ADDING ABRAHAM CLUBS AND α-PROPERNESS

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Abstract. For every indecomposable ordinal $\alpha < \omega_1$, we introduce a variant of Abraham forcing for adding a club in ω_1, which is $< \alpha$-proper but not α-proper.

§ 0. Introduction

To preserve $\aleph_1$, the properness property for a forcing notion was introduced by Shelah during his initial investigation of countable support iterations, see [4]. A poset $\mathbb{P}$ is called proper if forcing with $\mathbb{P}$ preserves stationary subsets of $[\lambda]^{\aleph_0}$, for all uncountable regular cardinals λ. If $\mathbb{P}$ is proper, then every countable set of ordinals in the extension is covered by a countable set of ordinals from the ground model, thus in particular forcing with $\mathbb{P}$ does not collapse $\aleph_1$. Shelah also introduced a characterization of properness, using $(N, \mathbb{P})$-generic conditions. The forcing notion $\mathbb{P}$ is proper if and only if for every large enough regular cardinals λ, and for club many countable elementary substructure N of $H(\lambda)$, for every condition $p \in N$, there is $q \leq p$ which is an $(N, \mathbb{P})$-generic condition, i.e., for every dense open subset $D \subseteq \mathbb{P}$ with $D \in N$, $D \cap N$ is predense below q.

Afterwards, Mitchell introduced strong properness using strongly $(N, \mathbb{P})$-generic conditions, which differs from $(N, \mathbb{P})$-generic conditions only in the point that every dense open $D \subseteq \mathbb{P} \cap N$ must be predense below that condition [2]. Obviously a notion of forcing is proper, if it is strongly proper.

Shelah also introduced, for each $\alpha < \omega_1$, a technical strengthening of properness known as "α-properness", and showed for every indecomposable ordinal α, there is a forcing notion which is $< \alpha$-proper but not α-proper.

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In this paper, for every indecomposable ordinal α, we introduce a variant of Abraham forcing from [1] for adding a club in ω_1, which is $< \alpha$-proper, but not α-proper, so giving a new example for separation α-properness.

In section 2 and inspired by Neeman's approach [3], we introduce a variant of Abraham forcing for adding a club in ω_1 with finite conditions and show that it is strongly proper, but not ω-proper. Then in section 3, for every indecomposable ordinal $\alpha < \omega_1$, we introduce a generalization $\mathbb{P}[\alpha]$ of our previous forcing which adds a club in ω_1, and such that $\mathbb{P}[\alpha]$ is $< \alpha$-proper but not α-proper. Indeed the forcing has the stronger property that if $\beta < \alpha$ and if $\mathcal{N} = \langle N_\xi : \xi \leq \beta \rangle$ is a β-tower with $\mathbb{P}[\alpha] \in N_0$, then every condition $p \in N_0$ has an extension q such that q is strongly $(N_\xi, \mathbb{P}[\alpha])$-generic, whenever $\xi \leq \beta$ is not a limit ordinal, and is $(N_\delta, \mathbb{P}[\alpha])$-generic, whenever $\delta \leq \beta$ is a limit ordinal.

§ 1. PRELIMINARIES

For a regular cardinal λ, let $H(\lambda)$ denote the collection of all sets x, whose transitive closure has size less than λ. We work with the structure $\langle H(\lambda), \in, \lhd^* \rangle$, where $\lhd^*$ is a fix well ordering of $H(\lambda)$. We use $N \prec H(\lambda)$ if $\langle N, \in, \lhd^* \rangle$ is an elementary substructure of $\langle H(\lambda), \in, \lhd^* \rangle$ and show the ordinal $N \cap \omega_1$ by δ_N. Let also $\mathcal{S}$ represents the collection of all countable elementary substructure of $H(\omega_1)$.

