathcal{L}^{fin}_{Boolean}(L)$, where L is $\mathcal{L}_{Denef-Pas}$, $\mathcal{L}_{Pas}$, or $\mathcal{L}_{Basarab}$. We shall equip the value group sort of these languages with the Presburger language to get decidabilty for the value group formulas.

Given an L-sentence ϕ, for sufficiently large p, ϕ is equivalent in $\mathbb{Q}_p$ to a sentence that involves only residue field ad value group sentences. The value group sentences can be decided and the residue field sentences can be decided as above.

For the finitely many exceptional p, ϕ is equivalent to a sentence that is quantifier free in the field sort relative to other sorts. The number of these other sorts is effectively computable, each such sort is finite (indeed the sorts are residue rings or higher residue rings in the case of $\mathcal{L}_{Denef-pas}$ or $\mathcal{L}_{Pas}$, or sorts from the

Axioms for Restricted Products,

Feferman-Vaught for Rings & Converse to Feferman-Vaught

1.1. **Atoms and Stalks.**

We follow as much as possible the development from Section 1 of [3].

Definition 1.2. *Let R be a commutative unital ring. The set $\{x : x = x^2\}$ of idempotents is a Boolean algebra, denoted by $\mathbb{B}$, with operations*

$$e \wedge f = ef,$$
$$\neg e = 1 - e,$$
$$0 = 0,$$
$$1 = 1,$$
$$e \vee f = 1 - (1-e)(1-f) = e + f - ef.$$

R carries a partial ordering defined by $e \leq f \Leftrightarrow ef = e$ (which is $\mathcal{L}_{rings}$-definable). The atoms of $\mathbb{B}$ are the minimal idempotents (with respect to the ordering) that are not equal to $0, 1$. (In fact we assume $0 \neq 1$).

Note that if R is a product, over an index set I, then $\mathbb{B}$ is isomorphic to the Boolean algebra of subsets of I via "characteristic functions".

Lemma 1.1. *For any e in $\mathbb{B}$ we have $R/(1-e)R \cong eR \cong R_e$, where R_e is the localization of R at $\{e^n : n \geq 0\}$.*

Proof. See Lemma 1 in [3]. □

We call R_e the stalk of R at e. Of special important are the R_e for *atoms e*.

Now we gradually impose axioms on R, in order to get a converse to Feferman-Vaught for restricted products.

Axiom 1. $\mathbb{B}$ is atomic.

Notes. This holds if R is a restricted product of *connected* rings. One does not even need the restricting formula $\varphi(x)$ to be definable. Moreover R and the unrestricted product have the same idempotents. The basic example is $\mathbb{A}_K$ embedded in $\prod_v K_v$.

Now we go through a series of consequences of the current axioms, and additions of new axioms.

Lemma 1.2. *If $f \in \mathbb{B}$, and $f \neq 0$, then $f = \bigvee\{e : e$ an atom, $e \leq f\}$, (where $\bigvee$ is union or supremum).*

Proof. This is Lemma 2 in [3]. □

We turn to Boolean values and follow 1.3 of [3].

Definition 1.3. *Let $\Theta(x_1, \ldots, x_n)$ be a formula of the language of rings, and $f_1, \ldots, f_n \in R$. Then $[[\Theta(f_1, \ldots, f_n)]]$ is defined as*

$$\bigvee_e \{e : e \text{ an atom, } R_e \models \Theta((f_1)_e, \ldots, (f_n)_e)\}$$

provided $\bigvee$ exists in $\mathbb{B}$. Here $(f)_e$ is the natural image of f in R_e (or, seen from perspective of Lemma 1.1, $f + (1-e)R$).

Axiom 2. $[[\Theta(f_1, \ldots, f_n)]]$ exists (as an element of $\mathbb{B}$). [FOR ALL F]

Notes. If R is a product of structures then $\mathbb{B}$ is complete, however completeness of a Boolean algebra is not a first-order property.
Axiom 2 is a substitute for completeness (and follows from it).
Axiom 2 is true in a restricted product of connected rings with respect to a given formula $\varphi(\bar{x})$.

1.2. **Boolean Values and Patching.**

The $[[\Phi(f_1, \ldots, f_n)]]$ are in $\mathbb{B}$, and occur in [10] in the context of products, with a different notation. The [[...]] notation comes from Boolean valued model theory [].
The next Lemmas come from 1.4 of [3].

