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THE EFFECTS OF ADDING A REAL TO MODELS OF SET THEORY

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To my parents

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Abstract

In chapter 1 we study Shelah's strong covering property and its applications to pairs (W, V) of models of ZFC with $V = W[R], R$ a real. The results in the first section of this chapter are due to Shelah [14]. The last section presents a result of Vanliere [16].

In chapter 2 we show that it is possible to violate GCH at all infinite cardinals by adding a single real to a model of GCH. Our assumption is the existence of an $H(\kappa^{+3})-$strong cardinal κ. By work of Gitik and Mitchell [10] more than an $H(\kappa^{++})-$strong cardinal is required.

In chapter 3 it is shown that it is possible to force Easton's theorem by adding a single real to a model of GCH. Our assumption is the existence of a proper class of measurable cardinals which is optimal by results of Chapter 1.

In chapter 4 we present a method for coding an arbitrary real by two Cohen reals in a cofinality preserving way. We use this result to prove another variant of the results of chapters 2 and 3.

In chapter 5 we study the effects of adding Cohen reals to models of set theory. We show that it is possible to have a pair (V, V_1) of models of ZFC with the same cofinalities so that adding one Cohen real over V_1 adds $\aleph_1-$many Cohen reals over V. We also show that if $V \subseteq V_1$ have the same cardinals and reals, then below the first fixed point of the $\aleph-$function adding $\aleph_\delta-$many Cohen reals over V_1 can not produce more than $\aleph_\delta-$many Cohen reals over V.

Contents

Chapter 1

Shelah's strong covering property and its applications

1.1 Shelah's strong covering property

In this chapter we study Shelah's strong covering property and give some of its applications. By a pair (W, V) we always mean a pair (W, V) of models of ZFC with the same ordinals such that $W \subseteq V$.

Let us give the main definition.

Definition 1.1.1. (1) (W, V) *satisfies the strong* $(\lambda, \alpha)-$*covering property, where* λ *is a regular cardinal of* V *and* α *is an ordinal, if for every model* $M \in V$ *with universe* α *(in a countable language) and* $a \subseteq \alpha, \mid a \mid < \lambda$ *(in* V*), there is* $b \in W$ *such that* $a \subseteq b \subseteq \alpha, b \prec M$, *and* $\mid b \mid < \lambda$ *(in* V*).* (W, V) *satisfies the strong* $\lambda-$*covering property if it satisfies the strong* $(\lambda, \alpha)-$*covering property for every* α.

(2) (W, V) *satisfies the strong* $(\lambda^*, \lambda, \kappa, \mu)-$*covering property, where* $\lambda^* \geq \lambda \geq \kappa$ *are regular cardinals of* V *and* μ *is an ordinal, if player one has a winning strategy in the following game, called the* $(\lambda^*, \lambda, \kappa, \mu)-$*covering game, of length* λ*:*

In the $i-th$ *move player I chooses* $a_i \in V$ *such that* $a_i \subseteq \mu, \mid a_i \mid < \lambda^*$ *(in* V*) and* $\bigcup_{j<i} b_j \subseteq a_i$*, and player II chooses* $b_i \in V$ *such that* $b_i \subseteq \mu, \mid b_i \mid < \lambda^*$ *(in* V*) and* $\bigcup_{j \leq i} a_j \subseteq$

b_i.

Player I wins if there is a club $C \subseteq \lambda$ *such that for every* $\delta \in C \cup \{\lambda\}, cf(\delta) = \kappa \Rightarrow \bigcup_{i<\delta} a_i \in W$. (W, V) *satisfies the strong* $(\lambda^*, \lambda, \kappa, \infty)-$*covering property, if it satisfies the strong* $(\lambda^*, \lambda, \kappa, \mu)-$*covering property for every* μ.

The following theorem shows the importance of the first part of this definition and plays an important role in the next section.

Theorem 1.1.2. *Suppose* $V = W[R], R$ *a real and* (W, V) *satisfies the strong* $(\lambda, \alpha)-$*covering property for* $\alpha < ([(2^{<\lambda})^W]^+)^V$. *Then* $(2^{<\lambda})^V = |(2^{<\lambda})^W|^V$.

Proof. Cf. [14, Theorem VII.4.5]. □

It follows from Theorem 1.1.2 that if $V = W[R], R$ a real and (W, V) satisfies the strong $(\lambda^+, ([(2^\lambda)^W]^+)^V)-$covering property, then $(2^\lambda)^V = |(2^\lambda)^W|^V$.

We are now ready to give the applications of the strong covering property. For a pair (W, V) of models of ZFC consider the following conditions:

(1_κ) : • $V = W[R], R$ a real,

• V and W have the same cardinals $\leq \kappa^+$,

• $W \models \ulcorner \forall \lambda \leq \kappa, 2^\lambda = \lambda^+ \urcorner$,

• $V \models \ulcorner 2^\kappa > \kappa^+ \urcorner$.

(2_κ) : $W \models \ulcorner GCH \urcorner$.

(3_κ) : V and W have the same cardinals.

Theorem 1.1.3. (1) *Suppose there is a pair* (W, V) *satisfying* $(1_{\aleph_0})$ *and* $(2_{\aleph_0})$. *Then* $\aleph_2^V$ *in inaccessible in* L.

(2) *Suppose there is a pair* (W, V) *as in* (1) *with* $V \models \ulcorner 2^{\aleph_0} > \aleph_2 \urcorner$. *Then* $0^\sharp \in V$.

(3) *Suppose there is a pair* (W, V) *as in* (1) *with* $CARD^W \cap (\aleph_1^V, \aleph_2^V) = \emptyset$. *Then* $0^\sharp \in V$.

(4) *Suppose* $\kappa > \aleph_0$ *and there is a pair* (W, V) *satisfying* (1_κ). *Then* $0^\sharp \in V$.

Before we give the proof of Theorem 1.2.1 we state some conditions which imply Shelah's strong covering property. Suppose that in $V, 0^\sharp$ does not exist. Then:

(α) If $\lambda^* \geq \aleph_2^V$ is regular in V, then (W, V) satisfies the strong λ^*-covering property.

(β) If $CARD^W \cap (\aleph_1^V, \aleph_2^V) = \emptyset$ then (W, V) satisfies the strong $\aleph_1^V-$covering property.

Remark 1.1.4. *For $\lambda^* \geq \aleph_3^V$, (α) follows from [14, Theorem VII.2.6], and (β) follows from [14, Theorem VII.2.8]. In order to obtain (α) for $\lambda^* = \aleph_2^V$ we can proceed as follows: As in the proof of [14, Theorem VII.2.6] proceed by induction on μ to show that (L, V) satisfies the strong $(\aleph_2^V, \aleph_1^V, \aleph_0^V, \mu)-$covering property. For successor μ (in L) use [14, Lemma VII.2.2] and for limit μ use [14, Remark VII.2.4](instead of [14, Lemma VII.2.3]). It then follows that (L, V) and hence (W, V) satisfies the strong $\aleph_2^V-$covering property.*

Proof of Theorem 1.2.1.

1. We may suppose that $0^\sharp \notin V$. Then by $(\alpha), (W, V)$ satisfies the strong $\aleph_2^V-$covering property. On the other hand by Jensen's covering lemma and [14, Claim VII.1.11], W has squares. By [14, Theorem VII.4.10], $\aleph_2^V$ is inaccessible in W, and hence in L.

2. Suppose not. Then by $(\alpha), (W, V)$ satisfies the strong $\aleph_2^V-$covering property. By Theorem 1.1.2, $(2^{\aleph_0})^V \leq (2^{\aleph_1})^V = |(2^{\aleph_1})^W|^V = |\aleph_2^W| = \aleph_2^V$, which is a contradiction.

3. Suppose not. Then by $(\beta), (W, V)$ satisfies the strong $\aleph_1^V-$covering property, hence by Theorem 1.1.2, $(2^{\aleph_0})^V = |(2^{\aleph_0})^W|^V = \aleph_1^V$, which is a contradiction.

4. Suppose not. Then by $(\alpha), (W, V)$ satisfies the strong κ^+-covering property. By Theorem 1.1.2, $(2^\kappa)^V = |(2^\kappa)^W|^V = \kappa^+$, and we get a contradiction.

Theorem 1.1.5. (1) *Suppose there is a pair (W, V) satisfying $(1_\kappa), (2_\kappa)$ and (3_κ). Then there is in V an inner model with a measurable cardinal.*

(2) *Suppose there is a pair (W, V) satisfying (1_κ), where $\kappa \geq \aleph_\omega$. Further suppose that $\kappa_W^{++} = \kappa_V^{++}$ and (W, V) satisfies the κ^+-covering property. Then there is in V an inner model with a measurable cardinal.*

Proof. 1. Suppose not. Then by [14, conclusion VII.4.3(2)], (W, V) satisfies the strong κ^+-covering property, hence by Theorem 1.1.2, $(2^\kappa)^V = |(2^\kappa)^W|^V = \kappa^+$, which is a contradiction.

2. Suppose not. Let $\kappa = \mu^{+n}$, where μ is a limit cardinal, and $n < \omega$. By [14, Theorem VII.2.6, Theorem VII.4.2(2) and Conclusion VII.4.3(3)], we can show that (W, V) satisfies the strong $(\kappa^+, \kappa, \aleph_1, \mu)-$covering property. On the other hand since (W, V) satisfies the κ^+-covering property and V and W have the same cardinals $\leq \kappa^+$, (W, V) satisfies the $\mu^{+i}-$covering property for each $i \leq n+1$. By repeatedly use of [14, Lemma VII.2.2], (W, V) satisfies the strong $(\kappa^+, \kappa, \aleph_1, \kappa^{++})-$covering property, and hence the strong $(\kappa^+, \kappa^{++})-$covering property. By Theorem 1.1.2, $(2^\kappa)^V = |(2^\kappa)^W|^V = \kappa^+$, which is a contradiction.

□

Remark 1.1.6. *In [14] (see also [15]), Theorem 1.2.3(1), for* $\kappa = \aleph_0$, *is stated under the additional assumption* $2^{\aleph_0} > \aleph_\omega$ *in* V.

1.2 On a theorem of Vanliere

In this section we prove the following result of Vanliere [16]:

Theorem 1.2.1. *Assume $V = L[X, R]$ where $X \subseteq \omega_n$ for some $n < \omega$, and $R \subseteq \omega$. If $L[X] \models \ulcorner ZFC + GCH \urcorner$ and the cardinals of $L[X]$ are the true cardinals, then GCH holds in V.*

Proof. Let κ be an infinite cardinal. We prove the following:

$$(*_\kappa) : \text{For any } Y \subseteq \kappa \text{ there is an ordinal } \alpha < \kappa^+ \text{ and}$$
$$\text{a set } Z \in L[X], Z \subseteq \kappa \text{ such that } Y \in L_\alpha[Z, R].$$

Then it will follow that $\mathcal{P}(\kappa) \subseteq \bigcup_{\alpha<\kappa^+} \bigcup_{Z \in \mathcal{P}^{L[X]}(\kappa)} L_\alpha[Z, R]$, and hence

$$2^\kappa \leq \sum_{\alpha<\kappa^+} \sum_{Z \in \mathcal{P}^{L[X]}(\kappa)} \mid L_\alpha[Z, R] \mid \leq \kappa^+ . (2^\kappa)^{L[X]} . \kappa = \kappa^+,$$

which gives the result. Now we return to the proof of $(*_\kappa)$.

Case 1. $\kappa \geq \aleph_n$.

Let $Y \subseteq \kappa$. Let θ be large enough regular such that $Y \in L_\theta[X, R]$. Let $N \prec L_\theta[X, R]$ be such that $\mid N \mid = \kappa, N \cap \kappa^+ \in \kappa^+$ and $\kappa \cup \{Y, X, R\} \subseteq N$. By the condensation lemma there are $\alpha < \kappa^+$ and π such that $\pi : N \cong L_\alpha[X, R]$. then $Y = \pi(Y) \in L_\alpha[X, R]$. Thus $(*_\kappa)$ follows.

Case 2. $\kappa < \aleph_n$.

We note that the above argument does not work in this case. Thus another approach is needed. To continue the work, we state a general result (again due to Vanliere) which is of interest in its own sake.

Lemma 1.2.2. *Suppose $\mu \leq \kappa < \lambda \leq \nu$ are infinite cardinals, λ regular. Suppose that $a \subseteq \mu, Y \subseteq \kappa, Z \subseteq \lambda$, and $X \subseteq \nu$ are such that $V = L[X, a], Z \in L[X], Y \in L[Z, a]$ and $\lambda^+_{L[X]} = \lambda^+$. Then there exists a proper initial segment Z' of Z such that $Z' \in L[X]$ and $Y \in L[Z', a]$.*

Proof. Let $\theta \geq \nu$ be regular such that $Y \in L_\theta[Z, a]$. Let $N \prec L_\theta[Z, a]$ be such that $\mid N \mid = \lambda, N \cap \lambda^+ \in \lambda^+$ and $\lambda \cup \{Y, Z, a\} \subseteq N$. By the condensation lemma we can find $\delta < \lambda^+$ and π such that $\pi : N \cong L_\delta[Z, a]$.

In V, let $\langle M_i : i < \lambda \rangle$ be a continuous chain of elementary submodels of $L_\delta[Z, a]$ with union $L_\delta[Z, a]$ such that for each $i < \lambda, M_i \supseteq \kappa$, $| M_i | < \lambda$ and $M_i \cap \lambda \in \lambda$.

In $L[Z]$ let $\langle W_i : i < \lambda \rangle$ be a continuous chain of elementary submodels of $L_\delta[Z]$ with union $L_\delta[Z]$ such that for each $i < \lambda, W_i \supseteq \kappa$, $| W_i | < \lambda$ and $W_i \cap \lambda \in \lambda$

Now we work in V. Let $E = \{i < \lambda : M_i \cap L_\delta[Z] = W_i\}$. Then E is a club of λ. Pick $i \in E$ such that $Y \in M_i$, and let $M = M_i$, and $W = W_i$. By the condensation lemma let $\eta < \lambda$ and $\bar{\pi}$ be such that $\bar{\pi} : M \cong L_\eta[Z^{'}, a]$ where $Z^{'} = \bar{\pi}[M \cap Z] = \bar{\pi}[(M \cap \lambda) \cap Z] = (M \cap \lambda) \cap Z$, a proper initial segment of Z. Then $Y = \bar{\pi}(Y) \in L_\eta[Z^{'}, a]$ and $Z^{'} \subseteq \eta < \lambda$. It remains to observe that $Z^{'} \in L[X]$ as $Z^{'}$ is an initial segment of Z. The lemma follows. □

We are now ready to complete the proof of Case 2. By Lemma 1.3.2 we can find a bounded subset X_n of ω_n such that $X_n \in L[X]$ and $Y \in L[X_n, R]$. Now trivially we can find a subset Z_{n-1} of ω_{n-1} such that $L[X_n] = L[Z_{n-1}]$, and hence $Z_{n-1} \in L[X]$ and $Y \in L[Z_{n-1}, R]$. Again by Lemma 1.3.2 we can find a bounded subset X_{n-1} of ω_{n-1} such that $X_{n-1} \in L[X]$ and $Y \in L[X_{n-1}, R]$, and then we find a subset Z_{n-2} of ω_{n-2} such that $L[X_{n-1}] = L[Z_{n-2}]$. In this way we can finally find a subset Z of κ such that $Z \in L[X]$ and $Y \in L[Z, R]$. Then as in case 1, for some $\alpha < \kappa^+, Y \in L_\alpha[Z, R]$ and $(*_\kappa)$ follows. □

Chapter 2

Killing the GCH everywhere with a single real

2.1 Killing the GCH everywhere with a single real

Shelah-Woodin [15] investigate the possibility of violating instances of GCH through the addition of a single real. In particular they show that it is possible to obtain a failure of CH by adding a single real to a model of GCH, preserving cofinalities. In this chapter we bring this work to its natural conclusion by showing that it is possible to violate GCH at all infinite cardinals by adding a single real to a model of GCH.

