Notes on Günther's Method and the Local Version of the Nash Isometric Embedding Theorem

Ralph Howard
Department of Mathematics
University of South Carolina
Columbia, SC, 29208
howard@math.scarolina.edu

Introduction

Let $U \subseteq \mathbf{R}^n$ be a connected open set and $f : U \to \mathbf{R}^N$ a smooth map. Then f is an ***immersion*** iff for all $x \in U$ the differential df_x is injective. If $x^1, \dots, x^n$ are the natural coordinates on $\mathbf{R}^n$ and $x = (x^1, \dots, x^n)$ then f can be written as a column vector of smooth functions

$$f(x^1, \dots, x^n) = \begin{bmatrix} f^1(x) \\ \vdots \\ f^N(x) \end{bmatrix}$$

and the condition that f be an immersion is just that the vectors $\partial_i f = \partial f / \partial x_i$ are pointwise linearly independent in U. The image of a smooth immersion can be thought of as a smooth n-dimensional submanifold of $\mathbf{R}^N$. To understand the geometry of the image a first step is to see what happens to the length of curves in U as they are mapped by f. Let $c(t) := (x^1(t), \dots, x^n(t))$, $a \le t \le b$ be a smooth curve in U so that the tangent vector to c is $c'(t) = (\dot{x}(t), \dots, \dot{x}^n)$. To simplify notation let

$$f_i := \partial_i f, \quad f_{ij} = f_{ji} := \partial_{ij} f, \quad \text{etc.}$$

Then the tangent vector to $\gamma(t) := f \circ c(t)$ is

$$\gamma'(t) = df(c'(t)) = \sum_{i=1}^{n} f_i \dot{x}^i.$$

Thus

$$\gamma'(t) \cdot \gamma'(t) = \sum_{i,j} f_i \cdot f_j \dot{x}^i \dot{x}^j$$

where $\cdot$ is the standaard inner product on $\mathbf{R}^N$. Therefore the length of γ is

$$\text{Length}[\gamma] = \int_a^b \sqrt{\gamma'(t) \cdot \gamma'(t)}\, dt = \int_a^b \sqrt{\sum_{i,j} g_{ij} \dot{x}^i \dot{x}^j}\, dt \tag{1}$$

where

$$g_{ij} = g_{ji} = f_i \cdot f_j.$$

This is usually expressed by saying "the element of arclength ds is given by"

$$ds^2 = \sum_{i,j} f_i \cdot f_j \, dx^i dx^j = \sum_{i,j} g_{ij} dx^i dx^j.$$

A ***Riemannian metric*** on $U \subset \mathbf{R}^n$ is a set of smooth functions $g_{ij} = g_{ji}$ so that the matrix $g(x) := [g_{ij}(x)]$ is positive definite for all $x \in U$. For geometric reasons Riemannian metrics are written as

$$ds^2 = g = \sum_{i,j} g_{ij} dx^i dx^j.$$

Given a Riemannian metric we can define the length of curve $\gamma(t) = (x^1, \ldots, x^n)$ by equation (1) and start to study the geometry of U with length measured in this manner. If a Riemannian metric g is of the form $g = df \cdot df$ for an immersion f we say that g is the ***Riemannian metric induced by*** f. Here are some examples. To start with the familiar let $f : (0, \infty) \times (-\infty, \infty) \to \mathbf{R}^2$ be "polar coordinates" $f(r, \theta) = (r \cos\theta, r \sin\theta)^t$ (where v^t is the transpose of the vector v). (Note that while f is an immersion, it is not injective.) Then

$$df = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} dr + \begin{bmatrix} -r\sin\theta \\ r\cos\theta \end{bmatrix} d\theta$$

and the induced Riemannian metric is

$$ds^2 = g = df \cdot df = dr^2 + r^2 d\theta^2$$

which is the usual formula for arclength in polar coordinates. For a less familiar example let $f : \mathbf{R}^2 \to \mathbf{R}^3$ be

$$f(u, v) = u \begin{bmatrix} \cos v \\ \sin v \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ v \end{bmatrix}.$$

(This is a parameterization of the helicoid $z = \arctan(y/x)$.) Then

$$df = \begin{bmatrix} \cos v \\ \sin v \\ 0 \end{bmatrix} du + u \begin{bmatrix} -\sin v \\ \cos v \\ 0 \end{bmatrix} dv + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} dv$$