Let $\mathbb{P}$ be a notion of forcing and let λ be a regular cardinal. We say λ is large enough (with respect to $\mathbb{P}$), if $\lambda > (2^{|\mathrm{tr}(\mathbb{P})|})^+$, where $\mathrm{tr}(\mathbb{P})$ is the transitive closure of $\mathbb{P}$. Note that if λ is large enough, then all interesting statements about $\mathbb{P}$ are absolute between $H(\lambda)$ and the ground model V.

Definition 1.1. Let P be a notion of forcing and let λ be a large enough regular cardinal. Let $N \prec H(\lambda)$ with $\mathbb{P} \in N$. A condition $q \in \mathbb{P}$ is said to be strongly $(N, \mathbb{P})$-generic if every dense open subset $D \subseteq \mathbb{P} \cap N$ is predense below q, i.e. for every $r \in \mathbb{P}$, if $r \leq q$ then there are $s \in \mathbb{P}$ and $t \in D \cap \mathbb{P}$ such that s extends both r and t.

Definition 1.2. A notion of forcing $\mathbb{P}$ is called strongly proper if for every large enough regular cardinal λ and club many countable $N \prec H(\lambda)$ with $\mathbb{P} \in N$, every $p \in \mathbb{P} \cap N$ has a strongly $(N, \mathbb{P})$-generic extension.

The next lemma is evident.

Lemma 1.3. *If $\mathbb{P}$ is strongly proper, then $\mathbb{P}$ is proper, in particular forcing with $\mathbb{P}$ does not collapse $\aleph_1$.*

Definition 1.4. Let $\alpha < \omega_1$. The sequence $\mathcal{N} = \langle N_\xi \colon \xi \leq \alpha \rangle$ is said to be an α-tower if for some regular cardinal λ,

(1) N_ξ is a countable elementary substructure of $H(\lambda)$ for all $\xi \leq \alpha$;
(2) $\alpha \in N_0$;
(3) $N_\zeta \in N_{\zeta+1}$ for all $\zeta < \alpha$;
(4) $N_\delta = \bigcup_{\xi<\delta} N_\xi$ for all limit ordinals $\delta \leq \alpha$;
(5) $\langle N_\zeta : \zeta \leq \xi \rangle \in N_{\xi+1}$ for every $\xi < \alpha$.

The notion of α-properness is defined as follows.

Definition 1.5. Assume $\alpha < \omega_1$, and $\mathbb{P}$ is a forcing notion. $\mathbb{P}$ is called α-proper if for every α-tower $\mathcal{N} = \langle N_\xi \colon \xi \leq \alpha \rangle$ with $\mathbb{P} \in N_0$, every condition $p \in \mathbb{P} \cap N_0$ has an extension q which is $(\mathcal{N}, \mathbb{P})$-generic, i.e., q is an $(N_\xi, \mathbb{P})$-generic condition for each $\xi \leq \alpha$. $\mathbb{P}$ is called $< \alpha$-proper, if it is β-proper for each $\beta < \alpha$.

§ 2. Adding a club in ω_1 by finite conditions

In this section we introduce a variant of Abraham forcing [1], and show that it is strongly proper but not ω-proper. We start by defining our forcing notion $\mathbb{P}$.

Definition 2.1. Let $\mathbb{P}$ consist of pairs $p = \langle \mathcal{M}_p, f_p \rangle$, where

(1) $\mathcal{M}_p = \langle M_i^p : i < n_p \rangle$ is a finite $\in$-increasing sequence of elements of $\mathcal{S}$, and
(2) the function $f_p : \mathcal{M}_p \longrightarrow H(\omega_1)$ is defined such that $f_p(M_i^p)$ is a finite subset of M_{i+1}^p if $i < n_p - 1$, and $f_p(M_{n_p-1}^p)$ is a finite subset of $H(\omega_1)$.

For $p, q \in \mathbb{P}$, we say $q \leq p$ if and only if $\mathcal{M}_p \subseteq \mathcal{M}_q$ and $f_p(M) \subseteq f_q(M)$ for every $M \in \mathcal{M}_p$.