Theorem 2.1.1. *([4]) Assume the consistency of an $H(\kappa^{+3})$-strong cardinal κ. Then there exists a pair (W, V) of models of ZFC such that*

(a) *W and V have the same cardinals,*

(b) *GCH holds in W,*

(c) *$V = W[R]$ for some real R,*

(d) *GCH fails at all infinite cardinals in V.*

The above Theorem answers an open question from [15]. The rest of this chapter is devoted to the proof of the above Theorem.

2.2 Prikry products

Assume GCH and suppose that S is a set of measurable cardinals which is *discrete*, i.e., contains none of its limit points. Fix normal measures U_α on α for α in S. Then $\mathbb{P}_S$ denotes the Prikry product of the forcings $\mathbb{P}_\alpha$, $\alpha \in S$, where $\mathbb{P}_\alpha$ is the Prikry forcing associated with the measure U_α. A $\mathbb{P}_S$-generic is uniquely determined by a sequence $(x_\alpha : \alpha \in S)$, where each x_α is an ω-sequence cofinal in α. With a slight abuse of terminology, we say that $(x_\alpha : \alpha \in S)$ is $\mathbb{P}_S$*-generic.*

Lemma 2.2.1. *(Fuchs [5], Magidor [12]) Suppose that $\langle x_\alpha : \alpha \in S\rangle$ is $\mathbb{P}_S$-generic over V.*
(a) V and $V[\langle x_\alpha : \alpha \in S\rangle]$ have the same cardinals.
(b) The sequence $\langle x_\alpha : \alpha \in S\rangle$ obeys the following "geometric property": If $\langle X_\alpha : \alpha \in S\rangle$ belongs to V and $X_\alpha \in U_\alpha$ for each $\alpha \in S$, then $\bigcup_{\alpha\in S} x_\alpha \setminus X_\alpha$ is finite.
(c) Conversely, suppose that $\langle y_\alpha : \alpha \in S\rangle$ is a sequence (in any outer model of V) satisfying the geometric property stated above. Then $\langle y_\alpha : \alpha \in S\rangle$ is $\mathbb{P}_S$-generic over V.
(d) Suppose $\alpha \in S$, $p \in \mathbb{P}_S$ and $\langle \Phi_\gamma : \gamma < \eta\rangle$ is a sequence of statements of the forcing language for $\mathbb{P}_S$ where $\eta < \alpha$. Then there exists $q \leq^* p$ such that $q \restriction \alpha = p \restriction \alpha$ and for each $\gamma < \eta$ if $r \leq q$ and r decides Φ_γ, then $(r \restriction \alpha) \cup (q \restriction [\alpha, \kappa))$ (where $\kappa = \sup(S)$) decides Φ_γ in the same way.

Theorem 2.2.2. *Suppose that κ is $H(\kappa^{+3})$-strong and S is a discrete set of measurable cardinals less than κ. Then after forcing with $\mathbb{P}_S$, κ remains $H(\kappa^{+3})$-strong.*

Proof. Suppose that $j : V \to M \supseteq H(\kappa^{+3})$, $crit(j) = \kappa$ is an elementary embedding witnessing the $H(\kappa^{+3})-$strength of κ. We can assume that j is derived from an extender $E = \langle E_a : a \in [\kappa^{+3}]^{<\omega}\rangle$. Then for each $a \in [\kappa^{+3}]^{<\omega}, E_a$ is a $\kappa-$complete ultrafilter on $[\kappa]^{|a|}$ and if $j_a : V \to M_a \cong Ult(V, E_a)$ is the corresponding elementary embedding then for all $B \subseteq [\kappa]^{|a|}$, we have $B \in E_a \Leftrightarrow a \in j_a(B)$. We also have an embedding $k_a : M_a \to M$ such that $k_a \circ j_a = j$.

We show that κ remains $H(\kappa^{+3})-$strong in the generic extension by $\mathbb{P}_S$. The proof uses ideas from [11] and [12]. Let G be $\mathbb{P}_S-$generic over V. Also let $\delta = min(j(S) - \kappa) > \kappa$.

Working in $V[G]$, we define for each $a \in [\kappa^{+3}]^{<\omega_1}, E^*_a$ as follows: Let $\xi = o.t(a)$, and let

$\dot{a}$ be a $\mathbb{P}_S-$name for a such that

$$\Vdash \ulcorner \dot{a} \subseteq \kappa^{+3} \text{ and } o.t(\dot{a}) = \xi \urcorner$$

For $p \in \mathbb{P}_S$ define $p \Vdash \ulcorner \dot{B} \in \dot{E}^*_a \urcorner$ iff

(1) $p \Vdash \ulcorner \dot{B} \subseteq [\kappa]^\xi \urcorner$,

(2) there exists $q \leq^* j(p)$ in $j(\mathbb{P}_S)$ such that $q \restriction \delta = j(p) \restriction \delta = p$, and $q \Vdash^M \ulcorner \dot{a} \in j(\dot{B}) \urcorner$.

Let $E^*_a = \dot{E}^*_a[G]$. It is easily seen that the above definition is well-defined.

Lemma 2.2.3. *(a) E^*_a is a $\kappa-$complete non-principal ultrafilter on $[\kappa]^\xi$,*

(b) If $a \in V$ is finite, then E^*_a extends E_a,

Proof. (a) We just prove that E^*_a is $\kappa-$complete. Suppose that $p \in \mathbb{P}_S$ and $p \Vdash \ulcorner [\kappa]^\xi = \bigcup\{\dot{B}_\gamma : \gamma < \eta\} \urcorner$ where $\eta < \kappa$. Then $j(p) \Vdash^M \ulcorner [j(\kappa)]^\xi = \bigcup\{j(\dot{B}_\gamma) : \gamma < \eta\} \urcorner$.

Working in M consider $\delta, j(p)$ and the sequence $(\Phi_\gamma : \gamma < \eta)$ of sentences where for each $\gamma < \eta, \Phi_\gamma$ is "$\dot{a} \in j(\dot{B}_\gamma)$" It then follows from Lemma 2.1.(d) that there is $q \leq^* j(p)$ in $j(\mathbb{P}_S)$ such that for each $\gamma < \eta$

- $q \restriction \delta = j(p) \restriction \delta = p$,
- if $r \leq q$ and r decides Φ_γ, then $(r \restriction \delta) \cup (q \restriction [\delta, j(\kappa))$ decides Φ_γ in the same way.

Now $q \Vdash^M \ulcorner \dot{a} \in [j(\kappa)]^\xi = \bigcup\{j(\dot{B}_\gamma) : \gamma < \eta\} \urcorner$ and hence we can find $r \leq q$ and $\gamma < \eta$ such that $r \Vdash \ulcorner \Phi_\gamma \urcorner$. Let $t = (r \restriction \delta) \cup (q \restriction [\delta, j(\kappa))$. It is now easy to show that $t \restriction \delta \leq p$ and $t \restriction \delta \Vdash \ulcorner \dot{B}_\gamma \in \dot{E}^*_a \urcorner$. This completes the proof of the $\kappa-$completeness of E^*_a.

(b) Suppose $a \in V$ is finite. Let $B \in E_a$ and $p \in \mathbb{P}_S$. We show that $p \Vdash \ulcorner B \in \dot{E}^*_a \urcorner$. Let $q = j(p)$. Then q has the required properties in the definition above which gives the result. □

In $V[G]$, for each $a \in [\kappa^{+3}]^{<\omega_1}$ let $j^*_a : V[G] \to M^*_a \simeq Ult(V[G], E^*_a)$ be the corresponding elementary embedding. Also for $a \subseteq b$ let $k_{a,b} : M^*_a \to M^*_b$ be the natural induced elementary embedding. Let

$$\langle M^*, \langle k^*_a : a \in [\kappa^{+3}]^{<\omega_1} \rangle\rangle = dirlim \langle\langle M^*_a : a \in [\kappa^{+3}]^{<\omega_1}\rangle, \langle k^*_{a,b} : a \subseteq b \rangle\rangle.$$

Also let $j^* : V[G] \to M^*$ be the induced embedding.

Lemma 2.2.4. *M^* is well-founded*

Proof. Suppose not. Then there is a sequence $(m_i : i < \omega)$ of elements of M^* such that

$$... \in^* m_2 \in^* m_1 \in^* m_0$$

where $\in^* = \in_{M^*}$. For each $i < \omega$ choose a_i and f_i such that $m_i = k^*_{a_i}([f_i]_{E^*_{a_i}})$. Let $a = \bigcup\{a_i : i < \omega\}$. Then $a \in [\kappa^{+3}]^{<\omega_1}$ and for some $g_i, m_i = k^*_a([g_i]_{E^*_a})$. It then follows from the elementarity of k^*_a that

$$... \in [g_2]_{E^*_a} \in [g_1]_{E^*_a} \in [g_0]_{E^*_a}.$$

This is in contradiction with Lemma 2.3 which implies M^*_a is well-founded. Thus M^* is well-founded and the lemma follows. □

If now we restrict ourself to E^*_a for finite a, then the smaller direct limit embeds into the full direct limit and is therefore well-founded. From now on, let M^* denote the smaller direct limit; accordingly each E^*_a is now given by the usual extender definition and j^* is the ultrapower embedding.

Note that $j^* : V[G] \to M^*$ is an elementary embedding with critical point κ. We show that it is an $H(\kappa^{+3})-$strong embedding. For this it suffices to show that $H(\kappa^{+3})^{V[G]} \subseteq M^*$. But since $H(\kappa^{+3})^{V[G]} = H(\kappa^{+3})[G]$, it suffices to show that $H(\kappa^{+3}) \subseteq M^*$ and $G \in M^*$.

For this purpose we introduce some special functions in V. Let $F : \kappa \to \kappa$ be defined by $F(\alpha) = \alpha^{+3}$. Then $j(F)(\kappa) = \kappa^{+3}$. Now for each $a \in [\kappa^{+3}]^{<\omega}$ with $\kappa \in a$ and $|a| = n$ define the function $G_a : [\kappa]^n \to \kappa$ by $G(\alpha_1, ..., \alpha_n) = \alpha_i^{+3}$ where κ is the $i-$th element of a. It is clear that $j(G_a)(a) = j(F)(\kappa) = \kappa^{+3}$. Also let $r : \kappa \to H(\kappa)$ be defined by $r(\alpha) = H(\alpha)$.

Suppose $f : [\kappa]^n \to H(\kappa)^{V[G]}$ is in $V[G]$ and a is a finite subset of κ^{+3} containing κ. We say the pair (f, a) has the property $(*)$ iff

$$\{\gamma : f(\gamma) \in r \circ G_a(\gamma)\} \in E^*_a. \text{ [1]}$$

We have the following easy lemma.

Lemma 2.2.5. *(a) If $j^*(f)(a) = j^*(g)(b)$ where κ is an element of both a and b, then (f, a) has the property $(*)$ iff (g, b) has the property $(*)$,*

[1] It can be shown that (f, a) has property $(*)$ iff $[f]_{E^*_a}$ represents an element of $H(\kappa^{+3})$ in M^*_a.

(b) If (f,a) has the property $(*)$ and $j^*(g)(b) \in j^*(f)(a)$ for some b containing κ, then (g,b) has the property $(*)$.

Lemma 2.2.6. *If (f,a) has the property $(*)$, then there is a function $h : [\kappa]^m \to H(\kappa)$ in V and a finite set $b \subseteq \kappa^{+3}$ such that $j^*(f)(a) = j^*(h)(b)$.*

Proof. Let $B = \{\gamma : f(\gamma) \in r \circ G_a(\gamma)\}$. Since (f,a) has the property $(*), B \in E^*_a$. Let $\dot{B}$ be a name for B and let $p \Vdash \ulcorner \dot{B} \in \dot{E}^*_a \urcorner$. This means that there is some $q \leq^* j(p)$ such that $q \restriction \delta = j(p) \restriction \delta = p$ and $q \Vdash^M \ulcorner a \in j(\dot{B}) \urcorner$. Hence we have $q \Vdash^M \ulcorner j(\dot{f})(a) \in j(r \circ G_a)(a) = H(\kappa^{+3}) \urcorner$.

For each $c \in H(\kappa^{+3})$ let Φ_c be the sentence "$j(\dot{f})(a) = c$". By applying Lemma 2.1.(d) we can find $r \leq^* q$ such that for every $c \in H(\kappa^{+3})$

- $r \restriction \delta = q \restriction \delta = p$,
- if $s \leq r$ and s decides Φ_c then $(s \restriction \delta) \cup (r \restriction [\delta, j(\kappa)))$ decides Φ_c in the same way.

Now $r \Vdash^M \ulcorner j(\dot{f})(a) \in j(r \circ G_a)(a) = H(\kappa^{+3}) \urcorner$, hence there are $s \leq r$ and $c \in H(\kappa^{+3})$ such that $s \Vdash \ulcorner \Phi_c \urcorner$. Let $t = (s \restriction \delta) \cup (r \restriction [\delta, j(\kappa)))$. By above, $t \Vdash^M \ulcorner \Phi_c \urcorner$.

Since $c \in H(\kappa^{+3})$, there is a function $h : [\kappa]^m \to H(\kappa)$ and a finite $b \subseteq \kappa^{+3}$ such that $c = j(h)(b)$. Thus $t \Vdash^M \ulcorner j(\dot{f})(a) = j(h)(b) \urcorner$ and the result follows. □

Define the sets X and X^* as follows

$$X = \{j(f)(a) : (f,a) \text{ is in } V \text{ and has the property } (*)\},$$
$$X^* = \{j^*(f)(a) : (f,a) \text{ is in } V[G] \text{ and has the property } (*)\}.$$

It follows from Lemma 2.5 that X and X^* are transitive.

Lemma 2.2.7. *If (f,a) has the property $(*)$ and $f \in V$, then $j^*(f)(a) = j(f)(a)$.*

Proof. Define $\Phi : X \to X^*$ by $\Phi(j(f)(a)) = j^*(f)(a)$. Then:
(1) Φ is well-defined: To see this suppose that $j(f)(a) = j(g)(b)$. We may further suppose that $a = b$. It then follows that $j(f)(a) = k_a([f]_{E_a}) = k_a([g]_{E_a}) = j(g)(b)$, and hence $B = \{x : f(x) = g(x)\} \in E_a$. By Lemma 2.3(b), $B \in E^*_a$ and hence $j^*(f)(a) = k^*_a([f]_{E^*_a}) = k^*_a([g]_{E^*_a}) = j^*(g)(b)$.
(2) Φ preserves the $\in$ relation: As in (1).