Thus

$$ds^2 = df \cdot df = du^2 + (1 + u^2) dv^2.$$

Exercise. (This example was brought to my attention by Margaret Reese [13].) For each $0 < \alpha < \pi/2$ define $f : (0, \infty) \times (-\infty, \infty) \to \mathbf{R}^3$ by

$$f(u, v) = \sqrt{u^2 + v^2} \begin{bmatrix} \sin(\alpha)\cos\left(\csc(\alpha)\arctan\left(\frac{v}{u}\right)\right) \\ \sin(\alpha)\sin\left(\csc(\alpha)\arctan\left(\frac{v}{u}\right)\right) \\ \cos(\alpha) \end{bmatrix}$$

Show f parameterizes part of the cone $z^2 = \cot^2(\alpha)(x^2 + y^2)$ and the induced Riemannian metric is the standard flat metric

$$ds^2 = g = df \cdot df = du^2 + dv^2.$$

That the metric on the cone is the flat metric is not surprising to anyone who has made a cone using paper and tape (the metric on the paper is flat and lengths of curves are not changed by bending the paper so long as it is not torn). This parameterization of the cone is usefully in solving the following problem: Given two points P_1 and P_2 on the cone, find the curve on the cone of shortest length connecting these points. Provided these points are in the image of f, say $P_1 = f(u_1, v_1)$ and $P_2 = f(u_2, v_2)$, and some other conditions are satisfied (which we leave to the reader to find and which hold if the points are close enough together) then the curve of shortest length is $c(t) = f((1-t)u_1 + tu_2, (1-t)v_1 + tv_2)$ with $0 \le t \le 1$ and the length of this curve is $\sqrt{(u_2 - u_1)^2 + (v_2 - v_1)^2}$.
Hint: Because of the rotational symmetry of the cone it makes sense to work in coordinates adapted to this symmetry. Thus change to polar coordinates $u = r\cos\theta$ and $v = r\sin\theta$. In these coordinates f is

$$f = r\begin{bmatrix} \sin(\alpha)\cos\left(\dfrac{\theta}{\sin(\alpha)}\right) \\ \sin(\alpha)\sin\left(\dfrac{\theta}{\sin(\alpha)}\right) \\ \cos(\alpha) \end{bmatrix}$$

and it is enough to show $df \cdot df = dr^2 + r^2 d\theta^2$.
These examples motivate:
Isometric Imbedding Problem. *Given a Riemannian metric $g = \sum_{i,j} g_{ij}dx^i dx^j$ on U then find (or prove the existence of) an immersion $f : U \to \mathbf{R}^N$ so that $df \cdot df = g$.*

The earliest result on this is the theorem of Janet-Cartan-Burstin [9, 2, 1] that when the functions g_{ij} are real analytic the Riemannian metric $g = \sum_{ij} g_{ij}dx^i dx^j$ is locally embeddable in $\mathbf{R}^N$ with $N = \frac{1}{2}n(n+1)$ (so that when $n = 2$, $N = 3$). In one sense this is very satisfying for $g_{ij} = f_i \cdot f_j$ represents $\frac{1}{2}n(n+1)$ equations and the number of unknown functions is N. Thus in the Janet-Cartan-Burstin theorem the number of unknown functions equals the number of equations. In the case of C^∞ metrics and for global results things are more complicated. The main result is due to Nash [12] who in 1956 proved every compact C^∞ Riemannian manifold can be globally embedded in $\mathbf{R}^N$ with $N = \frac{1}{2}n(3n + 11)$ (this number has since been improved see [5, Chap. 3]). The first, and by far the harder step, was to show that the space of embeddable metrics is open in the space of all metrics by use of a new type of implicit function theorem he invented for just this task. It is then shown that the set of embeddable metrics is also closed in the set of all metrics and as the space of metrics is connected this finishes the proof. The implicit function part of the argument was generalized and simplified by Moser [10, 11] into what has come to be called the Nash-Moser implicit function theorem. It has become one of the more important methods for dealing with nonlinear problems and has applications far beyond isometric embeddings. For more information on isometric embeddings see the books by Griffiths and Jensen [4] and Gromov [5, Chap 3] (the latter of these devotes 129 pages to isometric embeddings). For more on the Nash-Moser implicit function theorem see the article [8] of Hamilton.