Lemma 1.3. *Let $\Theta_1, \Theta_2, \Theta_3$ be $\mathcal{L}_{rings}$-formulas in the variables $x_1, \ldots, x_n$. Then for any $f_1, \ldots, f_n \in R$,*

- $[[(\Theta_1 \wedge \Theta_2)(f_1, \ldots, f_n)]] = [[\Theta_1(f_1, \ldots, f_n)]] \wedge [[\Theta_2(f_1, \ldots, f_n)]]$,

Axiom 3. For any atomic formula $\Theta(x_1, \ldots, x_n)$ of the language of rings,

$$R \models \Theta(f_1, \ldots, f_n) \Leftrightarrow \mathbb{B} \models [[\Theta(f_1, \ldots, f_n)]] = 1.$$

This is evidently true in restricted products, no matter what $\varphi(x)$ is.
Now we *fix* a $\varphi(x)$, and aim for axioms true in restricted direct products R with respect to $\varphi(x)$.
We come now to a fundamental point. Classically the notion of restricted product appeals to the absolute notion of finite which is not, of course, first-order. We are aiming for first-order axioms in some natural formalism. As already suggested, we are going to use an idea from Feferman -Vaught [10] of working with Boolean algebras $\mathbb{B}$ with a distinguished subset $\mathcal{F}in$, which in the case of the power set algebra is the ideal of finite sets. In the case of a Boolean algebra of idempotents, $\mathcal{F}in$ will be the set of finite idempotents, as explained earlier.
We will shortly be concerned with other interpretations of a predicate symbol for $\mathcal{F}in$, indispensable for understanding nonstandard models of our axioms (and in particular nonstandard models of the theory of the adeles.
But first we use provisional "axioms" where "finite" really means finite, and "finite idempotents " really mean finite idempotents, and "cofinite" really means cofinite.
We could avoid this ,and pass directly to the general case. But we prefer to discuss a provisional axiom connected to the kind of patching used in [10]

Axiom 4^+. For all $\Theta(x_1, \ldots, x_n, w)$, $f_1, \ldots, f_n$, there is a $g \in R$ such that if

$$[[\exists w(\varphi(w) \wedge \Theta(f_1, \ldots, f_n, w))]]$$

is cofinite in

$$[[\exists w \Theta(f_1, \ldots, f_n, w)]],$$

then $[[\exists w \Theta(f_1, \ldots, f_n, w)]]$ is cofinite in $[[\Theta(f_1, \ldots, f_n, g)]]$.
This is clearly true in restricted products with respect to $\varphi(x)$ (use Axiom of Choice).

Note. [3] has a simpler Axiom 4 for the unrestricted product case. That is not needed here.

Note. From now on, we will get involved with not only $\mathbb{B}$, but with the ideal $\mathcal{F}in$ in $\mathbb{B}$ consisting of finite elements of $\mathbb{B}$, i.e. finite unions of atoms.

We have to enrich the first-order language of Boolean algebras by a 1-ary predicate symbol $Fin(x)$. For our purposes $\mathbb{B}$ will be atomic as above, and $Fin(x)$ interpreted as the ideal of finite support idempotents. The interpretation of $Fin(x)$ in a Boolean algebra of sets, e.g. the power set $\mathcal{P}(I)$ of a set I is the (Boolean) ideal of finite sets.

However, note that Axiom 4^+ is *not* first-order. Any anxieties about this should be removed by considering the result that the theory of the class of all infinite atomic Boolean algebras in the language of Boolean algebras augmented by $Fin(x)$ is axiomatizable and complete (and admits quantifier elimination). This is proved first by Tarski but we give a new proof with explicit axioms in [9]. See also Section 1.4 below. [9] contains a unified treatment that includes further expansions by predicates for "congruence conditions on cardinality of finite sets".
We return to this matter later, reformulating Axiom 4^+ in terms of $Fin(x)$.

1.3. **Partitions.**

In order to give the proof of an analogue of the main result in [10] to our situation (rings R satisfying the axioms listed above) we need to review several notions of partition used in [10].