Lemma 2.2. *If $G \subseteq \mathbb{P}$ is a generic filter, then $C = \{\delta_M \colon M \in \mathcal{M}_p$ for some $p \in G\}$ is a club in ω_1.*

Proof. First, we need to prove the following claim:

Claim 2.3. *For every $p \in \mathbb{P}$ and $\gamma \in \omega_1$, there is $p' \leq p$ such that $\gamma < \delta_N$ for some $N \in \mathcal{M}_{p'}$.*

Proof of the Claim. Since $p, \gamma \in H(\omega_1)$, we can find $N \prec H(\omega_1)$ such that $p, \gamma \in N$. Let $p' = \langle \mathcal{M}_{p'}, f_{p'} \rangle$ be such that $\mathcal{M}_{p'} = \mathcal{M}_p \cup \{N\}$, $f_{p'}(M) = f_p(M)$ for every $M \in \mathcal{M}_p$ and $f_{p'}(N) = \emptyset$. It is easy to check that p' is a condition which extends p and has the desired property. □

Given any $\gamma \in \omega_1$, by Claim 2.3, the set

$$D_\gamma = \{q \in \mathbb{P} \colon \exists M \in \mathcal{M}_q(\gamma < \delta_M)\}$$

is dense in $\mathbb{P}$. This implies that C is unbounded in ω_1.

Now we claim that for every $p \in \mathbb{P}$ and $\gamma \in \omega_1$, if p forces γ to be a limit point of $\dot{C}$, then p also forces it is an element of $\dot{C}$. By contradiction suppose p does not hold in the statement. Hence there is no $M \in \mathcal{M}_p$ with $\delta_M = \gamma$. Using Claim 2.3 and the assumption that p forces γ is a limit point of $\dot{C}$, by extending p if necessary, we can assume that for some i with $i + 1 < n_p$, $\delta_{M_i^p} < \gamma < \delta_{M_{i+1}^p}$. Let $\xi < \delta_{M_{i+1}^p}$ be any ordinal greater than γ. Set $q = \langle \mathcal{M}_q, f_q \rangle$ where $\mathcal{M}_q = \mathcal{M}_p$ and $f_q(M) = f_p(M)$ for all $M \neq M_i^p$, and $f_q(M_i^p) = f_p(M_i^p) \cup \{\xi\}$. Now we have $q \leq p$ is a condition and every extension of q forces that γ is not a limit point of $\dot{C}$, indeed for any extension r of q, r forces $\dot{C}$ has empty intersection with the interval $(\delta_{M_i^p}, \xi)$, in particular r forces $\dot{C} \cap \gamma \subseteq \delta_{M_i^p} + 1$. □

Lemma 2.4. $\mathbb{P}$ *is strongly proper.*

Proof. Suppose that λ is a large enough regular cardinal, $N \prec H(\lambda)$ is countable with $\mathbb{P} \in N$, and $p \in \mathbb{P} \cap N$. Set $N' = N \cap H(\omega_1)$ and $p' = \langle \mathcal{M}_{p'}, f_{p'} \rangle$, where $\mathcal{M}_{p'} = \mathcal{M}_p \cup \{N'\}$ and $f_{p'}(M) = f_p(M)$ for all $M \in \mathcal{M}_p$ and $f_{p'}(N') = \emptyset$.

We demonstrate that p' serves as a strongly $(N, \mathbb{P})$-generic extension of p. It is clear that p' is a condition which extends p. Now we argue that, if $q \leq p'$, then $q{\restriction}_N = \langle \mathcal{M}_q \cap N, f_q{\restriction}_{\mathcal{M}_q \cap N} \rangle \in \mathbb{P} \cap N$.

Claim 2.5. $q{\restriction}_N \in \mathbb{P} \cap N$.