Recently Matthias Günther [6, 7] has greatly simplified the original version of Nash's proof of the embedding theorem by finding a method that avoids the use of the Nash-Moser theory and just uses the standard implicit function theorem from advanced calculus (although applied to a functional between Banach spaces). In this note we give most of the details in Günther's proof applied to the local imbedding problem for smooth Riemannian metrics. Our presentation owes a good deal to the unpublished note of Deane Yang [14].

Analytic Preliminaries

Here we collect the results form analysis needed in the proof. Let $U \subset \mathbf{R}^n$ be a bounded open set with smooth boundary. If $0 < \alpha < 1$ and $u : \overline{U} \to \mathbf{R}$ is a continuous function on $\overline{U}$ (the closure of U) then the Hölder semi-norm of u is defined by

$$[u]_{C^\alpha(\overline{U})} := \sup_{x,y \in \overline{U},\ x \neq y} \frac{|u(x) - u(y)|}{|x-y|^\alpha}.$$

Then for $k \geq 0$ the space $C^{k,\alpha}(\overline{U})$ is the set of functions u so that the norm

$$\|u\|_{C^{k,\alpha}(\overline{U})} := \sup_{x \in \overline{U}} |u(x)| + \sum_{|I|=k} [\partial_I u]_{C^\alpha(\overline{U})}$$

is finite (where $I = (i_1, \ldots, i_n)$ is a multi-index, $\partial_I := \partial_1^{i_1} \partial_2^{i_2} \cdots \partial_n^{i_n}$ and $|I| = i_1 + i_2 + \cdots + i_n$). Let $C_0^{k,\alpha}(\overline{U})$ be the subspace of $C^{k,\alpha}(\overline{U})$ of functions that vanish on ∂U. Let Δ be the usual Laplacian $\Delta := \sum_{i=1}^n \partial_i^2$. Then the following is a standard result from the theory of elliptic partial differential equations.

Global Schauder Estimates. *The map $\Delta : C_0^{2,\alpha}(\overline{U}) \to C^{0,\alpha}(\overline{U})$ is an isomorphism. That is ϕ is bijective and there is a constant C so that for all $\phi \in C_0^{2,\alpha}(\overline{U})$*

$$\frac{1}{C} \|\phi\|_{C_0^{2,\alpha}(\overline{U})} \leq \|\Delta \phi\|_{C^{0,\alpha}(\overline{U})} \leq C \|\phi\|_{C_0^{2,\alpha}(\overline{U})}$$

For a proof see Chapter 6 of [3]. This implies there is an inverse map $\Delta^{-1} : C^{0,\alpha}(\overline{U}) \to C_0^{2,\alpha}(\overline{U})$. The operator Δ^{-1} is of order -2 in the sense that it smoothes to order 2.

Let $C^{k,\alpha}(\overline{U}, \mathbf{R}^N)$ be the set of functions of class $C^{k,\alpha}$ with values in the vector space $\mathbf{R}^N$ with the obvious norm and let $C_0^{k,\alpha}(\overline{U}, \mathbf{R}^N)$ be the subspace of those function that vanish on ∂U. Then Δ is defined on functions of $C^{k,\alpha}(\overline{U}, \mathbf{R}^N)$ by just by letting $\Delta(u^1, \ldots, u_N)^t = (\Delta u^1, \ldots, \Delta u^N)$ and with this definition the last theorem is true for $\Delta : C_0^{k+2,\alpha}(\overline{U}, \mathbf{R}^N) \to C^{k,\alpha}(\overline{U}, \mathbf{R}^N)$. This is the from of the result we will use.

As in the scalar valued case this implies that there is a bounded inverse $\Delta^{-1} : C^{k,\alpha}(\overline{U}, \mathbf{R}^N) \to C_0^{k+2,\alpha}(\overline{U}, \mathbf{R}^N)$. The Laplacian Δ clearly commutes with the partial derivative ∂_i, thus the same is true of its inverse, that is $\partial_i \Delta^{-1} u = \Delta^{-1} \partial_i u$, a fact that will be used several times in the calculations below.