Notion 1. In a Boolean algebra B a partition is a finite sequence $< Y_1, \ldots, Y_m >$ of elements of B such that

$$Y_1 \vee \cdots \vee Y_m = 1$$

and $Y_i \wedge Y_j = 0$ if $i \neq j$. (We do not insist that each $Y_i \neq 0$, but do insist that the sequence is finite).

We note that in the definition of partition "finite" will always mean finite.

Axiom 4^{fin}. There is an ideal $\mathcal{F}in$ in $\mathbb{B}$ so that $(\mathbb{B}, \mathcal{F}in) \models T^{fin}$, and such that for all $\Theta(x_1, \dots, x_n, w)$, $f_1, \dots, f_n$ there is a $g \in R$ such that if

$$[[\exists w \Theta(f_1, \dots, f_n, w)]] \cap \neg[[\exists w(\varphi(w) \wedge \Theta(f_1, \dots, f_n, w))]] \in \mathcal{F}in,$$

then

$$[[\exists w \Theta(f_1, \dots, f_n, w)]] \cap \neg[[\Theta(f_1, \dots, f_n, g)]] \in \mathcal{F}in.$$

This is clearly true in classical products restricted by $\varphi(x)$ (use Axiom of Choice).

In Lemma 1.4 we need to change "finite" to "in Fin", and the proof goes through, getting

Lemma 1.5. *Suppose $Y_1, \dots, Y_m$ is a partition of $\mathbb{B}$. Suppose the sequence*

$$< \Theta_0(x_0, \dots, x_k, x_{k+1}), \dots, \Theta_m(x_0, \dots, x_k, x_{k+1}) >$$

is a partition. Suppose $f_1, \dots, f_k \in R$ and

$$Y_j \subseteq [[\exists x_{k+1} \Theta_j(f_0, \dots, f_k, x_{k+1})]]$$

for each j. Suppose in addition that for each j

$$Y_j \setminus [[\exists x_{k+1} \varphi(x_{k+1}) \wedge \Theta_j(f_1, \dots, f_k, x_{k+1})]] \in \mathcal{F}in$$

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Then there is a g in R so that

$$Y_j \subseteq [[\Theta_j(f_1, \dots, f_k, g)]]$$

for all j.

We now have Axioms 1-3 and Axiom 4^{fin}. Note that $\varphi(x)$, the restricting formula, is fixed.

There is one last Axiom 5.

Axiom 5. $\forall x(Fin([[\neg\varphi(x)]]))$.

Call the resulting axiom set $\mathcal{A}_\varphi$, axioms for φ-restricted products.

We have given axioms for pairs $(R, \mathcal{F}in)$, and we shall next prove that we have a Feferman-Vaught type theorem.

2. The Feferman-Vaught Theorem for Rings

2.1. The Main Theorem.

Theorem 2.1. *Let $\varphi(x)$ be an $\mathcal{L}$-formula. Let R a commutative unital ring satisfying the axioms $\mathcal{A}_\varphi$. Then for each $\mathcal{L}_{rings}$-formula $\Theta(x_0, \dots, x_m)$ there is, by an effective procedure, a partition*

$$< \Theta_0(x_0, \dots, x_m), \dots, \Theta_k(x_0, \dots, x_m) >$$

of $\mathcal{L}_{rings}$ formulas, and an $\mathcal{L}^{fin}_{Boolean}$ formula $\psi(y_0, \dots, y_k)$ such that for all $f_1, \dots, f_m$ in R

$$R \models \Theta(f_0, \dots, f_m) \Leftrightarrow$$
$$(\mathbb{B}, \mathcal{F}in) \models \psi([[\Theta_0(f_0, \dots, f_m), \dots, \Theta_k(f_0, \dots, f_m)]]).$$

Proof. In [10], there is a proof by induction on the complexity of Θ for products. That proof can be modified to go through for the case of restricted products with respect to a given formula φ (which is a substructure of a generalized product). This modification can be made to work in the case of our rings R. We give the details for the ring case. The interested reader can translate this to restricted products, extending [10].

If $\Theta(x_0, \dots, x_m)$ is a quantifier-free formula, then we can take the Boolean formula $[[\Theta(x_0, \dots, x_m)]] = 1$. Then for all $f_1, \dots, f_m \in R$,

$$R \models \Theta(f_0, \dots, f_m) \Leftrightarrow \mathbb{B} \models [[\Theta(f_0, \dots, f_m) = 1]].$$

Now suppose that Θ is of the form

$$\exists x_{m+1} \Theta^*(x_0, \dots, x_m, x_{m+1}),$$

assuming the result known for Θ^*.