Proof of the Claim. First we show that $\mathcal{M}_q \cap N$ is exactly the initial segment of $\mathcal{M}_q$ before N'. Since the sequence of elements of $\mathcal{M}_q$ before N' is a finite subset of $N' \subseteq N$, so it is also a finite subset of N. If there is $Q \in \mathcal{M}_q$ such that $N' \in Q$, then $Q \notin N$, since otherwise $Q \in N \cap H(\omega_1) = N'$ which gives a contradiction. The result follows immediately. □

Now let $D \subseteq \mathbb{P} \cap N$ dense open and let $r \in D$ be an extension of $q\restriction_N$. We have to find a condition s such that $s \leq r, q$.

Claim 2.6. *Let $\mathcal{M}_s = \mathcal{M}_r \cup \mathcal{M}_q$, $f_s(M) = f_r(M)$ if $M \in \mathcal{M}_r$ and $f_s(M) = f_q(M)$ if $M \in \mathcal{M}_q \setminus \mathcal{M}_r$. Then s is a common extension of q and r.*

Proof of the Claim. $\mathcal{M}_s$ is a finite increasing sequence of elements of $\mathcal{S}$, since $\mathcal{M}_q \cap N \subseteq \mathcal{M}_r \subseteq N'$ and $N' \in Q$ for every $Q \in \mathcal{M}_q \setminus \mathcal{M}_r$. It is also easily seen that f_s has the property that $f_s(M_i^s) \in M_{i+1}^s$ for every $i+1 < n_s$. Thus s is a condition, and it clearly extends both of q and r. □

Therefore, s is a witness for D to be predense below p', and $\mathbb{P}$ is strongly proper. □

Lemma 2.7. *$\mathbb{P}$ is not ω-proper.*

Proof. Assume that $\mathcal{N} = \langle N_i \colon i \leq \omega \rangle$ is an ω-tower of elements of $\mathcal{S}$ such that $\dot{C}, \mathbb{P} \in N_0$, where $\dot{C}$ is the canonical name for the club C, and let $p \in \mathbb{P} \cap N_0$. We show that there is no extension $q \leq p$ which is an $(\mathcal{N}, \mathbb{P})$-generic condition, meaning that q is $(N_i, \mathbb{P})$-generic for every $i \leq \omega$. Suppose by the way of contradiction that there is such a condition q. For each $i \leq \omega$, as q is $(N_i, \mathbb{P})$-generic and $\dot{C} \in N_i$, we have $q \Vdash$"$\dot{C} \cap \delta_{N_i}$ is unbounded in δ_{N_i}". As $q \Vdash$"$\dot{C}$ is a club in ω_1", hence $q \Vdash$"$\delta_{N_i} \in \dot{C}$".

On the other hand, by claim 2.3 and since $\mathcal{M}_q$ is finite, there are $q' \leq q$, $n \in \omega$, and $i+1 < n_{q'}$ such that $\delta_{N_i^{q'}} < \delta_{N_n} < \delta_{N_\omega} \leq \delta_{N_{i+1}^{q'}}$. Since $f_{q'}(N_i^{q'})$ is a finite subset of $N_{i+1}^{q'}$, there exist $m \in \omega$ and $\delta_{N_m} < \xi < \delta_{N_{i+1}^{q'}}$ such that $n < m$ and $\xi \notin f_{q'}(N_i^{q'})$. Let q'' be such that $\mathcal{M}_{q''} = \mathcal{M}_{q'}$ and $f_{q''}(M) = f_{q'}(M)$ for all $M \neq N_i^{q'}$ and $f_{q''}(N_i^{q'}) = f_{q'}(N_i^{q'}) \cup \{\xi\}$. As before, $q'' \in \mathbb{P}$ is an extension of q such that for all $r \leq q''(r \Vdash$ "$\delta_{N_m} \notin \dot{C}$") which leads us to a contradiction. □

§ 3. General case

In this section, we present a new proof for a theorem by Shelah, which states that for every indecomposable countable ordinal α, there exists a forcing notion $\mathbb{P}[\alpha]$, such that $\mathbb{P}[\alpha]$ is β-proper for every $\beta < \alpha$, but is not α-proper.