The Main Theorem

An immersion $f : U \to \mathbf{R}^N$ is ***free*** if and only if for all $x \in U$ the $n + \frac{1}{2}n(n+1)$ vectors f_i, $f_{ij}(x)$ are a basis of $\mathbf{R}^N$ (which implies $N = n + \frac{1}{2}n(n+1)$). Our goal is to show that the set of metrics induced by free immersions is open in the $C^{2,\alpha}$ topology in the set of all metrics. Let $f : U \to \mathbf{R}^N$ be a free immersion where U is a bounded open set in $\mathbf{R}^n$ and let $g = df \cdot df$ be the induced Riemann metric. Then let $h = \sum_{ij} h_{ij} dx^i dx^j$ where $h_{ij} = h_{ji}$ where it is no longer assumed the matrix $[h_{ij}]$ is positive definite. If h

is small in the uniform norm, then $g + h = \sum_{ij}(g_{ij} + h_{ij})dx^i dx^j$ will be positive definite and thus a Riemannian metric. Moreover every Riemannian matrix close to g will be of the form $g + h$ for some symmetric h. Likewise a small perturbation of f is of the form $f + u$ where $u : U \to \mathbf{R}^N$. If $g + h$ is the metric induced by $f + u$ then using $g = df \cdot df$ we have $g + h = (df + du) \cdot (df + du)$ implies

$$du \cdot df + df \cdot du + du \cdot du = h.$$

Define a non-linear operator $\mathcal{I}$ (which depends on f) from $C^{k+1,\alpha}(\overline{U}, \mathbf{R}^N)$ to $C^{k,\alpha}(\overline{U}, \mathrm{Sym})$ (where the second space is the set of all $C^{k,\alpha}$ functions form $\overline{U}$ to the vector space Sym of all symmetric $n \times n$ matrices) by

$$\mathcal{I}[u]_{ij} := \partial_i u \cdot f_j + \partial_j u \cdot f_i + \partial_i u \cdot \partial_j u. \tag{2}$$

Thus given h we wish to solve $\mathcal{I}[u] = h$ for u. Güther's wonderfully ingenious trick is to show when f is free the operator $\mathcal{I}$ can be factored as $\mathcal{I}[u] = L(u - Q(u, u))$ where L is a linear map with a right inverse and $Q(u, u)$ is a quadratic map of order zero (that it is it maps $C^{k,\alpha}$ functions to $C^{k,\alpha}$ functions). Then the proof of the advanced calculus version of the implicit function theorem implies that $\mathcal{I}[u] = h$ can be solved for u provided h is small (see below). We now put $\mathcal{I}[u]$ in the required form.

$$\begin{aligned}
\mathcal{I}[u]_{ij} &= \partial_i u \cdot f_j + \partial_j u \cdot f_i + \partial_i u \cdot \partial_j u \\
&= \partial_i(u \cdot f_j) + \partial_j(u \cdot f_i) - 2u \cdot f_{ij} + \partial_i u \cdot \partial_j u \\
&= \partial_i(u \cdot f_j) + \partial_j(u \cdot f_i) - 2u \cdot f_{ij} + \Delta^{-1}\Delta(\partial_i u \cdot \partial_j u) \\
&= \partial_i(u \cdot f_j) + \partial_j(u \cdot f_i) - 2u \cdot f_{ij} \\
&\qquad + \Delta^{-1}(\partial_i \Delta u \cdot \partial_j u) + \Delta^{-1}(\partial_i u \cdot \partial_j \Delta u) + 2\Delta^{-1} \sum_k \partial_{ik} u \cdot \partial_{jk} u \\
&= \partial_i(u \cdot f_j) + \partial_j(u \cdot f_i) - 2u \cdot f_{ij} \\
&\qquad + \Delta^{-1}\partial_i(\Delta u \cdot \partial_j u) + \Delta^{-1}\partial_j(\partial_i u \cdot \Delta u) - 2\Delta^{-1}(\partial_{ij} u \cdot \Delta u) + 2\Delta^{-1} \sum_k \partial_{ik} u \cdot \partial_{jk} u \\
&= \partial_i(u \cdot f_j + \Delta^{-1}(\Delta u \cdot \partial_j u)) + \partial_j(u \cdot f_i + \Delta^{-1}\partial_i u \cdot \Delta u) - 2u \cdot f_{ij} \\
&\qquad - 2\Delta^{-1}(\partial_{ij} u \cdot \Delta u) + 2\Delta^{-1} \sum_k \partial_{ik} u \cdot \partial_{jk} u \\
&= \partial_i(u \cdot f_j + Q_j(u, u)) + \partial_j(u \cdot f_i + Q_i(u, u)) - 2u \cdot f_{ij} + 2Q_{ij}(u, u)
\end{aligned}$$

