Questions that haunt my nights, while Shelah might answer by dawn

Mohammad Golshani (IPM)

Shelah's 80th Birthday Conference

July 14th-15th, 2025.

Table of Contents

Why such a talk

- Here is a conversation between Shelah and Alexander Soifer:

Why such a talk

- Here is a conversation between Shelah and Alexander Soifer:

  Shelah: Why do people attend conferences?

Why such a talk

- Here is a conversation between Shelah and Alexander Soifer:

  Shelah: Why do people attend conferences?
  Soifer: To show their latest results, to learn about achievements of others and to socialize.

Why such a talk

- Here is a conversation between Shelah and Alexander Soifer:

  Shelah: Why do people attend conferences?
  Soifer: To show their latest results, to learn about achievements of others and to socialize.
  Shelah: None of this makes any sense. People should attend conferences in order to solve together problems they could not solve on their own.

Why such a talk

- Here is a conversation between Shelah and Alexander Soifer:

  Shelah: Why do people attend conferences?
  Soifer: To show their latest results, to learn about achievements of others and to socialize.
  Shelah: None of this makes any sense. People should attend conferences in order to solve together problems they could not solve on their own.

- In this talk, I will present some of my favorite problems-some that I have thought about, and others that I would like to explore in the future (which means I may know very little about some of them!).

Set theory questions

- Is it consistent that for some singular cardinal κ, there are κ^+-Aronszajn trees and all of them are special?

Set theory questions

- Is it consistent that for some singular cardinal κ, there are κ^+-Aronszajn trees and all of them are special?
- Is there a model of ZFC in which $\diamondsuit$ holds and there is Σ^2_2-definable well-ordering of the reals?

Set theory questions

- Is it consistent that for some singular cardinal κ, there are κ^+-Aronszajn trees and all of them are special?
- Is there a model of ZFC in which $\diamondsuit$ holds and there is Σ^2_2-definable well-ordering of the reals?
- Is it consistent that there are no $\aleph_2$-Souslin trees, but there exists a non-special $\aleph_2$-Aronszajn tree?

Set theory questions

- Is it consistent that for some singular cardinal κ, there are κ^+-Aronszajn trees and all of them are special?
- Is there a model of ZFC in which $\diamondsuit$ holds and there is Σ^2_2-definable well-ordering of the reals?
- Is it consistent that there are no $\aleph_2$-Souslin trees, but there exists a non-special $\aleph_2$-Aronszajn tree?
- Is PFA for $\aleph_2$-c.c. proper posets consistent with large continuum?

Set theory questions

- Is it consistent that for some singular cardinal κ, there are κ^+-Aronszajn trees and all of them are special?
- Is there a model of ZFC in which $\diamondsuit$ holds and there is Σ^2_2-definable well-ordering of the reals?
- Is it consistent that there are no $\aleph_2$-Souslin trees, but there exists a non-special $\aleph_2$-Aronszajn tree?
- Is PFA for $\aleph_2$-c.c. proper posets consistent with large continuum?
- (Aspero) Is PFA for ω-proper posets consistent with large continuum?

Set theory questions

- (Woodin) Does GCH below a strongly compact cardinal imply GCH everywhere?

Set theory questions

- (Woodin) Does GCH below a strongly compact cardinal imply GCH everywhere?
- (Woodin) Is it consistent that there are models of ZFC $M \subseteq V$, such that $\aleph_\omega^V$ is a measurable cardinal in M, in V, GCH holds below $\aleph_\omega$ and $|\mathcal{P}(\aleph_\omega^V) \cap M| \geq \aleph_{\omega+3}^V$?

Set theory questions

- (Woodin) Does GCH below a strongly compact cardinal imply GCH everywhere?
- (Woodin) Is it consistent that there are models of ZFC $M \subseteq V$, such that $\aleph_\omega^V$ is a measurable cardinal in M, in V, GCH holds below $\aleph_\omega$ and $|\mathcal{P}(\aleph_\omega^V) \cap M| \geq \aleph_{\omega+3}^V$?
- Is it consistent that some singular cardinal $\kappa < \aleph_\kappa$ is a Jonsson cardinal? (the question for $\aleph_\omega$ is a longstanding open problem!)