Thus Φ is an isomorphism, and since both of X and X^* are transitive, it must be the identity. The lemma follows. $\square$

Lemma 2.2.8. $H(\kappa^{+3}) \subseteq M^*$.

Proof. We have $H(\kappa^{+3}) \subseteq X \subseteq X^* \subseteq M^*$. $\square$

Lemma 2.2.9. $G \in M^*$

Proof. First note that $\mathbb{P}_S \in H(\kappa^{+3}) \subseteq M^*$. Define $f : \kappa \to H(\kappa)^{V[G]}$ by $f(\alpha) = G_\alpha$, where $G_\alpha = G \cap H(\alpha)$ is $\mathbb{P}_S \cap H(\alpha)-$generic over V. Show that $G = j^*(f)(\kappa)$, and hence $G \in M^*$. By maximality of G it suffices to show that $G \subseteq j^*(f)(\kappa)$.

Let $p \in G$. Choose $h : [\kappa]^n \to H(\kappa)$ in V and a finite set $a \subseteq \kappa^{+3}$ containing κ such that $p = j(h)(a)$. Then by Lemma 2.7 $p = j^*(h)(a)$. Define $f_a(\alpha_1, ..., \alpha_n) = f(\alpha_i)$, where κ is the $i-$th element of a. Then $j^*(f_a)(a) = j^*(f)(\kappa)$. Now we have to prove that $j^*(h)(a) \in j^*(f_a)(a)$.

Let $\dot{f}_a$ be a $\mathbb{P}_S-$name for f_a such that $\Vdash_{\mathbb{P}_S} \ulcorner \dot{f}_a(\alpha_1, ..., \alpha_n) = \dot{G_{\alpha_i}} \urcorner$. Then $\Vdash_{j(\mathbb{P}_S)} \ulcorner j(\dot{f}_a)(a) = \dot{G} \urcorner$ and hence $\Vdash_{j(\mathbb{P}_S)} \ulcorner j(h)(a) \in j(\dot{f}_a)(a) \urcorner$. The lemma follows. $\square$

2.3 Coding

Friedman [3] presents a method for creating reals which are class-generic (but not set-generic) over a sufficiently L-like model, preserving Woodin cardinals. A similar method can be used to preserve strong cardinals. However the general problem of coding a predicate into a real while preserving large cardinal properties is open; we show here that this is possible if the predicate is a sequence which is generic for a discrete Prikry product.

Theorem 2.3.1. *Suppose that K is the canonical inner model for an $H(\kappa^{+3})$-strong cardinal κ. Suppose that S is the discrete set consisting of those measurable cardinals less than κ in K which are not limits of measurable cardinals in K. Also let $(x_\alpha : \alpha \in S)$ be $\mathbb{P}_S$-generic over K for the measures $(U_\alpha : \alpha \in S)$, where U_α is the unique normal measure on α in K. Then there is a cofinality-preserving set-forcing $\mathbb{P}$ for adding a real R over $K[(x_\alpha : \alpha \in S)]$ such that $K[(x_\alpha : \alpha \in S)][R] = K[R]$ and κ remains $H(\kappa^{+3})$-strong in $K[R]$.*

Proof. We will follow the proof of Jensen's coding theorem from [2], section 4.2, making use of Lemma 2.2.1 to argue that the relevant Σ_1 Skolem hulls taken with respect to certain initial segments of K are also Σ_1 elementary when the Prikry product generic is adjoined. We must impose some minor changes to the notion of "string s" and to the coding structures $\mathcal{A}^s$, $\tilde{\mathcal{A}}^s$, but for the most part the argument remains the same. The preservation of $H(\kappa^{+3})$-strength is based on ideas from [3].

We work in $L[E][(x_\alpha : \alpha \in S)]$ where $K = L[E]$ is a fine-structural inner model built from the sequence E of (partial) extenders. Abbreviate $(x_\alpha : \alpha \in S)$ as $\vec{x}$ and for any β let $\vec{x}(\leq \beta)$ denote $(x_\alpha : \alpha \in S,\ \alpha \leq \beta)$. We may also assume that for α in S, the min of x_α is greater than the supremum of $S \cap \alpha$, using the discreteness of the set S. Let A denote the union of the x_α, $\alpha \in S$.

Card denotes the class of infinite cardinals. For α in Card we define the ordinals $\mu^{<\eta}, \mu^\eta$ by induction on $\eta \in [\alpha, \alpha^+)$. An ordinal μ is a *ZF$^-$ ordinal* iff $L_\mu[E, \vec{x}(\leq \alpha)]$ is a model of ZF minus Power Set. Define: $\mu^{<\eta} = \cup\{\mu^\xi : \xi < \eta\} \cup \alpha$, $\mu^\eta =$ the least limit of ZF$^-$ ordinals μ such that μ is greater than $\mu^{<\eta}$ and, setting $\mathcal{A}^\eta = L_\mu[E, \vec{x}(\leq \alpha)]$ we have that $\mathcal{A}^\eta \models \alpha$ is the largest cardinal.

S_α, the set of *strings* at α consists of all $s : [\alpha, |s|) \to 2$, $\alpha \leq |s| < \alpha^+$, such that $|s|$ is a multiple of α and s belongs to $\mathcal{A}^{|s|}$. We write $s \leq t$ when t extends s and $s < t$ when t properly extends s. For $s \in S_\alpha$ we write $\mathcal{A}^s$ for $\mathcal{A}^{|s|}$ and μ^s for $\mu^{|s|}$.

For later use (see "Limit Precoding") we also define $\tilde{\mu}^s < \mu^s$ to be the least ZF$^-$ ordinal μ greater than $\mu^{<|s|}$ such that the structure $L_\mu[E, \vec{x}(\leq \alpha)]$ contains s and satisfies that α is the largest cardinal. The resulting structure $\tilde{\mathcal{A}}^s = L_{\tilde{\mu}^s}[E, x(\leq \alpha)]$ is a proper initial segment of $\mathcal{A}^s$ and, like $\mathcal{A}^s$, each element of $\tilde{\mathcal{A}}^s$ is Σ_1 definable in $\tilde{\mathcal{A}}^s$ from parameters in $\alpha \cup \{\vec{x}(\leq \alpha), s\}$. (We say that $\tilde{\mathcal{A}}^s$, $\mathcal{A}^s$ are Σ_1 *projectible to* α with parameters $\vec{x}(\leq \alpha), s$.)

To set up the coding we need the functions f^s, defined as follows: For α an uncountable cardinal, s in S_α and $i < \alpha$ let $H^s(i)$ denote the Σ_1 Skolem hull of $i \cup \{\vec{x}(\leq \alpha), s\}$ in $\mathcal{A}^s$. Then $f^s(i)$ is the ordertype of $H^s(i) \cap Ord$. For α a successor cardinal we define the coding set b^s to be the range of $f^s \restriction B^s$ where B^s consists of the successor elements of $\{i < \alpha : i$ is a limit of j such that $j = H^s(j) \cap \alpha\}$.

We describe a cofinality-preserving forcing which codes $K[\vec{x}]$ into $K[X]$ for some $X \subseteq \omega_1$, preserving the $H(\kappa^{+3})$-strength of κ. Then a simple *c.c.c* forcing can be used to code X into the desired real R.

We need a partition of the ordinals into four pieces: Let B, C, D, F denote the classes of ordinals which are congruent to $0, 1, 2, 3$ mod 4, respectively (The letters A and E are already used for other purposes). For any ordinal α, α^B denotes the α-th element of B and for any set Y of ordinals, Y^B denotes the set of α^B for α in Y (similarly for C, D, F).

The successor coding: Suppose $\alpha \in$ Card and $s \in S_{\alpha^+}$. A condition in R^s is a pair (t, t^*) where $t \in S_\alpha$, $t^* \subseteq \{b^{s \restriction \eta} : \eta \in [\alpha^+, |s|)\} \cup |t|$, $card(t^*) \leq \alpha$. Extension is defined by: $(t_0, t_0^*) \leq (t_1, t_1^*)$ iff t_0 extends t_1, t_0^* contains t_1^* and:

(1) If $|t_1| \leq \gamma^B < |t_0|$ and $\gamma \in b^{s \restriction \eta} \in t_1^*$ then $t_0(\gamma^B) = 0$ or $s(\eta)$.

(2) If $|t_1| \leq \gamma^C < |t_0|$ and $\gamma = \langle \gamma_0, \gamma_1 \rangle$ with $\gamma_0 \in A \cap t_1^*$ then $t_0(\gamma^C) = 0$ (where $\langle \cdot, \cdot \rangle$ is Gödel pairing of ordinals).

An R^s-generic over $\mathcal{A}^s$ adds (and is uniquely determined by) a function $T : \alpha^+ \to 2$ such that $s(\eta) = 0$ iff $T(\gamma^B) = 0$ for sufficiently large $\gamma \in B^{s \restriction \eta}$ and such that for $\gamma_0 < \alpha^+$, $\gamma_0 \in A$ iff $T(\langle \gamma_0, \gamma_1 \rangle^C) = 0$ for sufficiently large $\gamma_1 < \alpha^+$.

The limit precoding. Suppose that α is an infinite cardinal and s belongs to S_α. We say that $X \subseteq \alpha$ *precodes* s if X is the Σ_1 theory of $\tilde{\mathcal{A}}^s$ with parameters from $\alpha \cup \{\vec{x}(\leq \alpha), s\}$, viewed as a subset of α.

The limit coding. Suppose that α is an uncountable limit cardinal, $s \in S_\alpha$ and p is a sequence $((p_\beta, p^*_\beta) : \beta \in \text{Card} \cap \alpha)$ where $p_\beta \in S_\beta$ for each $\beta \in \text{Card} \cap \alpha$. We will define what it means for p to "code s". First define the sequence $(s_\gamma : \gamma \leq \gamma_0)$ of elements of S_α as follows: Let $s_0 = \emptyset$. For limit $\gamma \leq \gamma_0$, s_γ is the union of the s_δ, $\delta < \gamma$. Now suppose that s_γ is defined and for successor cardinals β less than α let $f^{s_\gamma}_p(\beta)$ be the least $\delta \geq f^{s_\gamma}(\beta)$ such that $p_\beta(\delta^D) = 1$, if such a δ exists. If $f^{s_\gamma}_p(\beta)$ is undefined for cofinally many successor cardinals $\beta < \alpha$ then set $\gamma_0 = \gamma$. Otherwise define $X \subseteq \alpha$ by: $\delta \in X$ iff $p_\beta((f^{s_\gamma}_p(\beta) + 1 + \delta)^D) = 1$ for sufficiently large successor cardinals $\beta < \alpha$. If $Even(X) = \{\delta : 2\delta \in X\}$ precodes an element t of S_α extending s_γ such that $\mathcal{A}^t$ contains X and the function $f^{s_\gamma}_p$, then set $s_{\gamma+1} = t$. Otherwise let $s_{\gamma+1}$ be $s_\gamma * X^F$ (i.e. the concatenation of s_γ with X^F viewed as a sequence of length α), provided $s_\gamma * X^F$ belongs to S_α and $f^{s_\gamma}_p$ belongs to $\mathcal{A}^{s_\gamma * X^F}$; if not, then again set $\gamma_0 = \gamma$. Now p *exactly codes* s if s equals one of the s_γ, $\gamma \leq \gamma_0$ and p *codes* s is an initial segment of some s_γ, $\gamma \leq \gamma_0$.

Finally we define the desired forcing. Let Card$'$ denote the class of uncountable limit cardinals. Also fix an extender ultrapower embedding $j : V = K[\vec{x}] \to M = K^*[\vec{x}^*]$ witnessing that κ is $H(\kappa^{+3})$-strong in $K[\vec{x}]$. I.e., j has critical point κ, $H(\kappa^{+3})$ of V is contained in M and every element of M is of the form $j(f)(\alpha)$ for some $f : \kappa \to V$ in V and $\alpha < \kappa^{+3}$.

The conditions. A condition in $\mathbb{P}$ is a sequence $p = ((p_\alpha, p^*_\alpha) : \alpha \in \text{Card}, \alpha \leq \alpha(p))$ where $\alpha(p) \leq \kappa^{+3}$ in Card and:

(1) $p_{\alpha(p)}$ belongs to $S_{\alpha(p)}$ and $p^*_{\alpha(p)} = \emptyset$.

(2) For $\alpha \in \text{Card} \cap \alpha(p)$, (p_α, p^*_α) belongs to $R^{p_{\alpha^+}}$.

(3) For $\alpha \in \text{Card}'$, $\alpha \leq \alpha(p)$, $p \restriction \alpha$ belongs to $\mathcal{A}^{p_\alpha}$ and exactly codes p_α.

(4) For $\alpha \in \text{Card}'$, $\alpha \leq \alpha(p)$, if α is inaccessible in $\mathcal{A}^{p_\alpha}$ then there exists a closed unbounded subset C of α, $C \in \mathcal{A}^{p_\alpha}$, such that for $\beta \in C$, $p^*_\beta = p^*_{\beta^+} = p^*_{\beta^{++}} = p_{\beta^+} = p_{\beta^{++}} = \emptyset$.

Conditions are ordered by: $p \leq q$ iff:

(a) $\alpha(p) \geq \alpha(q)$.

(b) $p(\alpha) \leq q(\alpha)$ in $R^{p_{\alpha^+}}$ for $\alpha \in \text{Card} \cap \alpha(p) \cap (\alpha(q) + 1)$.

(c) $p_{\alpha(p)}$ extends $q_{\alpha(q)}$ if $\alpha(p) = \alpha(q)$.

(d) If $\alpha(q) \geq \kappa^{++}$, $|q_{\kappa^{++}}| \leq \gamma < |p_{\kappa^{++}}|$, $\xi < |j(q)_{\kappa^{+3}}|$ is of the form $j(f)(i)$ for some $i < |q_{\kappa^{++}}|$ and function f with domain κ, $j(q)_{\kappa^{+3}}(\xi) = 0$ and γ belongs to $b^{j(q)_{\kappa^{+3}} \restriction \xi}$ (as defined in $K^*[\vec{x}^*]$, the ultrapower of $K[\vec{x}]$ by j) then $p_{\kappa^{++}}(\gamma^B) = 0$.

Clause (d) is to ensure that $G_{\kappa^{++}}$, the subset of κ^{+3} added by the generic G, codes the union of the $j(p)_{\kappa^{+3}}$ for p in G, a fact needed for the preservation of $H(\kappa^{+3})$-strength (see below).

This completes the definition of $\mathbb{P}$. The verification of cofinality and GCH preservation for $\mathbb{P}$ is as in [2], section 4.2, following the proofs of the Lemmas 4.3 – 4.6 found there. Here we only point out the added points to be made, taking into account that we are coding $\vec{x}$ over $K = L[E]$ and not over L. For this verification, requirement (4) above can be weakened to only require that $p^*_\beta = \emptyset$ for $\beta \in C$; the stronger form of (4) above is needed for the preservation of $H(\kappa^{+3})$-strength.