Definition 3.1. Let $\alpha < \omega_1$ be an indecomposable ordinal. The forcing notion $\mathbb{P}[\alpha]$ consists of conditions $p = \langle \mathcal{M}_p, f_p, \mathcal{W}_p \rangle$ such that:

- $\mathcal{M}_p = \langle M^p_\xi \colon \xi \leq \gamma_p \rangle$ is an $\in$-increasing sequence of elements of $\mathcal{S}$ for some $\gamma_p < \alpha$ which is continuous at limits;
- the function $f_p : \mathcal{M}_p \longrightarrow H(\omega_1)$ is defined such that $f_p(M^p_\xi)$ is a finite subset of $M^p_{\xi+1}$ for $\xi < \gamma$, and $f_p(M^p_\gamma)$ is a finite subset of $H(\omega_1)$; and
- the witness $\mathcal{W}_p$ is a subset of $\mathcal{M}_p$, such that for every $N \in \mathcal{W}_p$, $p{\restriction}_N = \langle \mathcal{M}_p \cap N, f_p{\restriction}_{\mathcal{M}_p \cap N}, \mathcal{W}_p \cap N \rangle \in N$.

For $p, q \in \mathbb{P}[\alpha]$, we say $q \leq p$ if and only if $\mathcal{M}_p \subseteq \mathcal{M}_q$, $f_p(M) \subseteq f_q(M)$ for every $M \in \mathcal{M}_p$ and $\mathcal{W}_p \subseteq \mathcal{W}_q$.

Lemma 3.2. *If $G \subseteq \mathbb{P}[\alpha]$ is a generic filter, then $C = \{\delta_M \colon M \in \mathcal{M}_p$ for some $p \in G\}$ is a club.*

Proof. Let $\gamma \in \omega_1$. Similar to the proof of claim 2.3, for every $p \in \mathbb{P}[\alpha]$ we can find $q \leq p$ such that for some $N \in \mathcal{M}_q$, $\gamma < \delta_N$, so the set $D_\gamma = \{q \in \mathbb{P}[\alpha] \colon \exists \xi \leq \gamma_q (\gamma \leq \delta_{M_\xi})\}$ is dense open, which guaranties that C is unbounded in ω_1.

Now assume that $p \Vdash$"$\gamma \notin \dot{C}$". We show that p forces that γ can not be a limit point of C. For simplicity write $M_\eta = M^p_\eta$, for all $\eta \leq \gamma_p$. First, by extending p let us assume that $\gamma < \delta_{M_{\gamma_p}}$. Let

$$\delta = \sup\{\delta_{M_\xi} \colon \xi \leq \gamma_p \wedge \delta_{M_\xi} < \gamma\}.$$

As $\mathcal{M}_p$ is continuous, there exists $\eta < \gamma_p$ such that $\delta_{M_\eta} = \delta$. By the assumption $p \Vdash$"$\gamma \notin \dot{C}$", $\delta < \gamma < \delta_{M_{\eta+1}}$, hence there is $\zeta \in M_{\eta+1}$ such that $\gamma < \zeta$. Let q be such that, $\mathcal{M}_q = \mathcal{M}_p$, $f_q(M_\xi) = f_p(M_\xi)$ for all $\xi \neq \eta$, $f_q(M_\eta) = f_p(M_\eta) \cup \{\zeta\}$, and $\mathcal{W}_q = \mathcal{W}_p$. It is easily seen that q is a condition, the main point is that $\mathcal{W}_q \subseteq \mathcal{M}_q = \mathcal{M}_p$, hence $\zeta \in N$ for all $N \in \mathcal{W}_q$ with $M_\eta \in N$, in particular, $q{\restriction}_N \in N$ for all $N \in \mathcal{W}_q$.