Corollary. R commutative ring with unit. $\varphi(x)$ given $\mathcal{L}$rings-formula

If R satisfies the axioms in $\mathcal{A}_\varphi$ Then

$R \equiv \prod^{\varphi}_{e \text{ an atom.}} R_e$, the restricted product of localizations

R_e over all atoms $e \in R$.

Elementary Invariants

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7.2. **The case of normal extensions.**

Theorem 7.2 (Derakhshan-Macintyre [38]). *Suppose that K is normal over $\mathbb{Q}$. If L is a number field such that $\mathbb{A}_L$ is elementarily equivalent to $\mathbb{A}_K$, then $L = K$.*

The proof uses a corollary of the Chebotarev density theorem that states that if K is a Galois extension of $\mathbb{Q}$, then K is completely determined by the rational primes that split completely in K (see [71], Corollary 13.10 page 548).

7.3. **Splitting types and arithmetical equivalence.**

Let p be a prime. We do not assume that p is unramified in K. The splitting type of p in K is a sequence

$$\Sigma_{p,K} = (f_1, \dots, f_r),$$

where $f_1 \le \dots \le f_r$ is such that $p\mathcal{O}_K = \mathcal{P}_1^{e_1} \dots \mathcal{P}_r^{e_r}$ and f_j is the residue degree of $\mathcal{P}_j$. Note that there can be repetitions and that the ramification indices e_i are not present.

For a splitting type A, define

$$P_K(A) = \{p : \Sigma_{p,K} = A\}.$$

Note that $P_K(A)$ is empty for all but finitely many A (since $\sum_{j=1}^r f_j \le [K : \mathbb{Q}]$).

Let K be a number field. The (Dedekind) zeta function of K is defined by $\zeta_K(s) = \sum_{\mathfrak{a} \in Spec(\mathcal{O}_K)} N(\mathfrak{a})^{-s}$, where $N(\mathfrak{a}) = [\mathcal{O}_K : \mathfrak{a}]$.

In [75, Theorem 1] Perlis proves that if K_1 and K_2 are number fields, then $\zeta_{K_1}(s) = \zeta_{K_2}(s)$ if and only if $P_{K_1}(A) = P_{K_2}(A)$ for all A. In this case K_1 and K_2 are said to be arithmetically equivalent.

By [75, Theorem 1], if K_1 and K_2 are arithmetically equivalent, then they have the same discriminant, the same number of real (resp. complex) absolute values, the same normal closure and unit groups.

It follows from Theorem 2.2 that if $\mathbb{A}_K \equiv \mathbb{A}_L$, then for each p,

$$\Sigma_{p,K_1} = \Sigma_{p,K_2}.$$

Applying Hermit's theorem that there are only finitely many number fields with discriminant bounded by any given positive integer (see [71]), one can deduce the following.

Theorem 7.3 (Derakhshan-Macintyre [38]). *For any given number field K, there are only finitely many number fields L such that are $\mathbb{A}_K$ and $\mathbb{A}_L$ are elementarily equivalent.*

This raises the question.

Problem 7.1. *Given a number field K, classify the number fields L such that $\mathbb{A}_L$ is elementarily equivalent to $\mathbb{A}_K$. What are the elementary invariants?*

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7.4. **Elementary equivalence of adele rings - a rigidity theorem.**

The question asking to what extent a number field is determined by its zeta function has a long history.

A number field K that is isomorphic to any number field L such that $\zeta_K(s) = \zeta_L(s)$ is called arithmetically solitary. Examples are any normal extension of $\mathbb{Q}$. The first nonsolitary field was discovered by Gassman in 1925 who gave two fields of degree 180 over $\mathbb{Q}$ which are arithmetically equivalent but not isomorphic (cf. [75]).

By a theorem of Uchida [83], two number fields L and K are isomorphic if and only if their absolute Galois groups G_{K_1} and G_{K_2} are isomorphic, a theorem in the realm of Grothendieck's anabelian conjectures.