where Q_i and Q_{ij} are quadratic functionals given by

$$Q_i(u, u) := \Delta^{-1}(\partial_i u \cdot \Delta u), \quad Q_{ij}(u, u) := \Delta^{-1}\left(\sum_k \partial_{ik} u \cdot \partial_{jk} u - \partial_{ij} u \cdot \Delta u\right).$$

Because Δ^{-1} is a bounded linear map from $C^{0,\alpha}(\overline{U})$ onto $C_0^{2,\alpha}(\overline{U})$ these quadratic functionals are bounded on $C_0^{2,\alpha}(\overline{U})$, that is there are constants C_i, C_{ij} so that

$$\|Q_i(u, u)\|_{C_0^{2,\alpha}(\overline{U})} \leq C_i \|u\|^2_{C_0^{2,\alpha}(\overline{U})}, \quad \|Q_{ij}(u, u)\|_{C_0^{2,\alpha}(\overline{U})} \leq C_{ij} \|u\|^2_{C_0^{2,\alpha}(\overline{U})}.$$

Because f is free there is a uniquely defined quadratic functional $Q(u, u)$ so that

$$Q(u, u) \cdot f_i = Q_i(u, u), \quad Q(u, u) \cdot f_{ij} = Q_{ij}(u, u). \tag{3}$$

As $Q(u,u)$ is just a linear combination with smooth coefficients of the Q_i and Q_{ij}'s it will also be a bounded quadratic functional on $C_0^{2,\alpha}(\overline{U})$. Thus if $Q(u,v)$ is the symmetric bilinear functional associated with $Q(u,u)$ there is a constant C_1 so that

$$\|Q(u,v)\|_{C_0^{2,\alpha}(\overline{U})} \le C_1 \|u\|_{C_0^{2,\alpha}(\overline{U})} \|v\|_{C_0^{2,\alpha}(\overline{U})}. \tag{4}$$

Using (3) in the rewritten version of $\mathcal{I}[u]$ gives

$$\begin{aligned}\mathcal{I}[u]_{ij} &= \partial_i(u\cdot f_j + Q(u,u)\cdot f_j) + \partial_j(u\cdot f_i + Q(u,u)\cdot f_i) - 2u\cdot f_{ij} + 2Q(u,u)\cdot f_{ij}\\ &= L(u - Q(u,u))_{ij}\end{aligned}$$

where L is the linear map

$$(Lv)_{ij} := \partial_j(u\cdot f_j) + \partial_j(u\cdot f_i) - 2u\cdot f_{ij}. \tag{5}$$

Summarizing:

Günther's Lemma. *If $f : U \to \mathbf{R}^N$ is free, the isometric embedding equation can be rewritten as*

$$\mathcal{I}[u] = L(u - Q(u,u)) = h$$

where L and Q are as (5) and (3) above.

We wish to solve this for u. As a first step we note (or rather Nash noted) that it is not hard to solve "linearized" problem $Lu = h$.

Nash's Lemma. *Let $C^{2,\alpha}(\overline{U}, \mathrm{Sym})$ be the space of $C^{2,\alpha}$ functions on $\overline{U}$ with values in the space* Sym *of symmetric $n\times n$ matrices. Let $C_0^{2,\alpha}(\overline{U}, \mathrm{Sym})$ be the subspace of those h that vanish on ∂U. Define $M : C^{2,\alpha}(\overline{U}, \mathrm{Sym}) \to C^{2,\alpha}(\overline{U}, \mathbf{R}^N)$ by*

$$Mh = v \qquad \text{if and only if} \qquad v\cdot f_i = 0, \quad -2v\cdot f_{ij} = h_{ij}$$

(such a v exists and is unique because f is free). Then M is a right inverse to L in that

$$LMh = h.$$

Also M maps $C_0^{2,\alpha}(\overline{U}, \mathrm{Sym})$ into $C_0^{2,\alpha}(\overline{U}, \mathbf{R}^N)$. Finally M is bounded:

$$\|Mh\|_{C^{2,\alpha}} \le C_2 \|h\|_{C^{2,\alpha}} \tag{6}$$

for some constant C_2.