Clearly q extends p, and every condition extending q forces that "$\dot{C}\cap(\delta_{M_\eta},\zeta]=\emptyset$" which guaranties that γ can not be a limit point of C. □

Theorem 3.3. $\mathbb{P}[\alpha]$ *is β-proper for every $\beta<\alpha$, but is not α-proper.*

The proof of theorem 3.3 will consist of a series of lemmas. In fact, for every $\beta<\alpha$, and every β-tower $\mathcal{N}=\langle N_\zeta:\zeta\leq\beta\rangle$ with $\mathbb{P}[\alpha]\in N_0$, and every condition $p\in\mathbb{P}[\alpha]\cap N_0$, we will find a condition $p'\leq p$ such that:

$(*)^{\beta,\mathcal{N}}_{p'}$: for every $\zeta\leq\beta$,

(a) if ζ is a successor ordinal, then p' is a strongly $(N_\zeta,\mathbb{P}[\alpha])$-generic condition, and

(b) if ζ is a limit ordinal, then p' is an $(N_\zeta,\mathbb{P}[\alpha])$-generic condition.

Also we show that for any α-tower $\mathcal{N}$ with $\mathbb{P}[\alpha],\dot{C}\in N_0$, no condition $q\in\mathbb{P}[\alpha]$ is an $(\mathcal{N},\mathbb{P}[\alpha])$-generic condition.

First, let us show that $\mathbb{P}[\alpha]$ is β-proper for every $\beta<\alpha$. Fix an arbitrary $\beta<\alpha$ and let $\mathcal{N}=\langle N_\zeta\colon\zeta\leq\beta\rangle$ be a β-tower of elements of $\mathcal{S}$ such that $\mathbb{P}[\alpha]\in N_0$. Let $p\in\mathbb{P}[\alpha]\cap N_0$. Set $p'=\langle\mathcal{M}_{p'},f_{p'},\mathcal{W}_{p'}\rangle$, where:

(1) $\mathcal{M}_{p'}=\mathcal{M}_p\cup\mathcal{N}$,
(2) $f_{p'}(M^{p'}_\xi)=f_{p'}(M^p_\xi)=f_p(M^p_\xi)$ for $\xi\leq\gamma_p$,
(3) $f_{p'}(M^{p'}_{\gamma_p+1+\zeta})=f_{p'}(N_\zeta)=\delta_{N_\zeta}+1$, for $\zeta\leq\beta$, and
(4) $\mathcal{W}_{p'}=\mathcal{W}_p\cup\{N_\zeta\in\mathcal{N}\colon\zeta\text{ is not a limit ordinal}\}$.

We assert that p' witnesses $(*)^{\beta,\mathcal{N}}_{p'}$ holds. By the construction, it is clear that $p'\leq p$, but we have to show that $p'\in\mathbb{P}[\alpha]$.

Lemma 3.4. p' *is a condition.*

Proof. Since $p\in N_0$, so in particular $M_{\gamma_p}\in N_0$ and $f_p(M^p_{\gamma_p})\in N_0$. Hence $\mathcal{M}_{p'}$ is an $\in$-increasing continuous sequence of elements of $\mathcal{S}$ of length $\gamma_p+\beta$. As α is indecomposable and γ_p and β are less than α, the length of $\mathcal{M}_{p'}$ which is equal to $\gamma_p+\beta$ is also less than α. Also by the construction, $f_{p'}(N_\zeta)\in N_{\zeta+1}$ for every $N_\zeta\in\mathcal{N}$.

It is enough to show that $\mathcal{W}_{p'}$ satisfies the desired requirement. Let $Q\in\mathcal{W}_{p'}$. If $Q\in\mathcal{W}_p$, then since $\mathcal{W}_p\in N_0$, we have $p'\restriction_Q=p\restriction_Q\in Q$. Now let $Q\in\mathcal{W}_{p'}\setminus\mathcal{W}_p$. Thus either $Q = N_0$ or $Q = N_{\zeta+1}$ for some $\zeta < \beta$. If $Q = N_0$, then $p'\restriction_{N_0} = p \in N_0$ and we are done. Now assume that $Q = N_{\zeta+1}$, for some $\zeta < \beta$. In this case, $\mathcal{M}_{p'} \cap N_{\zeta+1}$ is the initial segment of $\mathcal{M}_{p'}$ with the last element N_ζ, which belongs to $N_{\zeta+1}$. Also for all of the $Q' \in \mathcal{M}_{p'} \cap N_{\zeta+1}$, $f_{p'}(Q') \in N_{\zeta+1}$, since $\mathcal{M}_{p'}$ is an $\in$-increasing continuous sequence of countable elementary substructures. As $\mathcal{W}_p \in N_0$, and $\mathcal{N}$ is an $\in$-increasing continuous sequence of countable models, so $\mathcal{W}_{p'} \cap N_{\zeta+1} \in N_{\zeta+1}$. □

