Group topologies on automorphism groups of homogeneous structures

Zaniar Ghadernezhad
(joint work with Javier de la Nuez González)

Imperial College London

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Definition

Let (G, τ) is a topological group if both group operation and inverse function are continues functions.

- Every group G is topological group with discrete topology and trivial topology.
- Every group topology (G, τ) is determined by its neighbourhood at identity (follows from the fact that for every h the map $g \mapsto gh$ is a homeomorphism).
- Every T_1 group topology is regular.

Minimal topological groups

Definition

Let (G, τ) be a Hausdorff topological group.

- G is called *minimal* if G does not admit a strictly coarser Hausdorff group topology or equivalently every bijective continuous homomorphism from G to another another Hausdorff topological group is a homeomorphism.
- G is *totally minimal* if every continuous surjective homomorphism to a Hausdorff topological group is open.

- Compact topological groups are minimal.
- (Stephenson 1971) Locally compact abelian groups are minimal precisely when they are compact.
- All p-adic topologies on $\mathbb{Z}$ is minimal.

Markov Topology

Definition

Given a group G the Markov topology is the intersection of all Hausdorff group topologies on G.

- τ_M is T_1 but not necessarily a group topology.
- When τ_M is a group topology that would be totally minimal and the coarsest group topology on G.
- Extreme case is when τ_M is the discrete topology, so called non-topologizable groups.
- Under CH there exists infinite non-topologizable groups (Shelah), without CH (Hesse) and in ZFC countable non-topologizable groups (Ol'shanskii).

Symmetric group of infinite sets

Let Ω be an infinite set and $S_\infty := Sym(\Omega)$. Then (S_∞, τ_{st}) is a group topology where τ_{st} is the induced subspace topology on the product topology on Ω^Ω (i.e. point-wise convergence topology with the discrete topology on Ω).

Fact

When Ω is countable G is a Polish group (separable completely metrizable).
Fix an enumeration on Ω and define $d(g,h) = \frac{1}{2^i}$ where i is a small as possible with $g(i) \neq h(i)$ and $g^{-1}(i) \neq h^{-1}(i)$.

Theorem (Gaughan 1967)

Let Ω be a countable set then $\tau_{st} = \tau_M$ for S_∞.

Theorem (Banakh , Guran, Protasov 2012)

If H is a subgroup of S_∞ where $S_f \leqslant H \leqslant S_\infty$. Then the induced topology τ_{st} on H is totally minimal $\tau_M = \tau_{st}$.

Zariski Topology

Given a group G the Zariski topology τ_Z, is generated by the subbase consisting of the sets $\{x \in G \,|\, x^{\varepsilon_1} g_1 x^{\varepsilon_2} g_2 \cdots x^{\varepsilon_n} g_n \neq 1\}$, where $n \in \mathbb{N}$, $g_1, \ldots, g_n \in G$, and $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$.

Theorem (Gaughan - Banakh , Guran, Protasov)

If H is a subgroup of S_∞ where $S_f \leqslant H \leqslant S_\infty$. Then the induced topology τ_{st} on H is totally minimal and in fact $\tau_M = \tau_{st} = \tau_Z$.

Closed subgroups of S_∞

Fact

Let $G \leqslant S_\infty$. Then G is closed if and only if $G = \mathrm{Aut}(M)$ where M is a first-order structure.

Definition

A closed subgroup of S_∞ is oligomorphic if its diogonal action on Ω^n $(n \geq 1)$ has only finitely many orbits.

Theorem (Engeler, Ryll-Nardzewski, Svenonius)

Let M be a countable first order structure. TFAE

- $\mathrm{Aut}(M)$ *acts oligomorphically on* M;
- M *is an* ω*-categorical structure;*
- ...

General Questions

Question A

Let M be a countable ω-categorical (sufficiently nice e.g. ω-saturated, ...) first order structure and $G = \mathrm{Aut}(M)$. When (G, τ_{st}) is (totally) minimal?