Set theory questions

- (Woodin) Does GCH below a strongly compact cardinal imply GCH everywhere?
- (Woodin) Is it consistent that there are models of ZFC $M \subseteq V$, such that $\aleph_\omega^V$ is a measurable cardinal in M, in V, GCH holds below $\aleph_\omega$ and $|\mathcal{P}(\aleph_\omega^V) \cap M| \geq \aleph_{\omega+3}^V$?
- Is it consistent that some singular cardinal $\kappa < \aleph_\kappa$ is a Jonsson cardinal? (the question for $\aleph_\omega$ is a longstanding open problem!)
- Is it consistent that κ is an inaccessible non-weakly compact cardinal and there are no κ-souslin trees?

Set theory questions

- (Woodin) Is it consistent that $\aleph_\omega$ is strong limit, $2^{\aleph_\omega} > \aleph_{\omega+1}$ and there are no $\aleph_{\omega+1}$-Aronszajn trees?

Set theory questions

- (Woodin) Is it consistent that $\aleph_\omega$ is strong limit, $2^{\aleph_\omega} > \aleph_{\omega+1}$ and there are no $\aleph_{\omega+1}$-Aronszajn trees?
- (Shelah) Is the negation of continuum hypothesis consistent with the existence of a universal triangle-free graph on ω_1?

Set theory questions

- (Woodin) Is it consistent that $\aleph_\omega$ is strong limit, $2^{\aleph_\omega} > \aleph_{\omega+1}$ and there are no $\aleph_{\omega+1}$-Aronszajn trees?
- (Shelah) Is the negation of continuum hypothesis consistent with the existence of a universal triangle-free graph on ω_1?
- (Solovay, Sy Friedman) Is it consistent that the gap-2 two cardinal transfer principle holds but the gap-3 two cardinal transfer principle fails?

Set theory questions

- Let $\mathbb{A}(\aleph_n)$ be the statement: all trees of size and height $\aleph_n$ are special?
  Is it consistent that $\bigwedge_{n \geq 1} \mathbb{A}(\aleph_n)$?

Set theory questions

- Let $\mathbb{A}(\aleph_n)$ be the statement: all trees of size and height $\aleph_n$ are special?
  Is it consistent that $\bigwedge_{n \geq 1} \mathbb{A}(\aleph_n)$?
- Is it consistent that the first ω-many strongly compact cardinals coincide with the first ω-many measurable cardinals?

Set theory questions

- Let $\mathbb{A}(\aleph_n)$ be the statement: all trees of size and height $\aleph_n$ are special?
  Is it consistent that $\bigwedge_{n \geq 1} \mathbb{A}(\aleph_n)$?
- Is it consistent that the first ω-many strongly compact cardinals coincide with the first ω-many measurable cardinals?
- Is it consistent that there is no c.c.c. forcing notion adding a minimal real?

Set theory questions

- Let $\mathbb{A}(\aleph_n)$ be the statement: all trees of size and height $\aleph_n$ are special?
  Is it consistent that $\bigwedge_{n \geq 1} \mathbb{A}(\aleph_n)$?
- Is it consistent that the first ω-many strongly compact cardinals coincide with the first ω-many measurable cardinals?
- Is it consistent that there is no c.c.c. forcing notion adding a minimal real?
- (Prikry) Is it consistent that every non-trivial c.c.c. forcing notion adds a Cohen real or a random real?
  Talagrand's forcing is a candidate for a counterexample, it is known it adds no Cohen real, but it is open if it adds a random real or not.

Set theory questions

- Let $\mathbb{A}(\aleph_n)$ be the statement: all trees of size and height $\aleph_n$ are special?
  Is it consistent that $\bigwedge_{n \geq 1} \mathbb{A}(\aleph_n)$?
- Is it consistent that the first ω-many strongly compact cardinals coincide with the first ω-many measurable cardinals?
- Is it consistent that there is no c.c.c. forcing notion adding a minimal real?
- (Prikry) Is it consistent that every non-trivial c.c.c. forcing notion adds a Cohen real or a random real?
  Talagrand's forcing is a candidate for a counterexample, it is known it adds no Cohen real, but it is open if it adds a random real or not.
- Is there a forcing axiom, equivalent to the existence of black boxes?