Lemma 3.5. *p' is strongly $(N_\zeta, \mathbb{P}[\alpha])$-generic condition for every non-limit ordinal $\zeta \leq \beta$.*

Proof. We will proof this lemma in a series of claims. Thus suppose that $\zeta \leq \beta$ is not a limit ordinal.

Claim 3.6. *If $q \leq p'$, then $q\restriction_{N_\zeta} \in \mathbb{P}[\alpha] \cap N_\zeta$.*

Proof of the Claim. The proof is straightforward, the key point is that since $\mathcal{W}_{p'} \subseteq \mathcal{W}_q$, so $N_\zeta \in \mathcal{W}_q$, which in particular implies that $q\restriction_{N_\zeta} \in N_\zeta$. □

Claim 3.7. *Assume $D \subseteq \mathbb{P}[\alpha] \cap N_\zeta$ is dense open, and suppose that $r \in D$ is an extension of $q\restriction_{N_\zeta}$. Let $s = \langle \mathcal{M}_s, f_s, \mathcal{W}_s \rangle$ be such that $\mathcal{M}_s = \mathcal{M}_r \cup \mathcal{M}_q$, $f_s\restriction_{\mathcal{M}_r} = f_r$, $f_s\restriction_{\mathcal{M}_q \setminus \mathcal{M}_r} = f_q$ and $\mathcal{W}_s = \mathcal{W}_r \cup \mathcal{W}_q$. Then s is a condition which extends both of r and q.*

Proof of the Claim. By construction, it is easy to see s is a common extension of r and q, so we just have to show that $s \in \mathbb{P}[\alpha]$. Since the order types of both $\mathcal{M}_r$ and $\mathcal{M}_q$ are less than the indecomposable ordinal α, so is the order type of $\mathcal{M}_s$ (which is at most $\gamma_r + \gamma_q$). Furthermore, $\mathcal{M}_s$ is an $\in$-increasing continuous chain, of the form

$$\mathcal{M}_s = \mathcal{M}_r^\frown \langle N_\zeta \rangle^\frown \langle N \in \mathcal{M}_q : N_\zeta \in N \rangle.$$

Note that $r \in N_\zeta$ gives us $\mathcal{M}_r \in N_\zeta$, in particular $M^r_{\gamma_r} \in N_\zeta$ and $f_s(M^s_{\gamma_r}) = f_r(M^r_{\gamma_r}) \in N_\zeta$. Also, $f^s(N_\zeta) = f^q(N_\zeta) \in N$, for the least $N \in \mathcal{M}_q$ with $N_\zeta \in N$. It immediately follows that for every $\xi < \gamma_s$, $f_s(M^s_\xi) \in M^s_{\xi+1}$.

Now let $Q \in \mathcal{W}_s$. If $Q \in \mathcal{W}_r$, then $s\restriction_Q = r\restriction_Q \in Q$. If $Q \in \mathcal{W}_q \setminus \mathcal{W}_r$, then $s\restriction_Q = s\restriction_{N_\zeta} \cup s\restriction_{(N_\zeta, Q)} = r \cup q\restriction_{(N_\zeta, Q)} = r \cup q\restriction_Q$ which clearly is in Q. □

It follows from the above claim that p' is a strongly $(N_\zeta, \mathbb{P}[\alpha])$-generic condition. □

The next lemma is well-known, which takes care of the limit steps.