For more information about minimality, we refer the reader to the survey by Dikranjan and Megrelishvili.

Question B

For which (sufficiently homogeneous) structures is it true that $\tau_Z = \tau_{st}$. For which of them is the Zariski topology a group topology?

Problem C

Let M be a countable ω-categorical (or sufficiently nice) first order structure and $G = \mathrm{Aut}(M)$. Describe the lattice of all Hausdorff group topologies on G coarser than τ_{st}.

Stable ω-categorical structures

Theorem (Ben Yaacov, Tsankov. 2016)

Let G be the automorphism group of an ω-categorical, stable structure. Then G is totally minimal.

- *Roelcke precompact Polish groups and weakly almost preiodic functions.*
- Automorphism group of $(\mathbb{Q}, <)$ with τ_{st} is not minimal.

Urysohn space and Urysohn sphere

- Urysohn universal space $\mathscr{U}$ is a homogeneous space that contains all separable metric spaces. It is both ω-universal, i.e. it contains any finite metric space as a subspace and ω-homogeneous, i.e. any partial isometry between finite subspaces of $\mathscr{U}$ extends to an isometry of $\mathscr{U}$.

- Associated with the class of metric spaces with diameter at most 1 there is an object with similar properties $\mathscr{U}_1$, known as the Urysohn sphere.

Theorem (Uspenskij)

$Isom(\mathscr{U}_1)$ is Roelcke-precompact, topologically simple and minimal.

Definition

Recall that a topological group (G, τ) is Roelcke precompact if for any neighbourhood W of 1 there exists a finite $F \subset G$ such that $WFW = G$.

Z

Roelcke precompactness

Theorem (Ben Yaacov, Tsankov. 2016)

Let G be the automorphism group of an ω-categorical structure M. Then the following are equivalent:

- *$Th(M)$ is stable;*
- *Every Roelcke uniformly continuous function on G is weakly almost periodic.*

Both proofs rely on the assumption that the groups are Roelcke precompact and use a well behaved *independence* relation among (small) subsets of the structure to endow the Roeckle precompletion of the group with a topological semigroup structure.
Information on the topological quotients of the original group is then recovered from the latter via the functoriality of Roelcke compactification and Ellis lemma.

Fraïssé limits

Theorem (Fraïssé 1953)

Suppose $\mathscr{K}$ is a Fraïssé class. Then, there exists a unique, up to isomorphism, countable structure $\mathfrak{A}$ such that $\mathfrak{A}$ is ultra-homogeneous and $\mathscr{K} = \mathrm{Age}(\mathfrak{A})$.

Examples

- Class of all finite linearly ordered sets; Fraïssé limit is $(\mathbb{Q}, <)$.
- Class of all finite graphs; Fraïssé limit is called the random graph.
- Class of all finite metric spaces with rational distances; Fraïssé limit is Rational Urysohn space which the completion is the Urysohn universal space (metric space that contains all separable metric spaces).
- ...

Question A

More in general for each G-invariant $X \subseteq M$ there is an associated group topology τ_{st}^X generated by $\{G_A \,|\, A \subset_{fin} X\}$.

Theorem (Gh., de la Nuez)

Let M be the Fraïssé limit of a free amalgamation class in a countable relational structure. Let $G = \mathrm{Aut}(M)$. Then any group topology $\tau \subseteq \tau_{st}$ on G is of the form τ_{st}^X, where $X \subseteq M$ is some G-invariant set. In particular, if the action of G on M is transitive, then (G, τ_{st}) is totally minimal.

Question A

Theorem (Gh., de la Nuez)

Let M be a simple, ω-saturated countable structure with elimination of hyperimaginaries, locally finite algebraic closure and weak elimination of imaginaries. Assume furthermore that $Th(M)$ is one-based. Let $G = \mathrm{Aut}(M)$. Then

- *If G acts transitively on M, then (G, τ_{st}) is minimal.*
- *If all points are algebraically closed, then any group topology τ on G coarser than τ_{st} is of the form τ_{st}^X for some G-invariant $X \subseteq M$.*

Urysohn space

The collection $\{N_u(\varepsilon) \,|\, u \in \mathscr{U}, \varepsilon \in \mathbb{R} \setminus \{0\}\}$, where $N_u(\varepsilon) = \{g \in G \,|\, d(gu, u) \leq \varepsilon\}$ generates a group topology τ_m on the isometry group of $\mathscr{U}$.