A general fact that is needed throughout the proof is the following.

Lemma 2.3.2. *(Condensation) Suppose that α is an uncountable cardinal, $s \in S_\alpha$, $i < \alpha$ and as before let $H^s(i)$ denote the Σ_1 Skolem hull of $i \cup \{\vec{x}(\leq \alpha), s\}$ in $\mathcal{A}^s$.*

(a) If α is a successor cardinal then for sufficiently large $i < \alpha$, if i is a limit point of $\{j < \alpha : j = H^s(j) \cap j\}$ then the transitive collapse of $H^s(i)$ is of the form $\bar{K}[\vec{\bar{x}}]$ where $\bar{K}$ is an initial segment of K.

(b) If α is a limit cardinal then for sufficiently large cardinals $i < \alpha$ the transitive collapse of $H^s(i)$ is of the form $\bar{K}[\vec{\bar{x}}]$ where $\bar{K}$ is an initial segment of K.

The same holds with $\mathcal{A}^s$ replaced by any of its initial segments which contain s and have height equal to a ZF$^-$ ordinal.

Proof. Recall that s belongs to $\mathcal{A}^s = L_{\mu^{|s|}}[E, \vec{x}(\leq \alpha)]$. Now $x(\leq \alpha)$ is generic over K for the product $\mathbb{P}_{S(\leq\alpha)}$ of Prikry forcings at $\beta \leq \alpha$ in S. If α is in the closure of S then the intersection of $\mathbb{P}_{S(\leq\alpha)}$ with $L_\mu[E]$ is a class forcing in $L_\mu[E]$ whenever μ is a ZF$^-$ ordinal of size α such that α is the largest cardinal in $L_\mu[E]$. Nevertheless, all definable antichains in this forcing are sets. An examination of the proof of Lemma 2.2.1 in [5] reveals that any sequence which satisfies the geometric property of that lemma with respect to $L_\mu[E]$ for the forcing $\mathbb{P}_{S(\leq\alpha)} \cap L_\mu[E]$ is in fact generic for this forcing over $L_\mu[E]$. It follows that $x(\leq \alpha)$, which satisfies the geometric property with respect to the entire $L[E]$, is generic over $L_\mu[E]$ for this forcing. From this we infer the Σ_1 definability of the forcing relation for Δ_0 formulas for the forcing $\mathbb{P}_{S(\leq\alpha)} \cap L_{\mu^s}[E]$ and therefore that for $i \leq \alpha$, $H^s_0(i)$ = the Σ_1 Skolem hull of $i \cup \{\dot{s}\}$ in $\mathcal{A}^s_0$ $(= L_{\mu^{|s|}}[E])$ is equal to the intersection with $\mathcal{A}^s_0$ of $H^s(i)$ = the Σ_1 Skolem hull of $i \cup \{s\}$ in $\mathcal{A}^s$ (where $\dot{s}$ is a name for $s \in \mathcal{A}^s$). In particular, setting i equal to α, we see that $\mathcal{A}^s_0$ is Σ_1-projectible to α with parameter $\dot{s}$.

If i satisfies the requirements stated in (a) or (b) above, then the Σ_1 projectum of the transitive collapse of $H^s_0(i)$ is equal to i and if i is sufficiently large, then this transitive collapse is also sound. It follows that $\bar{K}$ = the transitive collapse of $H^s_0(i)$ is an initial segment of K for such i. The last statement of the lemma follows by the same argument, as any initial segment of $\mathcal{A}^s$ which contains s is Σ_1 projectible to α with parameter s. $\square$

Using Condensation as above, the proofs of Lemmas 4.3 – 4.6 from [2], section 4.2 can be carried out in the present setting:

In Lemma 4.3, one must take the α_i's to enumerate the first α sufficiently large elements of $\{\beta < \alpha^+ : \beta$ is a limit of $\bar{\beta}$ such that $\bar{\beta} = \alpha^+ \cap \Sigma_1$ Skolem hull of $(\bar{\beta} \cup \{x\})$ in $\mathcal{A}\}$ which are sufficiently large so that Condensation (a) guarantees that the transitive collapse of the associated Σ_1 hull is of the form $\bar{K}[\vec{x}]$ with $\bar{K}$ an initial segment of K. This facilitates the proof of the Claim in the proof of Lemma 4.3

In Lemma 4.4 one applies Condensation (b) to ensure that the Σ_1 Skolem hull H_β, when $\beta = \alpha \cap H_\beta$, transitively collapses to a structure built from an initial segment of K for sufficiently large cardinals $\beta < \alpha$; this is needed to argue that the resulting s_β is a string at β. The rest of the proof remains unchanged.

The proof of Lemma 4.5 (a) in the case of β inaccessible also uses Condensation (b) in the proof of the Claim, to verify that the p^λ_γ are strings (in S_γ). Also note that Jensen's subtle use of the assumption that $0^\#$ does not exist (referred to in the Note) has no counterpart here, as our structures $\mathcal{A}^s_0 = L_{\mu^s}[E]$, $s \in S_\alpha$ collapse $|s|$ to α without the use of s as an additional predicate (indeed, s is just a parameter in $L_{\mu^s}[E, \vec{x}(\leq \alpha)]$). The proofs of Lemma 4.5 in the case of singular β as well as Lemma 4.6 can be carried out as before.

We are left with the verification that κ remains $H(\kappa^{+3})$-strong after forcing with $\mathbb{P}$. Recall that $j : V = K[\vec{x}] \to M = K^*[\vec{x}^*]$ is the extender ultrapower embedding witnessing that κ is $H(\kappa^{+3})$-strong. Let G be $\mathbb{P}-$generic over V; in $V[G]$ we must produce a G^M which is $j(\mathbb{P})$-generic over M and which contains $j(p)$ for each p in G.

If $(D_i : i < \kappa)$ are dense subsets of $\mathbb{P}$ and p belongs to $\mathbb{P}$ then p has an extension q which "reduces each D_i below i^{+3}", i.e., any extension r of q can be further extended to meet D_i without changing $r(\beta)$ for $\beta \geq i^{+3}$. (This is a variant of Δ-distributivity, see page 30 of [2].) From this it follows that if we take the upward closure of $j[G]$, we obtain a compatible set of conditions which reduces each dense subset of $j(\mathbb{P})$ in M below κ^{+3}, using the ultrapower representation of M. Moreover, thanks to requirement (4) in the definition of $\mathbb{P}$, $j[G]$ contains no nontrivial information between κ and κ^{+3} (except for G_κ, the subset of κ^+ added by G), and therefore $j[G]$ is compatible with $G \cap H(\kappa^{+3})$. Moreover, thanks to condition (d) in the definition of extension of conditions, $G_{\kappa^{++}}$ will code the union of the $j(p)_{\kappa^{+3}}$, $p \in G$, and this coding is generic (using the fact that the $j(p)_{\kappa^{+3}}$ belong to $\mathcal{A}^\emptyset$; see Lemma 4.8 of [2]). So we can take G^M to be generated by the joins of conditions in $j[G]$ with those in $G \cap H(\kappa^{+3})$ to obtain the desired $j(\mathbb{P})$-generic over M. $\square$

2.4 Killing the GCH everywhere by a cardinal preserving forcing

In [13] the following is proved.

Theorem 2.4.1. *(Merimovich [13]) Suppose that GCH holds and κ is $H(\kappa^{+4})-$ strong. Then there exists a generic extension of the universe in which κ remains inaccessible and $\forall\lambda \leq \kappa, 2^\lambda = \lambda^{+3}$.*

Unfortunately in the Merimovich model a lot of cardinals are collapsed below κ. We show that a simple modification of his proof can give us the the total failure of the GCH below κ without collapsing any cardinals.

Theorem 2.4.2. *Suppose that GCH holds and κ is $H(\kappa^{+4})-$ strong. Then there exists a cardinal preserving generic extension of the universe in which κ remains inaccessible and $\forall\lambda \leq \kappa, 2^\lambda > \lambda^{+}$.*

Proof. We assume the reader has a copy of [13] at hand and we just mention the changes we need to prove the theorem.

- In page 372: replace R_U with $Add(\kappa^{+4}, i_U(\kappa)^{+3})_{N^*[G_{<\kappa}]}$. The arguments from [13] show that we can find the generics I_U, I_τ and $I_{\bar{E}}$ for this new R_U and the corresponding forcings R_τ and $R_{\bar{E}}$.

- In page 376, 3.2: in $N[I_U]$ all $N-$cardinals are preserved and the power function differs from the power function of N at the following point: $2^{\kappa^{+4}} = i_U(\kappa)^{+3}$.

- In page 379, 3.4: The forcing notion $\mathbb{P}_{\bar{E}}$, adds a club to κ. For each ν_1, ν_2 successive points in the club the cardinal structure and power function in the range $[\nu_1^+, \nu_2^{+3}]$ of the generic extension is the same as the cardinal structure and power function in the range $[\kappa^+, j_{\bar{E}}(\kappa)^{+3}]$ of $M_{\bar{E}}[I_{\bar{E}}]$.

- In page 411: replace Claim 10.6 with the following: Let G be $\mathbb{P}_{\bar{E}}$-generic with $p = p_l * ... * p_k * ... * p_0 \in G$ and $\bar{\epsilon}$ be such that $p_{l..k} \in \mathbb{P}_{\bar{\epsilon}}$ and $l(\bar{\epsilon}) = 0$. Let $\nu = \kappa(p_k^0)$.

Then, in $V[G]$, all cardinals in $[\nu^+, \kappa^0(\bar{\epsilon})^{+3}]$ are preserved and $2^{\nu^+} = \nu^{+4}$, $2^{\nu^{++}} = \nu^{+5}$, $2^{\nu^{+3}} = \nu^{+6}$, $2^{\nu^{+4}} = \kappa^0(\bar{\epsilon})^{+3}$,

- In page 412: replace $Col(\aleph_0, \lambda^+)_{V[G]}$ by $Add(\aleph_0, \lambda^{+3})_{V[G]}$ and let H be generic over $V[G]$ for this new forcing.

Now the proof of the theorem goes as follows: Let $p^* \in \mathbb{P}^*_{\bar{E}}$ such that $\kappa(p^{*0})$ is inaccessible and G be $\mathbb{P}_{\bar{E}}$-generic with $p^* \in G$. Set

$$M = \bigcup\{p_0^{\bar{E}_\kappa} : p \in G\},$$
$$C = \bigcup\{\kappa(p_0^{\bar{E}_\kappa}) : p \in G\}.$$

Note that M is a Radin generic sequence for the extender sequence $\bar{E}_\kappa$, hence $C \subset \kappa$ is a club. Also the first ordinal in this club is $\lambda = \kappa(p^{*0})$. We first investigate the range (λ, κ) in $V[G]$. Note that, by [13, Lemma 10.5], for $\bar{\epsilon} \in M$ it is enough to use $\mathbb{P}_{\bar{\epsilon}}$ in order to understand $V^{V[G]}_{\kappa^0(\bar{\epsilon})}$. So let $\mu \in (\lambda, \kappa)$.

- $\mu \in limC$: Then there is $\bar{\epsilon} \in M$ such that $l(\bar{\epsilon}) > 0$ and $\kappa(\bar{\epsilon}) = \mu$. By [13, Claim 10.7] μ remain a cardinal and by [13, Claim 10.3], $2^\mu = \mu^{+3}$,

- $\mu \in C \setminus \lim C$: Then there is $\bar{\epsilon} \in M$ such that $l(\bar{\epsilon}) = 0$ and $\kappa(\bar{\epsilon}) = \mu$. Let $\mu_2 \in C$ be the C-immediate predecessor of μ. By the above replacement of Claim 10.6 we have all cardinals in $[\mu_2^+, \mu^{+3}]$ are preserved and $2^{\mu_2^+} = \mu_2^{+4}$, $2^{\mu_2^{++}} = \mu_2^{+5}$, $2^{\mu_2^{+3}} = \mu_2^{+6}$, $2^{\mu_2^{+4}} = \mu^{+3}$. In particular $2^\mu \geq \mu^{+3}$.

- $\mu \notin C$: Then there are μ_2 and μ_1 two successive points in C such that $\mu \in (\mu_2, \mu_1)$. By above, if $\mu \in \{\mu_2^+, \mu_2^{++}, \mu_2^{+3}\}$ then $2^\mu = \mu^{+3}$, and if $\mu \in (\mu_2^{+3}, \mu_1)$ then $2^\mu \geq \mu_1^{+3} > \mu^+$.

We may note that the above argument also shows that all cardinals $> \lambda$ are preserved in $V[G]$, and since forcing with $\mathbb{P}_{\bar{E}}$ adds no new bounded subsets to λ, hence all cardinals are preserved in $V[G]$. It is now clear that in $V[G][H]$ all cardinals are preserved and that GCH fails everywhere below (and at) κ.

□

Note that in the above proof, we have a fixed gap 3 on a club of cardinals below κ. It is possible to weaken the hypotheses of Theorem 2.4.2 to κ being $H(\kappa^{+3})-$strong and get the same result as above. In this case we will get a fixed gap 2 on a club of cardinals below κ:

Theorem 2.4.3. *Suppose that GCH holds and* κ *is* $H(\kappa^{+3})-$ *strong. Then there exists a cardinal preserving generic extension of the universe in which* κ *remains inaccessible and* $\forall\lambda \leq \kappa, 2^{\lambda} > \lambda^{+}$.

See [4] for more details and the proof of the above theorem.

2.5 Proof of Theorem 2.1.1

*Suppose that K is the canonical inner model for a $H(\kappa^{+3})$-strong cardinal κ. Let S be a discrete set of measurable cardinals below κ of size κ, and for each $\alpha \in S$ fix a normal measure U_α over α. Consider the forcing $\mathbb{P}_S$ and let $(x_\alpha : \alpha \in S)$ be $\mathbb{P}_S$-*generic *over K. By Theorem 2.2.2, κ remains $H(\kappa^{+3})-$strong in $K[(x_\alpha : \alpha \in S)]$, thus we can apply Theorem 2.3.1 to find a cofinality-preserving forcing $\mathbb{P}$ which adds a real R over $K[(x_\alpha : \alpha \in S)]$ such that $K[(x_\alpha : \alpha \in S)][R] = K[R]$ and κ remains $H(\kappa^{+3})-$strong in $K[R]$. By Theorem 2.4.3 there exists a cardinal-preserving forcing $\mathbb{Q}$ and a subset $C \subseteq S, \mathbb{Q}-$generic over $K[R]$ such that in $K[R][C], \kappa$ remains inaccessible and for every $\lambda < \kappa, 2^\lambda > \lambda^+$. We now define a new sequence $(y_\alpha : \alpha \in S)$ by*

$$y_\alpha = \begin{cases} x_\alpha & \textit{if } \alpha \in C, \\ x_\alpha - \{min(x_\alpha)\} & \textit{otherwise }. \end{cases}$$

By Lemma 2.2.1, $(y_\alpha : \alpha \in S)$ is $\mathbb{P}_S-$generic over K. Let $W = V_\kappa^{K[(y_\alpha:\alpha\in S)]}$ and $V = W[R]$. Then

(1) W is a model of $ZFC + GCH$,

(2) $V = V_\kappa^{K[R][C]}$, and hence $V \models \ulcorner\forall\lambda, 2^\lambda > \lambda^{+}\urcorner$.