Iwasawa [56] proved that for number fields K and L, if $\mathbb{A}_K$ is isomorphic to $\mathbb{A}_L$, then $\zeta_K(s) = \zeta_L(s)$. The converse to Iwasawa's theorem relates to interesting questions. The converse is not true in general, but is true if the extensions are Galois, see [75].

From Perlis' [75, Theorem 1] and Theorem 2.2 it follows that if $\mathbb{A}_K$ and $\mathbb{A}_L$ are

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From Perlis' [75, Theorem 1] and Theorem 2.2 it follows that if $\mathbb{A}_K$ and $\mathbb{A}_L$ are elementarily equivalent, then $\zeta_K(s) = \zeta_L(s)$.

In [38] Macintyre and myself prove that elementary equivalence does determine the adele rings up to isomorphism, giving a converse to Iwasawa's theorem under a stronger hypothesis. This is a first-order "rigidity theorem " for adeles.

Theorem 7.4 (Derakhshan-Macintyre [38]). *Let K an L be number fields. If $\mathbb{A}_K$ and $\mathbb{A}_L$ are elementarily equivalent (as rings), then they are isomorphic.*

The proof uses a theorem of Iwasawa in [56, pages 331-356] that for number fields K and L, the adele rings $\mathbb{A}_K$ and $\mathbb{A}_L$ are isomorphic if and only there there is a bijection $\phi : V_K^{fin} \to V_L^{fin}$ such that the completions K_v and $L_{\phi(v)}$ are isomorphic for all $v \in V_K^{fin}$. This condition is also equivalent to the condition that the finite adeles $\mathbb{A}_K^{fin}$ and $\mathbb{A}_L^{fin}$ are isomorphic (cf. [38]).

Problem 7.2. *Find conditions under which adele rings are isomorphic.*

We also pose.

Problem 7.3. *Does Theorem 7.4 extend to algebraic groups G? Find algebraic groups G over $\mathbb{Q}$ such that if $G(\mathbb{A}_K)$ and $G(\mathbb{A}_L)$ are elementarily equivalent in the language of groups, then they are isomorphic. Is this true when G is a $\mathbb{Q}$-split semi-simple algebraic group over $\mathbb{Q}$?*

We note that one believes that for a $\mathbb{Q}$-split semi-simple algebraic group G, the field $\mathbb{Q}_p$ is definable in the group $G(\mathbb{Q}_p)$. It would be interesting to investigate adelic versions of this and use it to approach Problem 7.3.

Zeta functions of number fields (Dedekind)

$$\zeta_{\mathbb{Q}}(s) = \zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

converges $Re(s) > 1$ simple pole at $s = 1$

meromorphic contd to all $s \in \mathbb{C}$

functional equation

K # field

$$\zeta_K(s) = \sum_{I \trianglelefteq \mathcal{O}_K} \frac{1}{\mathrm{Norm}(I)^s} = \prod_{\substack{I \trianglelefteq \mathcal{O}_K \\ \text{prime}}} \frac{1}{1 - \mathrm{Norm}(I)^{-s}}$$

$$\mathrm{Norm}(I) = \left|\mathcal{O}_K / I\right|$$

similar analytic properties as $\zeta(s)$ – Hecke proved. meromorphic poles etc. functional eq.

Theorem (Iwasawa). If $\mathbb{A}_{K_1} \cong \mathbb{A}_{K_2}$, where K_1, K_2 are number fields. Then $\zeta_{K_1}(s) = \zeta_{K_2}(s)$

$\Rightarrow$ for any $n \geq 1$, The number of ideals of norm $\leq n$ in $\mathcal{O}_{K_1}$ and $\mathcal{O}_{K_2}$ is the same.

Note that the discriminants of K_1 & K_2 are the same

Thus $\mathbb{A}_{K_1} \equiv \mathbb{A}_{K_2} \xRightarrow{\text{By our Rigidity Thm}} \mathbb{A}_{K_1} \cong \mathbb{A}_{K_2} \Rightarrow K_1$ & K_2 same disc.