Set theory questions

- Is it consistent that there are (ω_1, ω_3)-superatomic Boolean algebras? what about (ω, ω_3)-superatomic Boolean algebras?

Set theory questions

- Is it consistent that there are (ω_1, ω_3)-superatomic Boolean algebras? what about (ω, ω_3)-superatomic Boolean algebras?
- (Kunen) Could MA first fails at a singular cardinal of cofinality above ω_1?

Set theory questions

- Is it consistent that there are (ω_1, ω_3)-superatomic Boolean algebras? what about (ω, ω_3)-superatomic Boolean algebras?
- (Kunen) Could MA first fails at a singular cardinal of cofinality above ω_1?
- (Foreman-Magidor-Shelah) Suppose that every constructible forcing adds a real over V. Must $0^\sharp$ exist?

Set theory questions

- Is it consistent that there are (ω_1, ω_3)-superatomic Boolean algebras? what about (ω, ω_3)-superatomic Boolean algebras?
- (Kunen) Could MA first fails at a singular cardinal of cofinality above ω_1?
- (Foreman-Magidor-Shelah) Suppose that every constructible forcing adds a real over V. Must $0^\sharp$ exist?
- Is $\Diamond_\kappa$ true for κ a weakly compact cardinal?

Set theory questions

- Is it consistent that there are (ω_1, ω_3)-superatomic Boolean algebras? what about (ω, ω_3)-superatomic Boolean algebras?
- (Kunen) Could MA first fails at a singular cardinal of cofinality above ω_1?
- (Foreman-Magidor-Shelah) Suppose that every constructible forcing adds a real over V. Must $0^\sharp$ exist?
- Is $\diamondsuit_\kappa$ true for κ a weakly compact cardinal?
- Is it consistent that every two Kurepa trees are club isomorphic? I heard this question from Hossein Lamei Ramandi.

Set theory questions

- Let X be an extremally disconnected (i.e. such that the closure of open sets is open) compact Hausdorff space. Then
  1. $C(X)$ is the space of continuous functions $f : X \to \mathbb{C}$,
  2. $C^+(X)$ is the space of continuous functions $f : X \to \mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$ such that the pre-image of ∞ is nowhere dense ($\mathbb{S}^2$ is the one point compactification of $\mathbb{C}$).

Set theory questions

- Let X be an extremally disconnected (i.e. such that the closure of open sets is open) compact Hausdorff space. Then
  1. $C(X)$ is the space of continuous functions $f : X \to \mathbb{C}$,
  2. $C^+(X)$ is the space of continuous functions $f : X \to \mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$ such that the pre-image of ∞ is nowhere dense ($\mathbb{S}^2$ is the one point compactification of $\mathbb{C}$).
- Suppose $\mathbb{B}$ is a complete Boolean algebra, and consider the Boolean valued universe $V^{\mathbb{B}}$. Let $St(\mathbb{B})$ be the Stone space of $\mathbb{B}$. Then there is a correspondence between:
  1. the family of $\mathbb{B}$-names for complex numbers in the boolean valued model $V^{\mathbb{B}}$:

     $$\dot{\mathbb{C}} = \{\tau \in V^{\mathbb{B}} : \| \tau \text{ is a complex number } \|_{\mathbb{B}} = 1_{\mathbb{B}}\}$$

     and
  2. $C^+(St(\mathbb{B}))$.

Set theory questions

- Let $\mathbb{B}$ be the complete boolean algebra given by Lebesgue measurable sets modulo Lebesgue null sets. Then $C(St(\mathbb{B}))$ is isomorphic to $L^\infty(\mathbb{R})$.

Set theory questions

- Let $\mathbb{B}$ be the complete boolean algebra given by Lebesgue measurable sets modulo Lebesgue null sets. Then $C(St(\mathbb{B}))$ is isomorphic to $L^\infty(\mathbb{R})$.
- Suppose $\mathbb{B}$ is a complete Boolean algebra. Is there a statement $RH(\mathbb{B})$ such that:

$$V^{\mathbb{B}} \models \text{"the Riemann hypothesis holds for } \dot{\mathbb{C}}\text{"}$$

if and only if

$$RH(\mathbb{B}) \text{ holds for } C^+(St(\mathbb{B})).$$

Set theory questions

- Is there a natural statement ϕ such that:
  1. ϕ is independent of $ZFC + V = L$,
  2. both ϕ and $\neg\phi$ are equiconsistent with ZFC?