Theorem 2.1.1 follows.

Chapter 3

Forcing Easton's theorem by adding a real

3.1 Forcing Easton's theorem by adding a real

In this chapter we show that assuming the existence of a proper class of measurable cardinals, it is possible to force Easton's theorem by adding a single real. More precisely:

Theorem 3.1.1. *([4]) Let M be a model of $ZFC + GCH+$ there exists a proper class of measurable cardinals. In M let $F : REG \longrightarrow CARD$ be an Easton function, i.e a definable class function such that*

- $\kappa \leq \lambda \longrightarrow F(\kappa) \leq F(\lambda)$, *and*
- $cf(F(\kappa)) > \kappa$.

Then there exists a pair (W, V) of cardinal preserving extensions of M such that

(a) $W \models \ulcorner GCH \urcorner$,

(b) $V = W[R]$ *for some real* R,

(c) $V \models \ulcorner \forall \kappa \in REG, 2^\kappa \geq F(\kappa) \urcorner$.

The reason that in (c) we do not require equality is that it might be possible that $F(\kappa)$ changes its cofinality in V to ω, and then clearly $2^\kappa \neq F(\kappa)$ in V. The rest of this chapter is devoted to the proof of the above Theorem.

3.2 A class version of the Prikry product

Let S be a class of measurable cardinals which is discrete. Fix normal measures U_α on α for α in S. We define a class version of the Prikry product as follows.

Conditions in $\mathbb{P}_S$ are triples $p = (X^p, S^p, H^p)$ such that

(1) X^p is a subset of S,

(2) $S^p \in \prod_{\alpha \in X^p} [\alpha \setminus sup(S \cap \alpha)]^{<\omega}$,

(3) $H^p \in \prod_{\alpha \in X^p} U_\alpha$,

(4) $supp(p) = \{\alpha : S^p(\alpha) \neq \emptyset\}$ is finite,

(5) $\forall \alpha \in X^p, maxS^p(\alpha) < minH^p(\alpha)$.

Let $p, q \in \mathbb{P}_S$. Then $p \leq q$ (p is an extension of q) iff

(1) $X^p \supseteq X^q$,

(2) $\forall \alpha \in X^q, S^p(\alpha)$ is an end extension of $S^q(\alpha)$,

(3) $\forall \alpha \in X^q, S^p(\alpha) \setminus S^q(\alpha) \subseteq H^q(\alpha)$,

(4) $\forall \alpha \in X^q, H^p(\alpha) \subseteq H^q(\alpha)$.

We also define an auxiliary relation $\leq^*$ on $\mathbb{P}_S$ as follows. Let $p, q \in \mathbb{P}_S$. Then $p \leq^* q$ (p is a direct or Prikry extension of q) iff

(1) $X^p \supseteq X^q$,

(2) $\forall \alpha \in X^q, S^p(\alpha) = S^q(\alpha)$,

(3) $\forall \alpha \in X^q, H^p(\alpha) \subseteq H^q(\alpha)$.

For $p \leq q$ in $\mathbb{P}_S$ we define the distance function $|p - q|$ to be a function on X^q so that for $\alpha \in X^q, |p - q|(\alpha) = l(S^p(\alpha)) - l(S^q(\alpha))$. Also let $\mathbb{P}_S \upharpoonright X = \{p \in \mathbb{P}_S : X^p \subseteq X\}$. It is clear that for any $X \subseteq S, \mathbb{P}_S \simeq (\mathbb{P}_S \upharpoonright X) \times (\mathbb{P}_S \upharpoonright S \setminus X)$.

Lemma 3.2.1. *$\mathbb{P}_S$ is pretame: Given $p \in \mathbb{P}_S$ and a definable sequence $(D_i : i < \alpha)$ of dense classes below p there exist $q \leq p$ and a sequence $(d_i : i < \alpha) \in V$ such that each $d_i \subseteq D_i$ is predense below q.*

Proof. Let $p_0 = p$ and let $\delta_0 > \alpha, \delta_0 \notin S$ be such that $X^{p_o} \subseteq \delta_0$. By repeatedly thinning the measure one sets above δ_0 we can find $p_1 \leq p_0$ and $\delta_1 > \delta_0, \delta_1 \notin S$ such that:

1. $X^{p_1} \subseteq \delta_1$,

2. p_1 agrees with p_0 below δ_0,

3. for any $q \leq p_0, q \in \mathbb{P}_S \restriction \delta_0$ and any $i < \alpha$ if q has an extension r meeting D_i which agrees with q below δ_0, then there is such an $r \in \mathbb{P}_S \restriction \delta_1$ whose measure one sets contain those of p_1.

Now repeat this $\omega-$times, producing $p_0, p_1, ...$ Let q be $\leq^* p_n$'s, $n < \omega$ with $X^q = \bigcup_{n<\omega} X^{p_n}$ obtained in the natural way. Also for each $i < \alpha$ set $d_i = D_i \restriction \delta_\omega = \{r \restriction \delta_\omega : r \in D_i\}$, where $\delta_\omega = sup_{n<\omega}\delta_n$. We show that q and the sequence $(d_i : i < \alpha)$ are as required.

Fix $i < \alpha$. Suppose $r \leq q, r \in D_i$. Let n be large enough so that $supp(r) \cap \delta_\omega \subseteq \delta_n$. At stage $n + 1$ we considered $r \restriction \delta_n$ and saw that it has an extension meeting D_i and agreeing with it below δ_n, so it must have such an extension whose measure one sets contain those of p_{n+1} and therefore those of q. This extension is compatible with r and therefore r has an extension which meets d_i, as required. □

It follows from [2, Theorem 2.18], and the above Lemma that the forcing relation is definable. The proof of the following lemma uses ideas from [12].

Lemma 3.2.2. $(\mathbb{P}_S, \leq, \leq^*)$ *has the Prikry property, i.e for each sentence* ϕ *of the forcing language of* $(\mathbb{P}_S, \leq)$*, and any* $p \in \mathbb{P}_S$ *there is* $q \leq^* p$ *which decides* ϕ.

Proof. Suppose ϕ is a sentence of the forcing language, $p \in \mathbb{P}_S$. Let $p = (X^p, S^p, H^p)$, let ϕ^0 denote $\neg\phi$ and ϕ^1 denote ϕ.

By reflection and by strengthening p in the sense of $\leq^*$, we may assume that $X^p = \gamma$, where it is dense in $\mathbb{P}_S \cap V_\gamma$ to decide ϕ.

For $\alpha < \gamma$, let $\mathcal{S}_\alpha$ denote the set of S^q where $q \in \mathbb{P}_{X_p \cap \alpha}$. For $s \in \mathcal{S}_\alpha$, set $F_{s,\alpha}(\delta_1, \ldots, \delta_n) = i$ iff there is $q \leq p$ such that $X^q = \gamma$, $S^q \restriction (X^p \setminus \{\alpha\}) = s$, $S^q(\alpha) = S^p(\alpha) * (\delta_1, \ldots \delta_n)$ and $q \Vdash \phi^i$. Set $F_{s,\alpha}(\delta_1, \ldots, \delta_n) = 2$ iff no such q exists.

Let $H(s,\alpha) \subseteq H^p(\alpha)$, $H(\alpha) \in U_\alpha$ be homogeneous for $F_{s,\alpha}$, and let $H(\alpha) = \bigcap_{s\in\mathcal{S}_\alpha)} H(s,\alpha)$. Then $H(\alpha) \in U_\alpha$ (as S is discrete) and we can set $q = (X^q, S^q, H^q)$, where $X^q = X^p$, $S^q = S^p$ and $H^q(\alpha) = H(\alpha)$ for $\alpha \in X^q$.

It is clear that $q \leq^* p$. We show that there is a $\leq^*$ extension of q which decides ϕ. Suppose not. Let $r \leq q$ be such that r decides ϕ. Suppose for example that $r \Vdash \phi$. We may further suppose that r is so that $|r-q|$ is minimal, and that $X^r = \gamma$. We note that $|r-q|$ is not the 0-funtion.

Let $\alpha < \gamma$ be the maximum of $\text{supp}(r)$, and let r_0 be obtained from r by replacing $S^r(\alpha)$ with $S^p(\alpha)$. We claim that r_0 already decides ϕ. For let $w \leq r_0$, such that $w \Vdash \neg\phi$. Let n denote $|S^r(\alpha)|$; We may assume that $|S^w(\alpha)| \geq n$. Let s denote S^{r_0} and $\delta_1, \dots \delta_k$ denote $S^w(\alpha)$. Then r witnesses that $F_{s,\alpha}$ has constant value 1 on $[H(s,\alpha)]^n$. Moreover, $\{\delta_1, \dots \delta_n\} \in [H(s,\alpha)]^n$. So there is r_1 such that $r_1 \Vdash \phi$, $S^{r_1} \restriction (X^p \setminus \{\alpha\}) = s$ and $S^{r_1}(\alpha) = \{\delta_1, \dots, \delta_n\}$. It is easily checked that S^{r_1} and $S^w \restriction \gamma$ are compatible, so r_1 and w are compatible, contradicting that they decide ϕ differently. Thus, r_0 already decides ϕ, contradicting the minimality of r. □

We can now easily show that $\mathbb{P}_S$ preserves cardinals and the GCH. Also as in the usual Prikry product a $\mathbb{P}_S-$generic is uniquely determined by a sequence $(x_\alpha : \alpha \in S)$ where each x_α is an $\omega-$sequence cofinal in α. As before, with a slight abuse of terminology, we say that $(x_\alpha : \alpha \in S)$ is $\mathbb{P}_S-$generic. The following is an analogue of Lemma 2.2.1 and its proof is essentially the same.

Lemma 3.2.3. *(a) The sequence $(x_\alpha : \alpha \in S)$ obeys the following "geometric property": if $(X_\alpha : \alpha \in S)$ is a definable class (in V) and $X_\alpha \in U_\alpha$ for each $\alpha \in S$ then $\bigcup_{\alpha \in S} x_\alpha \setminus X_\alpha$ is finite.*

(b) Conversely, suppose that $(y_\alpha : \alpha \in S)$ is a sequence (in any outer model of V) satisfying the geometric property stated above. Then $(y_\alpha : \alpha \in S)$ is $\mathbb{P}_S$-generic over V.

3.3 Proof of Theorem 3.1.1

Suppose M *is a model of* $ZFC + GCH+$ *there exists a proper class of measurable cardinals. Let* S *be a discrete class of measurable cardinals and for each* $\alpha \in S$ *fix a normal measure* U_α *over* α*. Consider the forcing* $\mathbb{P}_S$ *and let* $(x_\alpha : \alpha \in S)$ *be* $\mathbb{P}_S$-generic *over* M*. By Jensen's coding theorem (see [2]) there exists a cofinality-preserving forcing* $\mathbb{P}$ *which adds a real* R *over* $M[(x_\alpha : \alpha \in S)]$ *such that* $M[(x_\alpha : \alpha \in S)][R] = L[R]$*. In* $L[R]$ *define the function* $F^* : REG \to CARD$ *by*

$$F^*(\kappa) = \begin{cases} F(\kappa) & \textit{if } cf F(\kappa) \neq \omega, \\ F(\kappa)^+ & \textit{if } cf F(\kappa) = \omega. \end{cases}$$

Let $\mathbb{R}$ *be the Easton forcing corresponding to* F^* *for blowing up the power of each regular cardinal* κ *to* $F^*(\kappa)$ *and let* $C \subseteq S$ *be* $\mathbb{R}-$*generic over* $L[R]$.

We now define a new sequence $(y_\alpha : \alpha \in S)$ *by*

$$y_\alpha = \begin{cases} x_\alpha & \textit{if } \alpha \in C, \\ x_\alpha - \{min(x_\alpha)\} & \textit{otherwise }. \end{cases}$$

Using lemma 3.2.3, $(y_\alpha : \alpha \in S)$ *is* $\mathbb{P}_S-$*generic over* M*. Let* $W = M[(y_\alpha : \alpha \in S)]$*, and* $V = M[(y_\alpha : \alpha \in S), R]$*. Then the pair* (W, V) *is as required.*

Chapter 4

Coding a real by two Cohen reals in a cofinality preserving way

4.1 Coding a real by two Cohen reals

In this chapter we present a method for coding an arbitrary real by two Cohen reals in a cofinality preserving way.

Theorem 4.1.1. *([1]) Suppose that R is a real in V. Then there are two reals a and b such that*

(a) *a and b are Cohen generic over V,*

(b) *all of the models $V, V[a], V[b]$ and $V[a, b]$ have the same cofinalities,*

(c) *$R \in L[a, b]$.*

Proof. Working in V, let a^* be $Add(\omega, 1)-$generic over V and let b^* be $Add(\omega, 1)-$generic over $V[a^*]$, where $Add(\omega, 1)$ is the Cohen forcing for adding a new real. Note that $V[a^*]$ and $V[a^*, b^*]$ are cofinality preserving generic extensions of V. Working in $V[a^*, b^*]$ let $\langle k_N : N < \omega\rangle$ be an increasing enumeration of $\{N : a^*(N) = 0\}$ and let $a = a^*$ and $b = \{N : b^*(N) = a^*(N) = 1\} \cup \{k_N : R(N) = 1\}$. Then clearly $R \in L[\langle k_N : N < \omega\rangle, b] \subseteq L[a, b]$ as $R = \{N : k_N \in b\}$.

We show that b is $Add(\omega, 1)-$generic over V. It suffices to prove the following

(*) For any $(p,q) \in Add(\omega,1) * \underset{\sim}{Add}(\omega,1)$ and any dense open subset $D \in V$ of $Add(\omega,1)$ there exists $(\bar{p},\bar{q}) \leq (p,q)$ such that $(\bar{p},\bar{q}) \Vdash \underset{\sim}{b}$ extends some element of D.

Let (p,q) and D be as above. By extending one of p or q if necessary, we can assume that $lh(p) = lh(q)$. Let $\langle k_N : N < M \rangle$ be an increasing enumeration of $\{N < lh(p) : p(N) = 0\}$. Let $s : lh(p) \to 2$ be such that considered as a subset of ω,

$$s = \{N < lh(p) : p(N) = q(N) = 1\} \cup \{k_N : N < M, R(N) = 1\}.$$

Let $t \in D$ be such that $t \leq s$. Extend p, q to $\bar{p}, \bar{q}$ of length $lh(t)$ so that for i in the interval $[lh(s), lh(t))$

- $\bar{p}(i) = 1$,
- $\bar{q}(i) = 1$ iff $i \in t$.

Then

$$t = \{N < lh(t) : \bar{p}(N) = \bar{q}(N) = 1\} \cup \{k_N : N < M, R(N) = 1\}.$$

Thus $(\bar{p},\bar{q}) \Vdash \ulcorner \underset{\sim}{b}$ extends t $\urcorner$ and $(*)$ follows. The theorem follows. □

The following theorems can be proved easily using Theorem 4.1.1 and the main results of chapters 2 and 3.