Proof: A straight forward exercise in (pointwise) linear algebra and chasing through the definition of the norm $\|\cdot\|_{C^{2,\alpha}}$. ❑

Theorem (Nash, Günther) *Let $U \subset \mathbf{R}^n$ be a bounded domain with smooth boundary and $f : U \to \mathbf{R}^N$ ($N = n + \frac{1}{2}n(n+1)$) a smooth free immersion. Then there is a $\delta > 0$ (only depending on f and α) so that if g is a $C^{2,\alpha}$ Riemannian that agrees with $df\cdot df$ on ∂U and $\|g - df\cdot df\|_{C^{2,\alpha}} < \delta$ then there is an immersion $v : U \to \mathbf{R}^N$ with $g = dv\cdot dv$ and that agrees with f on ∂U. The dependence of v on g is continuous and if g is $C^{k,\alpha}$, $2 \le k \le \infty$ then so is v.*

Proof: Every g that agrees with $df\cdot df$ on ∂U is of the form $df\cdot df + h$ where vanishes on ∂U, that is $h \in C_0^{2,\alpha}(\overline{U}, \mathrm{Sym})$. Let $v = f + u$ where $u \in C_0^{2,\alpha}(\overline{U}, \mathbf{R}^N)$ (then u vanishes on ∂U and thus v agrees with f on ∂U). As above we want to solve the system $\mathcal{I}[u] = h$. By Günther's lemma this can be rewritten as $L(u - Q(u, u)) = h$. Let M be the right inverse to L given by Nash's lemma. Then it is enough to find u so that

$$u = Mh + Q(u, u).$$

(Apply L to this equation and use $LMh = h$ to see that any such u solves our problem.) Define $\Phi_h : u \in C_0^{2,\alpha}(\overline{U}, \mathbf{R}^N) \to C_0^{2,\alpha}(\overline{U}, \mathbf{R}^N)$ by

$$\Phi_h(u) = Mh + Q(u, u).$$

Then we are looking for fixed points of Φ_h. We will show that Φ_h is a contraction on a closed ball. Let C_1 and C_2 be as in (4) and (6). Using the bound (4) and bilinearity

$$\begin{aligned}\|\Phi_h(u_2) - \Phi_h(u_1)\|_{C^{2,\alpha}} &= \|Q(u_2, u_2) - Q(u_1, u_1)\|_{C^{2,\alpha}} \\ &\leq \|Q(u_2, u_2 - u_1)\|_{C^{2,\alpha}} + \|Q(u_2 - u_1, u_1)\|_{C^{2,\alpha}} \\ &\leq C_1(\|u_1\|_{C^{2,\alpha}} + \|u_2\|_{C^{2,\alpha}})\|u_2 - u_1\|_{C^{2,\alpha}}\end{aligned}$$

Therefore

$$\|u_1\|_{C^{2,\alpha}},\ \|u_2\|_{C^{2,\alpha}} \leq \frac{1}{4C_1} \quad \text{implies} \quad \|\Phi_h(u_2) - \Phi_h(u_1)\|_{C^{2,\alpha}} \leq \frac{1}{2}\|u_2 - u_1\|_{C^{2,\alpha}}.$$

Using (4) and (6)

$$\|\Phi_h(u)\|_{C^{2,\alpha}} \leq \|Mh\|_{C^{2,\alpha}} + \|Q(u, u)\|_{C^{2,\alpha}} \leq C_2\|h\|_{C^{2,\alpha}} + C_1\|u\|^2_{C^{2,\alpha}}$$

so that

$$\|u\|_{C^{2,\alpha}} \leq \frac{1}{4C_1},\ \|h\|_{C^{2,\alpha}} \leq \frac{3}{16C_1C_2} \quad \text{implies} \quad \|\Phi_h\|_{C^{2,\alpha}} \leq \frac{1}{4C_1}.$$

Define

$$r := \frac{1}{4C_1}, \quad \delta = \frac{3}{16C_1C_2}$$

then for any $h \in C_0^{2,\alpha}(\overline{U}, \mathrm{Sym})$ with $\|h\|_{C^{2,\alpha}} \leq \delta$ we see Φ_h maps the closed ball $B(r) := \{u \in C_0^{2,\alpha}(\overline{U}, \mathbf{R}^N) : \|u\|_{C^{2,\alpha}} \leq r\}$ into its self and is a contraction with Lipschitz constant $\leq \frac{1}{2}$. By the contraction mapping principle (for example see [3, Page 69]) Φ_h has a unique fixed point u in $B(r)$. This proves the existence of a solution.