Thus $\mathbb{A}_{K_1} \equiv \mathbb{A}_{K_2} \overset{\text{By our Rigidity Thm}}{\Rightarrow} \mathbb{A}_{K_1} \cong \mathbb{A}_{K_2} \Rightarrow K_1$ & K_2 same disc
$\Rightarrow$ by Hermite's Theorem only finitely many K_2 s.t. $\mathbb{A}_{K_1} \equiv \mathbb{A}_{K_2}$

Dreams & Challenges

- What are imaginaries in $Th(\mathbb{A}_K)$?
- Use Hrushovski-Rideau-Martin elimination of imaginaries for $\mathbb{Q}_p$ in Language with geometric sorts $Gl_n(\mathbb{Q}_p)/Gl_n(\mathbb{Z}_p)$
This is uniform in p

$\mathbb{A}_K/K^*$ (Cohen-Connes)-space of adele classes.

In his 1999 Selecta paper, Connes proves a local trace formula for this space and proves a conrep global trace formula is equivalent to Riemann Hypothesis (RH)

Connes-Consani consider $\hat{\mathbb{Z}} \backslash \mathbb{A}_\mathbb{Q}/\mathbb{Q}^*$ This relates to RH Spectral interpretation of zeros

Zilber asked if this set is interpretable in $\mathbb{A}_\mathbb{Q}$? We proved Yes.

- Study $\mathbb{A}K/K^*$. This is a hyperring (with multi-valued addition).
Develop theory of adelic integration and Fourier transform

We proved model theoretic results for restricted products of hyperring $K_v/1+\mathcal{M}_v^n$ over v:
namely $\prod'_{v \in V_K^{fin}} K_v/1+\mathcal{M}_v^n$. The hyperrings were introduced by Krasner in connection to correspondence between infinitely ramified extensions of $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$.
$\sim$ P. Scholtze's Perfectoid spaces

- Model Theory for ∞-ramified ext^n of $\mathbb{Q}_p$?

- Study $G(\hat{\mathbb{Z}}) \backslash G(\mathbb{A})/G(\mathbb{Q})$, G a nice alg group. If G reductive, this involves Shimura varieties
For other groups G it relates to Langlands Program on correspondence between Representations of Galois groups and automorphic representations

- Develop theory of integration & Fourier transform

- Let $X \subseteq \mathbb{A}_K^m$ be definable. By quantifier-elimination X is measurable

- Let $X \subseteq \mathbb{A}_K^m$ be definable. By quantifier-elimination X is measurable.

Study adelic integrals $\int_X |f|^s d\mu$, where f a nice definable function and μ a measure (e.g. Tamagawa meas)

These adelic integrals are usually Euler products over $p \leq \infty$ of Denef-Loeser type "definable" integrals

$$\prod_{p \leq \infty} \int_X |f|^s d\mu_p, \quad X \subseteq \mathbb{Q}_p^m \text{ definable set, } f \text{ def function}$$

$(\mathbb{Q}_\infty = \mathbb{R})$.

(D 2020 Euler products) proves results on meromorphic continuation of such Euler products and gives information on poles of extended function

(D 2020 Model Theory of adeles & Number Theory) uses our model theory of restricted products to propose a framework for model theoretic study of automorphic representations and Langlands Program. & related topics including $\mathbb{Q}_p$ and $\mathbb{A}_K$ in continuous logic & Artin reciprocity

I define notions of adelic constructible integrals and prove that the global Zeta integrals that arise in work of Jacquet-Langlands for GL_2-automorphic forms are constructible.

In General

In general, the constructible integrals have

o-minimal structures on the $\mathbb{R}$-factors

e.g. $\mathbb{R}_{an}$, $\mathbb{R}_{exp, log}$ of

van den Dries - Miller / van den Dries - Macintyre - Marker

- Study adelic constructible functions
  prove the class is closed under Fourier transform and integration.

- Explore connections of the o-minimal components of adelic constructible functions to

  Hodge theory (a la Tsimerman et al)

  & definability of period mappings

Greatest Challenge

& Deepest Problem

The Conjecture of Birch and Swinnerton-Dyer (Millenium Problem) as a question about adelic integrals over adelic spaces of Elliptic curves

$$\int_{E(\mathbb{A}_{\mathbb{Q}})} f \, d\tau \quad , \quad \tau = \text{Tamagawa measure on adeles}$$

f = suitable "definable" function

This relates to analytic properties of certain Euler products

The conjecture is about $L(E,s)$

L-series of Elliptic curves

and its relation to rank of E.

There are many fascinating intermediate problems.