Theorem 4.1.2. *([4]) Assume the consistency of an $H(\kappa^{+3})$-strong cardinal κ. Then there exist a model W of ZFC and two reals a and b such that*

(a) The models $W, W[a], W[b]$ and $W[a,b]$ have the same cardinals,

(b) $W[a]$ and $W[b]$ satisfy GCH,

(c) GCH fails at all infinite cardinals in $W[a,b]$.

Theorem 4.1.3. *([4]) Let M be a model of $ZFC + GCH +$ there exists a proper class of measurable cardinals. In M let $F : REG \longrightarrow CARD$ be an Easton function. Then there exist a cardinal preserving generic extension W of M and two reals a and b such that*

(a) The models $W, W[a], W[b]$ and $W[a,b]$ have the same cardinals,

(b) $W[a]$ and $W[b]$ satisfy GCH,

(c) $W[a,b] \models \ulcorner \forall\kappa \in REG, 2^\kappa \geq F(\kappa) \urcorner$.

Chapter 5

Adding a lot of Cohen reals by adding a few

5.1 Adding $\aleph_1$−many Cohen reals by adding one

A basic fact about Cohen reals is that adding λ−many Cohen reals cannot produce more than λ−many of Cohen reals. More precisely, if $\langle r_\alpha : \alpha < \lambda \rangle$ are λ−many Cohen reals over V, then in $V[\langle r_\alpha : \alpha < \lambda \rangle]$ there are no λ^+−many Cohen reals over V.

But if instead of dealing with one universe V we consider two, then the above may no longer be true. In this section we prove the following:

Theorem 5.1.1. *([8]) Suppose that V satisfies GCH. Then there is a cofinality preserving generic extension V_1 of V satisfying GCH so that adding a Cohen real over V_1 produces a generic for the finite support product of $\aleph_1$−many copies of Cohen forcing over V, and hence adds $\aleph_1$−many Cohen reals over V.*

Proof. The basic idea of the proof will be to split ω_1 into ω sets such that none of them will contain an infinite set of V. It turned out however that just not containing an infinite set of V is not enough. We will use a stronger property. As a result the forcing turns out to be more complicated. We are now going to define the forcing sufficient for proving the theorem. Fix a nonprincipal ultrafilter U over ω.

Definition 5.1.2. *Let $(\mathbb{P}_U, \leq, \leq^*)$ be the Prikry (or in this context Mathias) forcing with U, i.e.*

- $\mathbb{P}_U = \{\langle s, A\rangle \in [\omega]^{<\omega} \times U : max s < \min A\}$,
- $\langle t, B\rangle \leq \langle s, A\rangle \Longleftrightarrow t$ *end extends* s *and* $(t \backslash s) \cup B \subseteq A$,
- $\langle t, B\rangle \leq^* \langle s, A\rangle \Longleftrightarrow t = s$ *and* $B \subseteq A$.

We call $\leq^*$ a direct or $*-$extension. The following are the basic facts on this forcing that will be used further.

Lemma 5.1.3. *(1) The generic object of $\mathbb{P}_U$ is generated by a real,*

(2) $(\mathbb{P}_U, \leq)$ satisfies the c.c.c,

(3) If $\langle s, A\rangle \in \mathbb{P}_U$ and $b \subseteq \omega \backslash (max s + 1)$ is finite, then there is a $*-$extension of $\langle s, A\rangle$, forcing the generic real to be disjoint to b.

Proof. 1. If G is $\mathbb{P}_U-$generic over V, then let $r = \bigcup\{s : \exists A, \langle s, A\rangle \in G\}$. r is a real and $G = \{\langle s, A\rangle \in \mathbb{P}_U : r$ end extends s and $r \backslash s \subseteq A\}$.

2. Trivial using the fact that for $\langle s, A\rangle, \langle t, B\rangle \in \mathbb{P}_U$, if $s = t$ then $\langle s, A\rangle$ and $\langle t, B\rangle$ are compatible.

3. Consider $\langle s, A \backslash (max b + 1)\rangle$.

$\square$

We now define our main forcing notion.

Definition 5.1.4. *$p \in \mathbb{P}$ iff $p = \langle p_0, \underset{\sim}{p}_1\rangle$ where*

(1) $p_0 \in \mathbb{P}_U$,

(2) $\underset{\sim}{p}_1$ is a $\mathbb{P}_U-$name such that for some $\alpha < \omega_1$, $p_0 \Vdash \ulcorner \underset{\sim}{p}_1 : \alpha \longrightarrow \omega \urcorner$ and such that the following hold

(2a) For every $\beta < \alpha$, $\underset{\sim}{p}_1(\beta) \subseteq \mathbb{P}_U \times \omega$ is a $\mathbb{P}_U-$name for a natural number such that

- $\underset{\sim}{p}_1(\beta)$ *is partial function from $\mathbb{P}_U$ into ω,*
- *for some fixed $l < \omega$, $dom \underset{\sim}{p}_1(\beta) \subseteq \{\langle s, \omega \backslash max s + 1\rangle : s \in [\omega]^l\}$,*

- *for all* $\beta_1 \neq \beta_2 < \alpha, ran\underset{\sim}{p}_1(\beta_1) \cap ran\underset{\sim}{p}_1(\beta_2)$ *is finite.*

(2b) for every countable $I \subseteq \alpha$, $I \in V$, $p_0' \leq p_0$ *and finite* $J \subseteq \omega$ *there is a finite set* $a \subseteq \alpha$ *such that for every finite set* $b \subseteq I \backslash a$ *there is* $p_0'' \leq^* p_0'$ *such that* $p_0'' \Vdash \ulcorner (\forall \beta \in b, \forall\ \kappa \in J, \underset{\sim}{p}_1(\beta) \neq k) \& (\forall \beta_1 \neq \beta_2 \in b, \underset{\sim}{p}_1(\beta_1) \neq \underset{\sim}{p}_1(\beta_2)) \urcorner$.

Notation 5.1.5. *(1) Call* α *the length of* p *(or* $\underset{\sim}{p}_1$*) and denote it by* $lh(p)$ *(or* $lh(\underset{\sim}{p}_1)$*).*

(2) For $n < \omega$ *let* $\underset{\sim}{I}_{p,n}$ *be a* $\mathbb{P}_U-$*name such that* $p_0 \Vdash \ulcorner \underset{\sim}{I}_{p,n} = \{\beta < \alpha : \underset{\sim}{p}_1(\beta) = n\} \urcorner$. *Then we can coincide* $\underset{\sim}{p}_1$ *with* $\langle \underset{\sim}{I}_{p,n} : n < \omega \rangle$.

Remark 5.1.6. *(2a) will guarantee that for* $\beta < \alpha$, $p_0 \Vdash \ulcorner \underset{\sim}{p}_1(\beta) \in \omega \urcorner$. *The last condition in (2a) is a technical fact that will be used in several parts of the argument. The condition (2b) appears technical but it will be crucial for producing numerous Cohen reals.*

Definition 5.1.7. *For* $p = \langle p_0, \underset{\sim}{p}_1 \rangle, q = \langle q_0, \underset{\sim}{q}_1 \rangle \in \mathbb{P}$, *define*

- $p \leq q$ *iff*

  1. $p_0 \leq_{\mathbb{P}_U} q_0$,
  2. $lh(q) \leq lh(p)$,
  3. $p_0 \Vdash \ulcorner \forall n < \omega, \underset{\sim}{I}_{q,n} = \underset{\sim}{I}_{p,n} \cap lh(q) \urcorner$.

- $p \leq^* q$ *iff*

  1. $p_0 \leq^*_{\mathbb{P}_U} q_0$,
  2. $p \leq q$.

we call $\leq^*$ a direct or $*-$extension.

Remark 5.1.8. *In the definition of* $p \leq q$, *we can replace (3) by* $p_0 \Vdash \ulcorner \underset{\sim}{q}_1 = \underset{\sim}{p}_1 \restriction lh(q) \urcorner$.

Lemma 5.1.9. *Let* $\langle p_0, \underset{\sim}{p}_1 \rangle \Vdash \ulcorner \alpha$ *is an ordinal* $\urcorner$. *Then there are* $\mathbb{P}_U-$*names* $\underset{\sim}{\beta}$ *and* $\underset{\sim}{q}_1$ *such that* $\langle p_0, \underset{\sim}{q}_1 \rangle \leq^* \langle p_0, \underset{\sim}{p}_1 \rangle$ *and* $\langle p_0, \underset{\sim}{q}_1 \rangle \Vdash \ulcorner \underset{\sim}{\alpha} = \underset{\sim}{\beta} \urcorner$.

Proof. Suppose for simplicity that $\langle p_0, \underset{\sim}{p}_1 \rangle = \langle \langle <>, \omega \rangle, \phi \rangle$. Let θ be large enough regular and let $\langle N_n : n < \omega \rangle$ be an increasing sequence of countable elementary submodes of H_θ such that $\mathbb{P}$, $\underset{\sim}{\alpha} \in N_0$ and $N_n \in N_{n+1}$ for each $n < \omega$. Let $N = \bigcup\limits_{n<\omega} N_n$, $\delta_n = N_n \cap \omega_1$ for

$n < \omega$ and $\delta = \bigcup_{n<\omega} \delta_n = N \cap \omega_1$. Let $\langle J_n : n < \omega \rangle \in N_0$ be a sequence of infinite subsets of $\omega \backslash \{0\}$ such that $\bigcup_{n<\omega} J_n = \omega \backslash \{0\}$, $J_n \subseteq J_{n+1}$, and $J_{n+1} \backslash J_n$ is infinite for each $n < \omega$. Also let $\langle \alpha_i : 0 < i < \omega \rangle$ be an enumeration of δ such that for every $n < \omega$, $\{\alpha_i : i \in J_n\} \in N_{n+1}$ is an enumeration of δ_n and $\{\alpha_i : i \in J_{n+1}\} \cap \delta_n = \{\alpha_i : i \in J_n\}$.

We define by induction a sequence $\langle p^s : s \in [\omega]^{<\omega} \rangle$ of conditions such that

- $p^s = \langle p_0^s, \underset{\sim}{p}{}_1^s \rangle = \langle \langle s, A_s \rangle, \underset{\sim}{p}{}_1^s \rangle$,
- $p^s \in N_{s(lhs-1)+1}$,
- $lh(p^s) = \delta_{s(lhs-1)+1}$,
- if t does not contradict p_0^s (i.e if t end extends s and $t \backslash s \subseteq A_S$) then $p^t \leq p^s$.

For $s = <>$, let $p^{<>} = \langle \langle <>, \omega \rangle, \phi \rangle$. Suppose that $<> \neq s \in [\omega]^{<\omega}$ and $p^{s \restriction lhs-1}$ is defined. We define p^s. First we define $t^{s \restriction lhs-1} \leq^* p^{s \restriction lhs-1}$ as follows: If there is no $*-$extension of $p^{s \restriction lhs-1}$ deciding $\underset{\sim}{\alpha}$ then let $t^{s \restriction lhs-1} = p^{s \restriction lhs-1}$. Otherwise let $t^{s \restriction lhs-1} \in N_{s(lhs-2)+1}$ be such an extension. Note that $lh(t^{s \restriction lhs-1}) \leq \delta_{s(lhs-2)+1}$.

Let $t^{s \restriction lhs-1} = \langle t_0, \underset{\sim}{t}_1 \rangle, t_0 = \langle s \restriction lhs-1, A \rangle$. Let $C \subseteq \omega$ be an infinite set almost disjoint to $\langle ran \underset{\sim}{t}_1(\beta) : \beta < lh(\underset{\sim}{t}_1) \rangle$. Split C into ω infinite disjoint sets C_i, $i < \omega$. Let $\langle c_{ij} : j < \omega \rangle$ be an increasing enumeration of C_i, $i < \omega$. We may suppose that all of these is done in $N_{s(lhs-1)+1}$. Let $p^s = \langle p_0^s, \underset{\sim}{p}{}_1^s \rangle$, where

- $p_0^s = \langle s, A \backslash (maxs + 1) \rangle$,
- for $\beta < lh(\underset{\sim}{t}_1)$, $\underset{\sim}{p}{}_1^s(\beta) = \underset{\sim}{t}_1(\beta)$,
- for $i \in J_{s(lhs-1)}$ such that $\alpha_i \in \delta_{s(lhs-1)} \backslash lh(\underset{\sim}{t}_1)$

$$\underset{\sim}{p}{}_1^s(\alpha_i) = \left\{ \langle \langle s * \langle r_1, ..., r_i \rangle, \omega \backslash (r_i + 1) \rangle, c_{ir_i} \rangle : r_1 > \max s, \langle r_1, ..., r_i \rangle \in [\omega]^i \right\}.$$

Trivially $p^s \in N_{s(lhs-1)+1}$, $lh(p^s) = \delta_{s(lhs-1)}$, and if $s(lhs-1) \in A$, then $p^s \leq t^{s \restriction lhs-1}$.

Claim 5.1.10. $p^s \in \mathbb{P}$.

Proof. We check conditions in Definition 5.1.4.

(1) i.e. $p_0^s \in \mathbb{P}_U$ is trivial.

(2) It is clear that $p_0^s \Vdash \ulcorner \underset{\sim}{p}_1^s : \delta_{s(lhs-1)} \longrightarrow \omega \urcorner$ and that (2a) holds. Let us prove (2b). Thus suppose that $I \subseteq \delta_{s(lhs-1)}$, $I \in V$, $p \leq p_0^s$ and $J \subseteq \omega$ is finite. First we apply (2b) to $\langle p, \underset{\sim}{t}_1 \rangle, I \cap lh(\underset{\sim}{t}_1)$, p and J to find a finite set $a' \subseteq lh(\underset{\sim}{t}_1)$ such that

(*) For every finite set $b \subseteq I \cap lh(\underset{\sim}{t}_1) \backslash a'$ there is $p' \leq^* p$ such that p' $\Vdash \ulcorner (\forall \beta \in b, \forall k \in J, \underset{\sim}{t}_1(\beta) \neq k) \& (\forall \beta_1 \neq \beta_2 \in b, \underset{\sim}{t}_1(\beta_1) \neq \underset{\sim}{t}_1(\beta_2)) \urcorner$.