Lemma 3.8. *Suppose $\zeta \leq \beta$ is a limit ordinal and for every ordinal η less than ζ, p' is $(N_\eta, \mathbb{P}[\alpha])$-generic condition. Then p' is an $(N_\zeta, \mathbb{P}[\alpha])$-generic condition.*

Proof. Let $\dot{\tau} \in N_\zeta$ be a name of an ordinal. Since $N_\zeta = \bigcup_{\eta<\zeta} N_\eta$, so $\dot{\tau} \in N_\eta$ for some $\eta < \zeta$. By assumption, $p' \Vdash$ "$N_\eta \cap \mathrm{Ord} = N_\eta[\dot{G}] \cap \mathrm{Ord}$", i.e. $p' \Vdash$ "$\dot{\tau} \in N_\eta$" which implies $p' \Vdash$ "$\dot{\tau} \in N_\zeta$. Since $\dot{\tau}$ is arbitrary, so p' is $(N_\zeta, \mathbb{P}[\alpha])$-generic condition. □

We now show that $\mathbb{P}[\alpha]$ is not α-proper.

Lemma 3.9. $\mathbb{P}[\alpha]$ *is not α-proper.*

Proof. Let $\mathcal{N} = \langle N_\zeta \colon \zeta \leq \alpha \rangle$ be an arbitrary α-tower,with $\mathbb{P}[\alpha], \dot{C} \in N_0$, where $\dot{C}$ is the canonical name for the club added by the forcing. Let $p \in \mathbb{P}[\alpha] \cap N_0$, and assume towards contradiction that $q \leq p$ is an $(\mathcal{N}, \mathbb{P}[\alpha])$-generic condition. Without loss of generality, assume $\delta_{N_\alpha} < \delta_{M^q_{\gamma_q}}$. For every $\zeta \leq \alpha$, as $\dot{C} \in N_\zeta$ and q is an $(N_\zeta, \mathbb{P}[\alpha])$-generic condition, q forces $\dot{C} \cap \delta_{N_\zeta}$ is unbounded in δ_{N_ζ}, hence $q \Vdash$"$\delta_{N_\zeta} \in \dot{C}$".

As α is indecomposable, and the set $\mathcal{M}_q$ has order type less than α, we can find some ordinal $\zeta < \alpha$ such that

$$\{\delta_M : M \in \mathcal{M}_q\} \cap [\delta_{N_\zeta}, \delta_{N_{\zeta+2}}) = \emptyset.$$

Let $\eta < \gamma_q$ be maximal such that $\delta_{M^q_\eta} < \delta_{N_\zeta}$. It then follows that $\delta_{M^q_{\eta+1}} \geq \delta_{N_{\zeta+2}}$. As $q \Vdash$"$\delta_{N_\zeta} \in \dot{C}$", we can find some model $\tilde{M}$ such that:

- $\tilde{M} \in M^q_{\eta+1}$,
- $M^q_\eta \in \tilde{M}$,
- $\delta_{\tilde{M}} = \delta_{N_\zeta}$.

let r be such that $\mathcal{M}_r = \mathcal{M}_q \cup \{\tilde{M}\}$, $f_r(M) = f_q(M)$ for all $M \in \mathcal{M}_q$, $f_r(\tilde{M}) = \{\delta_{N_{\zeta+1}} + 1\}$ and $\mathcal{W}_r = \mathcal{W}_q$. Then r is a condition. The main point is that for all $N \in \mathcal{W}_r$, if $\tilde{M} \in N$, then $\delta_{N_{\zeta+1}} + 1 \in N$, as $\delta_{N_{\zeta+1}} + 1 < \delta_{N_{\zeta+2}} \leq \delta_N$. Clearly r is an extension of q and it forces that $\delta_{N_{\zeta+1}} \notin \dot{C}$, which is a contradiction. □

Theorem 3.3 follows.

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