Theorem (Gh., de la Nuez)

Then τ_m is the coarsest non-trivial group topology on G coarser than the stabilizer topology τ_{st}. In particular, (G, τ_m) is totally minimal.

One can prove the following

Theorem (Gh., de la Nuez)

Let $G = Isom(\mathscr{U})$. Then there are exactly 4 group topologies on G coarser than τ_{st}:

$$\tau_{st} \supsetneq \tau_{0^+,0} \supsetneq \tau_m \supsetneq \{\emptyset, G\}$$

where $\tau_{0^+,0}$ is the topology whose system of neighbourhoods of the identity is generated by the collection $\{\mathscr{N}^{sp}_{u,v} \,|\, u, v \in \mathscr{U}, d(u,v) > 0\}$, where $\mathscr{N}^{sp}_{u,v} := \{g \in G \,|\, d(gu, v) \leq d(u,v)\}$ for any $u, v \in \mathscr{U}$ with $d(u,v) > 0$.

Question B

Theorem (Gh., de la Nuez)

The Zariski topology on $\mathrm{Aut}(M)$ *is not a group topology if* $M = \mathrm{Flim}(\mathscr{K})$ *for a Fraïssé class* $\mathscr{K}$ *in a relational language* $\mathfrak{L}$ *in each of the following cases:*

- $\mathscr{K}$ *is a non-trivial free amalgamation class and the action of* G *on* M *is transitive;*
- M *is the rational Urysohn space;*
- M *is the random tournament;*
- $\mathscr{K}$ *is of the form* $\mathscr{K}_1 \otimes \mathscr{K}_2$ *for strong amalgamation classes* $\mathscr{K}_1$ *and* $\mathscr{K}_2$ *where:*
  - $\mathscr{K}_1$ *is non-trivial or the action of* $\mathrm{Aut}(\mathrm{Flim}(\mathscr{K}_1))$ *on the set* $M^2 \setminus \{(a,a)\}_{a \in M}$ *is transitive.*
  - $\mathrm{Flim}(\mathscr{K}_2)$ *is the countable dense meet tree, the cyclic tournament* $S(2)$ *or* $(\mathbb{Q}, <)$*.*

Question B

Theorem (Gh., de la Nuez)

The Zariski topology τ_Z on $\mathrm{Aut}(M)$ is a group topology in case M is one of the following:

- *A countable dense meet-tree or the lexicographically ordered dense meet-tree, in which case $\tau_Z = \tau_{st}$;*
- *The cyclic tournament $S(2)$.*

Minimality of the automorphsim group of Random graph

Let $\mathscr{K}$ be the class of all finite graphs and $R := \mathrm{Flim}(\mathscr{K})$. Let $G = \mathrm{Aut}(R)$. Suppose τ is a non-trivial group topology on G coarser than τ_{st}.

Denote by $\mathscr{N}$ the collection of all open neighbourhoods of 1.

Let $\mathscr{X}$ be the collection of all non-empty families of isomorphisms between finite substructures of R.

Each $V \in \mathscr{N}$ can be written as $V_{\mathscr{F}} := \bigcup_{f \in \mathscr{F}} G_f$ for some $\mathscr{F} \in \mathscr{X}$ (assume that for any finite partial isomorphism f either $f \in \mathscr{F}$ and no extension of f is in $\mathscr{F}$ or else $G_f \nsubseteq V$).

Denote by $\mathscr{Y}$ the collection of all the families $\mathscr{F}$ obtained as we let V range in $\mathscr{N}$.