Let $p = \langle s * \langle r_1, ..., r_m \rangle, B \rangle$. Suppose that $\delta_{s(lhs-1)} \backslash lh(\underset{\sim}{t}_1) = \{\alpha_{J_1}, ..., \alpha_{J_i}, ...\}$ where $J_1 < J_2 < ...$ are in $J_{s(lhs-1)}$. Let

$$a = a' \cup \{\alpha_{J_1}, ..., \alpha_{J_m}\}.$$

We show that a is as required. Thus suppose that $b \subseteq I \backslash a$ is finite. Apply (*) to $b \cap lh(\underset{\sim}{t}_1)$ to find $p' = \langle s * \langle r_1, ..., r_m \rangle, B' \rangle \leq^* p$ such that

$$p' \Vdash \ulcorner (\forall \beta \in b \cap lh(\underset{\sim}{t}_1), \forall k \in J, \underset{\sim}{t}_1(\beta) \neq k) \& (\forall \beta_1 \neq \beta_2 \in b \cap lh(\underset{\sim}{t}_1), \underset{\sim}{t}_1(\beta_1) \neq \underset{\sim}{t}_1(\beta_2)) \urcorner.$$

Also note that

$$p' \Vdash \ulcorner \forall \beta \in b \cap lh(\underset{\sim}{t}_1), \underset{\sim}{p}_1^s(\beta) = \underset{\sim}{t}_1(\beta) \urcorner.$$

Pick $k < \omega$ such that

$$\forall \beta \in b \cap lh(\underset{\sim}{t}_1), \forall \alpha_i \in b \backslash lh(\underset{\sim}{t}_1), ran \underset{\sim}{p}_1^s(\beta_1) \cap (ran \underset{\sim}{p}_1^s(\alpha_i) \backslash k) = \phi.$$

Let $q = \langle s * \langle r_1, ..., r_m \rangle, B \rangle = \langle s * \langle r_1, ..., r_m \rangle, B' \backslash (\max J + k + 1) \rangle$. Then $q \leq^* p' \leq^* p$. We show that q is as required. wee need to show that

1. $q \Vdash \ulcorner \forall \beta \in b \backslash lh(\underset{\sim}{t}_1), \forall k \in J, \underset{\sim}{p}_1^s(\beta) \neq k \urcorner$,

2. $q \Vdash \ulcorner \forall \beta_1 \neq \beta_2 \in b \backslash lh(\underset{\sim}{t}_1), \underset{\sim}{p}_1^s(\beta_1) \neq \underset{\sim}{p}_1^s(\beta_2) \urcorner$,

3. $q \Vdash \ulcorner \forall \beta_1 \in b \cap lh(\underset{\sim}{t}_1), \forall \beta_2 \in b \backslash lh(\underset{\sim}{t}_1), \underset{\sim}{p}_1^s(\beta_1) \neq \underset{\sim}{p}_1^s(\beta_2) \urcorner$.

Now (1) follows from the fact that $q \Vdash \ulcorner \underset{\sim}{p}_1^s(\alpha_i) \geq (i-m)-th$ element of $B > \max J \urcorner$. (2) follows from the fact that for $i \neq j < \omega$, $C_i \cap C_j = \emptyset$, and $ran \underset{\sim}{p}_1^s(\alpha_i) \subseteq C_i$. (3) follows from the choice of k. The claim follows. $\square$

This completes our definition of the sequence $\langle p^s : s \in [\omega]^{<\omega} \rangle$. Let

$$\underset{\sim}{q}_1 = \{\langle p_0^s, \langle \beta, \underset{\sim}{p}_1^s(\beta)\rangle\rangle : s \in [\omega]^{<\omega}, \beta < lh(p^s)\}.$$

Then $\underset{\sim}{q}_1$ is a $\mathbb{P}_U$−name and for $s \in [\omega]^{<\omega}$, $p_0^s \Vdash \ulcorner \underset{\sim}{p}_1^s = \underset{\sim}{q}_1 \restriction lh(\underset{\sim}{p}_1^s)\urcorner$.

Claim 5.1.11. $\langle\langle <>, \omega\rangle, \underset{\sim}{q}_1\rangle \in \mathbb{P}$.

Proof. We check conditions in Definition 5.1.4.

(1) i.e. $\langle <>, \omega\rangle \in \mathbb{P}_U$ is trivial.

(2) It is clear from our definition that

$$\langle <>, \omega\rangle \Vdash \ulcorner \underset{\sim}{q}_1 \text{ is a well-defined function into } \omega\urcorner.$$

Let us show that $lh(\underset{\sim}{q}_1) = \delta$. By the construction it is trivial that $lh(\underset{\sim}{q}_1) \leq \delta$. We show that $lh(\underset{\sim}{q}_1) \geq \delta$. It suffices to prove the following

(*) For every $\tau < \delta$ and $p \in \mathbb{P}_U$ there is $q \leq p$ such that $q \Vdash \ulcorner \underset{\sim}{q}_1(\tau)$ is defined $\urcorner$.

Fix $\tau < \delta$ and $p = \langle s, A\rangle \in \mathbb{P}_U$ as in (*). Let t be an end extension of s such that $t \backslash s \subseteq A$ and $\delta_{t(lht-1)} > \tau$. Then p_0^t and p are compatible and $p_0^t \Vdash \ulcorner \underset{\sim}{q}_1(\tau) = \underset{\sim}{p}_1^t(\tau)$ is defined $\urcorner$. Let $q \leq p_0^t, p$. Then $q \Vdash \ulcorner \underset{\sim}{q}_1(\tau)$ is defined $\urcorner$ and (*) follows. Thus $lh(\underset{\sim}{q}_1) = \delta$.

(2a) is trivial. Let us prove (2b). Thus suppose that $I \subseteq \delta$, $I \in V$, $p \leq \langle <>, \omega\rangle$ and $J \subseteq \omega$ is finite. Let $p = \langle s, A\rangle$.

First we consider the case where $s =<>$. Let $a = \emptyset$. We show that a is as required. Thus let $b \subseteq I$ be finite. Let $n \in A$ be such that $n > \max J + 1$ and $b \subseteq \delta_n$. Let $t = s * \langle n\rangle$. Note that

$$\forall \beta_1 \neq \beta_2 \in b,\ ran\, \underset{\sim}{p}_1^t(\beta_1) \cap ran\, \underset{\sim}{p}_1^t(\beta_2) = \emptyset.$$

Let $q = \langle <>, B\rangle = \langle <>, A \backslash (\max J + 1)\rangle$. Then $q \leq^* p$ and q is compatible with p_0^t. We show that q is as required. We need to show that

1. $q \Vdash \ulcorner \forall \beta \in b, \forall k \in J, \underset{\sim}{q}_1(\beta) \neq k\urcorner$,
2. $q \Vdash \ulcorner \forall \beta_1 \neq \beta_2 \in b, \underset{\sim}{q}_1(\beta_1) \neq \underset{\sim}{q}_1(\beta_2)\urcorner$.

For (1), if it fails, then we can find $\langle r, D\rangle \leq q, p_0^t$, $\beta \in b$ and $k \in J$ such that $\langle r, D\rangle \leq^* p_0^r$ and $\langle r, D\rangle \Vdash \ulcorner \underset{\sim}{q}_1(\beta) = k\urcorner$. But $\langle r, D\rangle \Vdash \ulcorner \underset{\sim}{q}_1(\beta) = \underset{\sim}{p}_1^r(\beta) = \underset{\sim}{p}_1^t(\beta)\urcorner$, hence $\langle r, D\rangle \Vdash \ulcorner \underset{\sim}{p}_1^t(\beta) = k\urcorner$. This is impossible since $minD \geq minB > maxJ$. For (2), if it fails, then we can find

$\langle r,D\rangle \le q, p_0^t$ and $\beta_1 \ne \beta_2 \in b$ such that $\langle r,D\rangle \le^* p_0^r$ and $\langle r,D\rangle \Vdash \ulcorner \underset{\sim}{q}_1(\beta_1) = \underset{\sim}{q}_1(\beta_2)\urcorner$. As above it follows that $\langle r,D\rangle \Vdash \ulcorner \underset{\sim}{p}_1^t(\beta_1) = \underset{\sim}{p}_1^t(\beta_2)\urcorner$. This is impossible since for $\beta_1 \ne \beta_2 \in b$, $ran\,\underset{\sim}{p}_1^t(\beta_1) \cap ran\,\underset{\sim}{p}_1^t(\beta_2) = \emptyset$. Hence q is as required and we are done.

Now consider the case $s \ne <>$. First we apply (2b) to t^s, $I \cap lh(t^s)$, p and J to find a finite set $a' \subseteq lh(t^s)$ such that

(**) For every finite set $b \subseteq I \cap lh(t^s) \backslash a'$ there is $p' \le^* p$ such that p'
$\Vdash \ulcorner (\forall \beta \in b, \forall k \in J,\ \underset{\sim}{p}_1^s(\beta) \ne k) \& (\forall \beta_1 \ne \beta_2 \in b, \underset{\sim}{p}_1^s(\beta_1) \ne \underset{\sim}{p}_1^s(\beta_2))\urcorner$

Let $t^s = \langle t_0, \underset{\sim}{t}_1\rangle, \delta_{s(lhs-1)+1} \backslash \delta_{s(lhs-1)} = \{\alpha_{J_1}, \alpha_{J_2}, ...\}$, where $J_1 < J_2 < ...$ are in $J_{s(lhs-1)+1}$. Define

$$a = a' \cup \{\alpha_1, \alpha_2, ..., \alpha_{J_{lhs+1}}\}.$$

We show that a is as required. First apply (**) to $b \cap lh(t^s)$ to find $p' = \langle s, A'\rangle \le^* p$ such that

$$p' \Vdash \ulcorner (\forall \beta \in b \cap lh(t^s), \forall k \in J, \underset{\sim}{t}_1(\beta) \ne k) \& (\forall \beta_1 \ne \beta_2 \in b \cap lh(t^s), \underset{\sim}{t}_1(\beta_1) \ne \underset{\sim}{t}_1(\beta_2))\urcorner.$$

Pick $n \in A'$ such that $n > maxJ + 1$ and $b \subseteq \delta_n$ and let $r = s * \langle n\rangle$. Then

$$\forall \beta_1 \ne \beta_2 \in b \backslash lh(t^s), ran\,\underset{\sim}{p}_1^r(\beta_1) \cap ran\,\underset{\sim}{p}_1^r(\beta_2) = \emptyset.$$

Pick $k < \omega$ such that $k > n$ and

$$\forall \beta_1 \in b \cap lh(t^s), \forall \beta_2 \in b \backslash lh(t^s), ran\,\underset{\sim}{p}_1^r(\beta_1) \cap (ran\,\underset{\sim}{p}_1^r(\beta_2) \backslash k) = \emptyset.$$

Let $q = \langle s, B\rangle = \langle s, A' \backslash (maxJ + k + 1) \cup \{n\}\rangle$. Then $q \le^* p' \le^* p$ and q is compatible with p_0^r (since $n \in B$). We show that q is as required. We need to prove the following

1. $q \Vdash \ulcorner \forall \beta \in b, \forall k \in J, \underset{\sim}{q}_1(\beta) \ne k\urcorner$,
2. $q \Vdash \ulcorner \forall \beta_1 \ne \beta_2 \in b \backslash lh(t^s), \underset{\sim}{q}_1(\beta_1) \ne \underset{\sim}{q}_1(\beta_2)\urcorner$,
3. $q \Vdash \ulcorner \forall \beta_1 \in b \cap lh(t^s), \forall \beta_2 \in b \backslash lh(t^s), \underset{\sim}{q}_1(\beta_1) \ne \underset{\sim}{q}_1(\beta_2)\urcorner$.

The proofs of (1) and (2) are as in the case $s = <>$. Let us prove (3). Suppose that (3) fails. Thus we can find $\langle u, D\rangle \le q, p_0^r$, $\beta_1 \in b \cap lh(t^s)$ and $\beta_2 \in b \backslash lh(t^s)$ such that $\langle u, D\rangle \le^* p_0^u$

and $\langle u,D\rangle\Vdash\ulcorner \underset{\sim}{q}_1(\beta_1)=\underset{\sim}{q}_1(\beta_2)\urcorner$. But $\langle u,D\rangle\Vdash\ulcorner \underset{\sim}{q}_1(\beta)=\underset{\sim}{p}^u_1(\beta)=\underset{\sim}{p}^r_1(\beta)\urcorner$ for $\beta\in b$, hence $\langle u,D\rangle\Vdash\ulcorner \underset{\sim}{p}^r_1(\beta_1)=\underset{\sim}{p}^r_1(\beta_2)\urcorner$. Now note that $\beta_2=\alpha_i$ for some $i>lhs+1$, $minD\geq n$ and $min(D\backslash\{n\})>k$, hence by the construction of p^r

$$\langle u,D\rangle\Vdash\ulcorner \underset{\sim}{p}^r_1(\beta_2)\geq(i-lhs)\text{−th element of } D>k\urcorner.$$

By our choice of k, $ran\underset{\sim}{p}^r_1(\beta_1)\cap(ran\underset{\sim}{p}^r_1(\beta_2)\backslash k)=\emptyset$ and we get a contradiction. (3) follows. Thus q is as required, and the claim follows. $\square$

Let

$$\underset{\sim}{\beta}=\{\langle p^s_0,\delta\rangle : s\in[\omega]^{<\omega},\exists\gamma(\delta<\gamma,p^s\Vdash\ulcorner\underset{\sim}{\alpha}=\gamma\urcorner)\}.$$

Then $\underset{\sim}{\beta}$ is a $\mathbb{P}_U$−name of an ordinal.

Claim 5.1.12. $\langle\langle<>,\omega\rangle,\underset{\sim}{q}_1\rangle\Vdash\ulcorner\underset{\sim}{\alpha}=\underset{\sim}{\beta}\urcorner$.

Proof. Suppose not. There are two cases to be considered.

Case 1. There are $\langle r_0,\underset{\sim}{r}_1\rangle\leq\langle\langle<>,\omega\rangle,\underset{\sim}{q}_1\rangle$ and δ such that $\langle r_0,\underset{\sim}{r}_1\rangle\Vdash\ulcorner\delta\in\underset{\sim}{\alpha}$ and $\delta\notin\underset{\sim}{\beta}\urcorner$. We may suppose that for some ordinal α, $\langle r_0,\underset{\sim}{r}_1\rangle\Vdash\ulcorner\underset{\sim}{\alpha}=\alpha\urcorner$. Then $\delta<\alpha$. Let $r_0=\langle s,A\rangle$. Consider $p^s=\langle p^s_0,\underset{\sim}{p}^s_1\rangle$. Then p^s_0 is compatible with r_0 and there is a $*$−extension of p^s deciding $\underset{\sim}{\alpha}$. Let $t\in N_{s(lhs-1)+1}$ be the $*$−extension of p^s deciding $\underset{\sim}{\alpha}$ chosen in the proof of Claim 5.1.10. Let $t=\langle t_0,\underset{\sim}{t}_1\rangle, t_0=\langle s,B\rangle$, and let γ be such that $\langle t_0,\underset{\sim}{t}_1\rangle\Vdash\ulcorner\underset{\sim}{\alpha}=\gamma\urcorner$. Let $n\in A\cap B$. Then

- $p^{s*\langle n\rangle}_0, t_0$ and p^s_0 are compatible and $\langle s*\langle n\rangle, A\cap B\cap A_{s*\langle n\rangle}\rangle$ extends them,
- $p^{s*\langle n\rangle}\leq t$.

Thus $p^{s*\langle n\rangle}\Vdash\ulcorner\underset{\sim}{\alpha}=\gamma\urcorner$. Let $u=\langle s*\langle n\rangle, A\cap B\cap A_{s*\langle n\rangle}\backslash(n+1)\rangle$.