  Harvey Friedman has given some statements removing item (2).

Some other questions

- (Harvey Friedman) Is there a polynomial $f(x, y) \in \mathbb{Q}[x, y]$ such that $f : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ is a bijection? (some number theoretic conjectures give a negative answer to the question).

Some other questions

- (Harvey Friedman) Is there a polynomial $f(x, y) \in \mathbb{Q}[x, y]$ such that $f : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ is a bijection? (some number theoretic conjectures give a negative answer to the question).
- Shelah showed that the Hales-Jewett number $HJ(r, n)$, and hence the Van der Waerden number $W(r, n)$ belongs to the class $\mathcal{E}^5$ of Grzegorczyk hierarchy. Gowers has shown $W(r, n)$ belongs to the class $\mathcal{E}^3$. Can we improve Shelah's bound for $HJ(r, n)$?

Some other questions

-
- (Harvey Friedman) Is there a polynomial $f(x, y) \in \mathbb{Q}[x, y]$ such that $f : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ is a bijection? (some number theoretic conjectures give a negative answer to the question).
- Shelah showed that the Hales-Jewett number $HJ(r, n)$, and hence the Van der Waerden number $W(r, n)$ belongs to the class $\mathcal{E}^5$ of Grzegorczyk hierarchy. Gowers has shown $W(r, n)$ belongs to the class $\mathcal{E}^3$. Can we improve Shelah's bound for $HJ(r, n)$?
- The density Hales-Jewett theorem states that: for every $k \geq 2$ and $0 < \delta \leq 1$ there is an integer N such that if $n \geq N$ and A is a subset of $[k]^n$ of density δ, then A contains a combinatorial line of $[k]^n$. Let $DHJ(k, \delta)$ denote the least integer N with this property. Are there primitive recursive upper bounds for $DHJ(k, \delta)$?

Some other questions

- Suppose $(X_n : n < \omega)$ is a $\subseteq$-decreasing sequence of subsets of $\mathbb{R}^2$, each homeomorphic to the unit disk. Does $\bigcap_n X_n$ have the fixed point property?

Some other questions

- Suppose $(X_n : n < \omega)$ is a $\subseteq$-decreasing sequence of subsets of $\mathbb{R}^2$, each homeomorphic to the unit disk. Does $\bigcap_n X_n$ have the fixed point property?
- Let $\mathcal{L}$ be a class of modules and M be an R-Module. A homomorphism $f \in Hom(A, M)$, with $A \in \mathcal{L}$, is called an $\mathcal{L}$-precover of M if the induced map $Hom(A_0, A) \to Hom(A_0, M)$ is surjective for all $A_0 \in \mathcal{L}$. An $\mathcal{L}$-precover $f \in Hom(A, M)$ is a covering class (with respect to $\mathcal{L}$) provided that each $g \in Hom(A, A)$ satisfying $f = fg$ is an automorphism of A.

  (Enochs limit conjecture) Let $\mathcal{L}$ be a covering class. Then $\mathcal{L}$ is closed under taking direct limits.

Some other questions

- Suppose $(X_n : n < \omega)$ is a $\subseteq$-decreasing sequence of subsets of $\mathbb{R}^2$, each homeomorphic to the unit disk. Does $\bigcap_n X_n$ have the fixed point property?
- Let $\mathcal{L}$ be a class of modules and M be an R-Module. A homomorphism $f \in Hom(A, M)$, with $A \in \mathcal{L}$, is called an $\mathcal{L}$-precover of M if the induced map $Hom(A_0, A) \to Hom(A_0, M)$ is surjective for all $A_0 \in \mathcal{L}$. An $\mathcal{L}$-precover $f \in Hom(A, M)$ is a covering class (with respect to $\mathcal{L}$) provided that each $g \in Hom(A, A)$ satisfying $f = fg$ is an automorphism of A.
  (Enochs limit conjecture) Let $\mathcal{L}$ be a covering class. Then $\mathcal{L}$ is closed under taking direct limits.
- (Scott) Is every Scott set the standard system of a nonstandard model of PA? Known to be true for Scott sets of size $\leq \aleph_1$.