Fact

$\mathscr{Y}, \mathscr{N}$ are invariant under the action of G by conjugation.

Lemma

Assume that for some $\mathscr{F} \in \mathscr{X}$ *there is some* $a \in R$ *and a finite set* $A \subset R$ *such that for all* $f \in \mathscr{F}$ *there is some* $y \in A$ *such that* $f(a) = y$*. Then* $\tau = \tau_{st}$*.*

Proof.

Since singletons are algebraically closed, there is some $h \in G_a$ such that $h(A) \cap A = \{a\}$. Every $\sigma \in V_{\mathscr{F}} \cap V_{\mathscr{F}}^{h^{-1}} = V_{\mathscr{F}} \cap V_{\mathscr{F}^{h^{-1}}}$ has to fix the element a. □

Assume there do not exist a, A and $\mathscr{F}$ as in the lemma above.

Zig-zag property

Since τ is not trivial then there is $\mathscr{F} \in \mathscr{Y}$ such that $V = V_{\mathscr{F}}$ satisfies $V^4 \neq G$. Since $1 \in N_{\mathscr{F}}$, we have $id_C \in \mathscr{F}$ for some finite subset $C \neq \emptyset$.

Lemma

Let A, B tuples of elements from R for which there is a chain $A = A_0, B_0, \ldots, B_{n-1}, A_n = g(A)$ such that $A_i B_i \cong A_{i+1} B_i \cong AB$ for $0 \leq i < n$. Then $g \in (G_A G_B)^n G_A$.

Then we can prove

Lemma

Let A, B be finite subsets of R. Then $G_A G_B G_A G_B = G_{A \cap B}$.

Choose $\mathscr{F}_0 \in \mathscr{X}$ such that $V^k_{\mathscr{F}_0} \subset V$. We will assume that $\mathscr{F}$ is closed under taking inverses. Just as above, there is some $\emptyset \neq B \subset R$ such that $Id_B \in \mathscr{F}_0$. Let $c_1, c_2 \cdots c_r$ be an enumeration of the vertices of C.

Lemma

Let $K : \mathbb{N} \to \mathbb{N}$ such that $K(0) = 0$ and $K(j+1) \geq 2K(j) + 16$. For each j there is some finite $B_j \subset R$ such that $B_j \cap C \leq \{c_j, c_{j+1} \cdots c_r\}$ and $G_{B_j} \subset V^{K(j)}_{\mathscr{F}_0}$.

Indeed, if we let $B' = B_{r+1}$, then $B' \cap C = \emptyset$ and by the previous lemma $G_C G_{B'} G_C G_{B'} = G$, contradicting the assumption that $V^4 \neq G$.

Proof of Lemma

The claim is proven by induction on j. Let $V_j := V_{\mathscr{F}_0}^{K(j)}$. Notice that $G_{f^{-1}} G_f \supset G_{dom(f)}$ for any finite partial isomorphism f. Suppose we have already found B_j for some $j \geq 0$.
Now, our assumption implies the existence of some $f \in \mathscr{F}_0$ such that $(c_j, b) \notin f$ for all $b \in B$. Let $\hat{f} \in G$ be some extension of f and set $B_{j+1} = B_j \cap f^{-1}(B)$. Clearly $B_{j+1} \cap C \subset \{c_{j+1}, c_{j+2}, \cdots c_r\}$.

$$V_{\mathscr{F}_0}^8 \supseteq \hat{f}\, G_B G_{im(f)} G_B \hat{f}^{-1} \supseteq \hat{f}\, G_{B \cap im(f)}\, \hat{f}^{-1} = G_{f^{-1}(B)}$$

and hence

$$V_{j+1} \supseteq V_j V_{\mathscr{F}_0}^8 V_j V_{\mathscr{F}_0}^8 \supseteq G_{B_j} G_{f^{-1}(B)} G_{B_j} G_{f^{-1}(B)} \supseteq G_{B_j \cap f^{-1}(B)} = G_{B_{j+1}}$$

Thank you very much for your attention!