To see the dependence of u on h is continuous let $u_1, u_2 \in B(r)$ and $\|h_1\|_{C^{2,\alpha}}, \|h_2\|_{C^{2,\alpha}} \leq \delta$ with $\Phi_{h_1}(u_1) = u_1$ and $\Phi_{h_2}(u_1) = u_2$. Noting that $\Phi_{h_2} = M(h_2 - h_1) + \Phi_{h_1}$ and using (6) and that Φ_{h_2} is a contraction

$$\begin{aligned}\|u_2 - u_1\|_{C^{2,\alpha}} &= \|\Phi_{h_2}(u_2) - \Phi_{h_1}(u_1)\|_{C^{2,\alpha}} \\ &\leq \|M(h_2 - h_1)\|_{C^{2,\alpha}} + \|\Phi_{h_2}(u_2) - \Phi_{h_2}(u_1)\|_{C^{2,\alpha}} \\ &\leq C_2\|h_2 - h_1\|_{C^{2,\alpha}} + \frac{1}{2}\|u_2 - u_1\|_{C^{2,\alpha}}\end{aligned}$$

so that $\|u_2 - u_1\|_{C^{2,\alpha}} \leq 2C_2\|h_2 - h_1\|_{C^{2,\alpha}}$. This shows the dependence of u on h is not only continuous, but Lipschitz continuous.

This only leaves the regularity question. For fixed $u \in B(r)$ the linear operator $\mathcal{Q}_u := I - Q(u, \cdot)$ is elliptic of order zero. Thus if h is in $C^{k,\alpha}$, then so is Mh and $\Phi_h(u) = u$ iff $\mathcal{Q}_u u = Mh$. Therefore the regularity statement of the theorem follows form standard arguments. ❑

References

[1] C. Burstin, Ein Beitragzum Problem der Enbettung der Riemannschen Räume in Euklidischen Räume, Rec. Math. (Math. Sbornik), **38**, (1931) pp 74–85

[2] E. Cartan, Sur la possibilité de plonger un espace riemannian donné un espace euclidean, Ann. Soc. Pol. Math. **6**, (1927) pp 1–7

[3] D. Gilbarg and N. S. Trudinder, Elliptic Partial Differential Equations of the Second Order, Springer-Verlag, New York, 1977

[4] P. A. Griffiths and G. R. Jensen, Differential Systems and Isometric Embeddings, Princeton University Press, New Jersey , (1987)

[5] M. Gromov, Partial Differential Relations, Springer-Verlag, New York, 1986

[6] M. Günther, On the perturbation problem associated to isometric embeddings of Riemannian manifolds, Ann. Global Geom., **7**, (1989) pp 69–77

[7] M. Günther, Isometric Embeddings of Riemannian Manifolds, *in* Proceedings of the International Congress of Mathematicians, Kyoto 1990 (Vol. 2), Springer-Verlag, New York (1991) pp 1137–1143

[8] R. Hamilton, The inverse function theorem of Nash and Moser, Bull. Amer. Soc., **17**, (1982) pp 65–222

[9] M. Janet, Sur la possibilité de plonger un espace riemannian donné das un espace euclidean, Ann. Soc. Math. Pol., **5**, (1931) pp 74–85

[10] J. Moser, A new technique for the construction of solutions of nonlinear differential equations, Proc. Nat. Acad. Sci. U.S.A., **47**, (1963) pp 1824–1831

[11] J. Moser, A rapidly convergent iteration method and non-linear differential equations, Ann. Scuola Norm. Sup. Pisa, **20**, (1966) pp 499–535

[12] J. Nash, The imbedding problem for Riemannian manifolds, Ann. Math., **63**, (1956) pp 20–63

[13] M. Reese, e-mail message, 24 October 1994

[14] D. Yang, Günther's proof of the Nash isometric embedding theorem, Unpublished manuscript