Some other questions

- For a real number β let $\| \beta \|$ denote the distance of β to the nearest integer.
  (Hardy) Are there a transcendental real number α and a real number $\epsilon > 0$ such that $\| \epsilon\alpha^n \|$ tends to 0 as n tends to infinity?

Some other questions

- For a real number β let $\parallel \beta \parallel$ denote the distance of β to the nearest integer.
  (Hardy) Are there a transcendental real number α and a real number $\epsilon > 0$ such that $\parallel \epsilon\alpha^n \parallel$ tends to 0 as n tends to infinity?
- What is the algebra (if any) corresponding to the continuous model theory?

Some other questions

- For a real number β let $\parallel \beta \parallel$ denote the distance of β to the nearest integer.
  (Hardy) Are there a transcendental real number α and a real number $\epsilon > 0$ such that $\parallel \epsilon\alpha^n \parallel$ tends to 0 as n tends to infinity?
- What is the algebra (if any) corresponding to the continuous model theory?
- Let $V[G]$ be obtained from V by adding a Cohen (resp. random) real. Let $C \in V[G]$ be the set of reals which are Cohen (resp. random) generic over V.
  Is $\mathbb{Z}$ definable in the structure $(\mathbb{R}^{V[G]}, +, \cdot, <, 0, 1, C)$?

Some other questions

- For a real number β let $\parallel \beta \parallel$ denote the distance of β to the nearest integer.
  (Hardy) Are there a transcendental real number α and a real number $\epsilon > 0$ such that $\parallel \epsilon\alpha^n \parallel$ tends to 0 as n tends to infinity?
- What is the algebra (if any) corresponding to the continuous model theory?
- Let $V[G]$ be obtained from V by adding a Cohen (resp. random) real. Let $C \in V[G]$ be the set of reals which are Cohen (resp. random) generic over V.
  Is $\mathbb{Z}$ definable in the structure $(\mathbb{R}^{V[G]}, +, \cdot, <, 0, 1, C)$?
- By a theorem of Keisler, For every countable theory T with an infinite model, the stability function of T is one of the following six functions:
  $\kappa, \kappa + 2^\omega, \kappa^\omega, \mathrm{ded}(\kappa), \mathrm{ded}(\kappa)^\omega, 2^\kappa$.
  Is is consistent that for some cardinal κ, we have $\mathrm{ded}(\kappa) < \mathrm{ded}(\kappa)^\omega < 2^\kappa$?

Some other questions

- By a theorem of Shepherdson, models of Open Induction are exactly discretely ordered semirings that are integer parts of their real closures. By Latter results, countable real closed fields whose integer parts are models of PA are exactly the recursively saturated ones. Also if an integer part is a model of $I\Delta_0$ and there is no x such that x, x^2, x^3 is unbounded, then the real closed field is recursively saturated.
  Call a real closed field *short* if there is an x such that the powers of x are cofinal.
  1. Are there any short real closed fields with no $I\Delta_0$ integer parts?
  2. Is the set of real closed fields with $I\Delta_0$ integer parts complete Σ^1_1?

Some other questions

- As is know, Vaught's conjecture is a special case of topological Vaught's conjecture. On the other hand, Vaught's conjecture is true for some theories, like stable theories (Shelah), superstable theories of finite rank (Buechler), O-minimal theories (Mayer), and so on. What is the analogue of the topological Vaught's conjecture for the above theories?
  Also the topological Vaught conjecture holds for some special cases, like continuous actions of nilpotent Polish groups on Polish spaces (Hjorth-Solecki) and so on. Is there a case of topological Vaught's conjecture which is know to be true and which implies the Vaught's conjecture for the above theories (ω-stable, superstble, ...).

Some other questions

- Let FO+Maj be first-order logic extended with the majority quantifier, $\mathrm{Maj}_x \phi(x)$, which holds in a structure if the formula $\phi(x)$ is true for more than half of the elements of the domain.
  Mostafa Mirabi Does FO+Maj satisfy 0-1 law?

Long live with joyful and productive life

Thank You for Your Attention!