Then $u\leq p^{s*\langle n\rangle}_0$ and $u\Vdash\ulcorner\underset{\sim}{r}_1$ extends $\underset{\sim}{p}^{s*\langle n\rangle}_1$ which extends $\underset{\sim}{t}_1\urcorner$. Thus $\langle u,\underset{\sim}{r}_1\rangle\leq t,\langle r_0,\underset{\sim}{r}_1\rangle,p^{s*\langle n\rangle}$. It follows that $\alpha=\gamma$. Now $\delta<\gamma$ and $p^{s*\langle n\rangle}\Vdash\ulcorner\underset{\sim}{\alpha}=\gamma\urcorner$. Hence $\langle p^{s*\langle n\rangle}_0,\delta\rangle\in\underset{\sim}{\beta}$ and $p^{s*\langle n\rangle}\Vdash\ulcorner\delta\in\underset{\sim}{\beta}\urcorner$. This is impossible since $\langle r_0,\underset{\sim}{r}_1\rangle\Vdash\ulcorner\delta\notin\underset{\sim}{\beta}\urcorner$.

Case 2. There are $\langle r_0,\underset{\sim}{r}_1\rangle\leq\langle\langle<>,\omega\rangle,\underset{\sim}{q}_1\rangle$ and δ such that $\langle r_0,\underset{\sim}{r}_1\rangle\Vdash\ulcorner\delta\in\underset{\sim}{\beta}$ and $\delta\notin\underset{\sim}{\alpha}\urcorner$. We may further suppose that for some ordinal α, $\langle r_0,\underset{\sim}{r}_1\rangle\Vdash\ulcorner\underset{\sim}{\alpha}=\alpha\urcorner$. Thus $\delta\geq\alpha$. Let $r=\langle s,A\rangle$. Then as above p^s_0 is compatible with r and there is a $*$−extension of p^s deciding $\underset{\sim}{\alpha}$. Choose t as in Case 1, $t = \langle t_0, \underset{\sim}{t}_1\rangle, t_0 = \langle s, B\rangle$ and let γ be such that $\langle t_0, \underset{\sim}{t}_1\rangle \Vdash \ulcorner \underset{\sim}{\alpha} = \gamma \urcorner$. Let $n \in A \cap B$. Then as in Case 1, $\alpha = \gamma$ and $p^{s*\langle n\rangle} \Vdash \ulcorner \underset{\sim}{\alpha} = \gamma \urcorner$. On the other hand since $\langle r_0, \underset{\sim}{r}_1\rangle \Vdash \ulcorner \delta \in \underset{\sim}{\beta} \urcorner$, we can find $\bar{s}$ such that $\bar{s}$ does not contradict $p_0^{s*\langle n\rangle}$, $\langle p_0^{\bar{s}}, p_1^{\bar{s}}\rangle \Vdash \ulcorner \underset{\sim}{\alpha} = \bar{\gamma} \urcorner$ for some $\bar{\gamma} > \delta$ and $\langle p_0^{\bar{s}}, \delta\rangle \in \underset{\sim}{\beta}$. Now $\bar{\gamma} = \gamma = \alpha > \delta$ which is in contradiction with $\delta \geq \alpha$. The claim follows. $\square$

This completes the proof of Lemma 5.1.9. $\square$

Lemma 5.1.13. *Let $\langle p_0, \underset{\sim}{p}_1\rangle \Vdash \ulcorner \underset{\sim}{f} : \omega \longrightarrow 0n \urcorner$. Then there are $\mathbb{P}_U-$names $\underset{\sim}{g}$ and $\underset{\sim}{q}_1$ such that $\langle p_0, \underset{\sim}{q}_1\rangle \leq^* \langle p_0, \underset{\sim}{p}_1\rangle$ and $\langle p_0, \underset{\sim}{q}_1\rangle \Vdash \ulcorner \underset{\sim}{f} = \underset{\sim}{g} \urcorner$.*

Proof. For simplicity suppose that $\langle p_0, \underset{\sim}{p}_1\rangle = \langle\langle <>, \omega\rangle, \emptyset\rangle$. Let θ be large enough regular and let $\langle N_n : n < \omega\rangle$ be an increasing sequence of countable elementary submodels of H_θ such that $\mathbb{P}, \underset{\sim}{f} \in N_0$ and $N_n \in N_{n+1}$ for every $n < \omega$. Let $N = \bigcup_{n<\omega} N_n$, $\delta_n = N_n \cap \omega_1$ for $n < \omega$ and $\delta = \bigcup_{n<\omega} \delta_n = N \cap \omega_1$. Let $\langle J_n : n < \omega\rangle \in N_0$ and $\langle \alpha_i : 0 < i < \omega\rangle$ be as in Lemma 5.1.9.

We define by induction a sequence $\langle p^s : s \in [\omega]^{<\omega}\rangle$ of conditions and a sequence $\langle \underset{\sim}{\beta}_s : s \in [\omega]^{<\omega}\rangle$ of $\mathbb{P}_U-$names for ordinals such that

- $p^s = \langle p_0^s, \underset{\sim}{p}_1^s\rangle = \langle\langle s, \omega \backslash (\max s + 1)\rangle, \underset{\sim}{p}_1^s\rangle$,
- $p^s \in N_{s(lhs-1)+1}$,
- $lh(p^s) \geq \delta_{s(lhs-1)}$,
- $p^s \Vdash \ulcorner \underset{\sim}{f}(lhs - 1) = \underset{\sim}{\beta}_s \urcorner$,
- if t end extends s, then $p^t \leq p^s$.

For $s = <>$, let $p^{<>} = \langle\langle <>, \omega\rangle, \emptyset\rangle$. Now suppose that $s \neq <>$ and $p^{s\restriction lhs-1}$ is defined. We define p^s. Let $C_{s\restriction lhs-1}$ be an infinite subset of ω almost disjoint to $\langle ran \underset{\sim}{p}_1^{s\restriction lhs-1}(\beta) : \beta < lh(p^{s\restriction lhs-1})\rangle$. Split $C_{s\restriction lhs-1}$ into ω infinite disjoint sets $\langle C_{s\restriction lhs-1,t} : t \in [\omega]^{<\omega}$ and t end extends $s \restriction lhs-1\rangle$. Again split $C_{s\restriction lhs-1,s}$ into ω infinite disjoint sets $\langle C_i : i < \omega\rangle$. Let $\langle c_{ij} : j < \omega\rangle$ be an increasing enumeration of C_i, $i < \omega$. We may suppose that all of these is done in $N_{s(lhs-1)+1}$. Let $q^s = \langle q_0^s, \underset{\sim}{q}_1^s\rangle$, where

- $q_0^s = \langle s, \omega \backslash (\max s + 1) \rangle$,
- for $\beta < lh(p^{s \restriction lhs-1})$, $\underset{\sim}{q}{}_1^s(\beta) = \underset{\sim}{p}{}_1^{s \restriction lhs-1}(\beta)$,
- for $i \in J_{s(lhs-1)}$ such that $\alpha_i \in \delta_{s(lhs-1)} \backslash lh(p^{s \restriction lhs-1})$

$$\underset{\sim}{q}{}_1^s(\alpha_i) = \{\langle \langle s * \langle r_1, ..., r_i \rangle, \omega \backslash (r_i + 1) \rangle, c_{ir_i} \rangle : r_1 > \max s, \langle r_1, ..., r_i \rangle \in [\omega]^i\}.$$

Then $q^s \in N_{s(lhs-1)+1}$ and as in the proof of claim 5.1.10, $q^s \in \mathbb{P}$. By Lemma 5.1.9, applied inside $N_{s(lhs-1)+1}$, we can find $\mathbb{P}_U-$names $\underset{\sim}{\beta}_s$ and $\underset{\sim}{p}{}_1^s$ such that $\langle q_0^s, \underset{\sim}{p}{}_1^s \rangle \leq \langle q_0^s, \underset{\sim}{q}{}_1^s \rangle$ and $\langle q_0^s, \underset{\sim}{p}{}_1^s \rangle \Vdash \ulcorner \underset{\sim}{f}(lhs-1) = \underset{\sim}{\beta}_s \urcorner$. Let $p^s = \langle p_0^s, \underset{\sim}{p}{}_1^s \rangle = \langle q_0^s, \underset{\sim}{p}{}_1^s \rangle$. Then $p^s \leq p^{s \restriction lhs-1}$ and $p^s \Vdash \ulcorner \underset{\sim}{f} \restriction lhs = \{\langle i, \underset{\sim}{\beta}_{s \restriction i+1} \rangle : i < lhs\} \urcorner$.

This completes our definition of the sequences $\langle p^s : s \in [\omega]^{<\omega} \rangle$ and $\langle \underset{\sim}{\beta}_s : s \in [\omega]^{<\omega} \rangle$. Let

$$\begin{aligned} \underset{\sim}{q}{}_1 &= \{\langle p_0^s, \langle \beta, \underset{\sim}{p}{}_1^s(\beta) \rangle \rangle : s \in [\omega]^{<\omega}, \beta < lh(p^s)\}, \\ \underset{\sim}{g} &= \{\langle p_0^s, \langle i, \underset{\sim}{\beta}_{s \restriction i+1} \rangle \rangle : s \in [\omega]^{<\omega}, i < lhs\}. \end{aligned}$$

Then $\underset{\sim}{q}{}_1$ and $\underset{\sim}{g}$ are $\mathbb{P}_U-$names.

Claim 5.1.14. $\langle \langle <>, \omega \rangle, \underset{\sim}{q}{}_1 \rangle \in \mathbb{P}$.

Proof. We check conditions in Definition 5.1.4.

(1) i.e $\langle <>, \omega \rangle \in \mathbb{P}_U$ is trivial.

(2) It is clear by our construction that

$$\langle <>, \omega \rangle \Vdash \ulcorner \underset{\sim}{q}{}_1 \text{ is a well-defined function } \urcorner$$

and as in the proof of claim 5.1.11, we can show that $lh(\underset{\sim}{q}{}_1) = \delta$. (2a) is trivial. Let us prove (2b). Thus suppose that $I \subseteq \delta$, $I \in V$, $p \leq \langle <>, \omega \rangle$ and $J \subseteq \omega$ is finite. Let $p = \langle s, A \rangle$. If $s = <>$, then as in the proof of 5.1.11, we can show that $a = \emptyset$ is a required. Thus suppose that $s \neq <>$. First we apply (2b) to p^s, $I \cap lh(p^s)$, p and J to find $a' \subseteq lh(p^s)$ such that

(*) For every finite $b \subseteq I \cap lh(p^s) \backslash a'$ there is $p' \leq^* p$ such that p' $\Vdash \ulcorner (\forall \beta \in b, \forall k \in J, \underset{\sim}{p}{}_1^s(\beta) \neq k) \& (\forall \beta_1 \neq \beta_2 \in b, \underset{\sim}{p}{}_1^s(\beta_1) \neq \underset{\sim}{p}{}_1^s(\beta_2)) \urcorner$.

Let $\delta_{s(lhs-1)+1} \backslash \delta_{s(lhs-1)} = \{\alpha_{J_1}, ..., \alpha_{J_i}, ...\}$ where $J_1 < J_2 < ...$ are in $J_{s(lhs-1)+1}$. Let

$$a = a' \cup \{\alpha_1, \alpha_2, ..., \alpha_{J_{lhs}}\}.$$

We show that a is as required. Let $b \subseteq I \backslash a$ be finite. First we apply (*) to $b \cap lh(p^s)$ to find $p' = \langle s, A' \rangle \leq^* p$ such that

$$p' \Vdash \ulcorner (\forall \beta \in b \cap lh(p^s), \forall k \in J, \underset{\sim}{p}{}^s_1(\beta) \neq k) \& (\forall \beta_1 \neq \beta_2 \in b \cap lh(p^s), \underset{\sim}{p}{}^s_1(\beta_1) \neq \underset{\sim}{p}{}^s_1(\beta_2)) \urcorner.$$

Also note that for $\beta \in b \cap lh(p^s)$, $p' \Vdash \ulcorner \underset{\sim}{q}{}_1(\beta) = \underset{\sim}{p}{}^s_1(\beta)) \urcorner$. Pick m such that $\max s + \max J + 1 < m < \omega$ and if t end extends s and $m < \max t$, then $C_{s,t}$ is disjoint to J and to $ran \underset{\sim}{p}{}^s_1(\beta)$ for $\beta \in b \cap lh(p^s)$. Then pick $n > m, n \in A'$ such that $b \subseteq \delta_n$, and let $t = s * \langle n \rangle$. Then

- $\forall \beta_1 \neq \beta_2 \in b \backslash lh(p^s)$, $ran \underset{\sim}{p}{}^t_1(\beta_1) \cap ran \underset{\sim}{p}{}^t_1(\beta_2) = \emptyset$,
- $\forall \beta_1 \in b \cap lh(p^s), \forall \beta_2 \in b \backslash lh(p^s)$, $ran \underset{\sim}{p}{}^t_1(\beta_1) \cap ran \underset{\sim}{p}{}^t_1(\beta_2) = \emptyset$,
- $\forall \beta \in b \backslash lh(p^s), ran \underset{\sim}{p}{}^t_1(\beta) \cap J = \emptyset$.

Let $q = \langle s, B \rangle = \langle s, A' \backslash (n+1) \rangle$. Then $q \leq^* p' \leq^* p$ and using the above facts we can show that

$$q \Vdash \ulcorner (\forall \beta \in b, \forall k \in J, \underset{\sim}{q}{}_1(\beta) = \underset{\sim}{p}{}^t_1(\beta) \neq k) \& (\forall \beta_1 \neq \beta_2 \in b, \underset{\sim}{q}{}_1(\beta_1) = \underset{\sim}{p}{}^t_1(\beta_1) \neq \underset{\sim}{p}{}^t_1(\beta_2) = \underset{\sim}{q}{}_1(\beta_2)) \urcorner.$$

Thus q is as required and the claim follows. $\square$

Claim 5.1.15. $\langle \langle <>, \omega \rangle, \underset{\sim}{q}{}_1 \rangle \Vdash \ulcorner \underset{\sim}{f} = \underset{\sim}{g} \urcorner$.

Proof. Suppose not. Then we can find $\langle r_0, \underset{\sim}{r}{}_1 \rangle \leq \langle \langle <>, \omega \rangle, \underset{\sim}{q}{}_1 \rangle$ and $i < \omega$ such that $\langle r_0, \underset{\sim}{r}{}_1 \rangle \Vdash \ulcorner \underset{\sim}{f}(i) \neq \underset{\sim}{g}(i) \urcorner$. Let $r_0 = \langle s, A \rangle$. Then r_0 is compatible with p^s_0 and $r_0 \Vdash \ulcorner \underset{\sim}{r}{}_1$ extends $p^s_1 \urcorner$. Hence $\langle r_0, \underset{\sim}{r}{}_1 \rangle \leq \langle p^s_0, \underset{\sim}{p}{}^s_1 \rangle = p^s$. Now $p^s \Vdash \ulcorner \underset{\sim}{g}(i) = \underset{\sim}{\beta}{}_{s \restriction i+1} = \underset{\sim}{f}(i) \urcorner$ and we get a contradiction. The claim follows. $\square$

This completes the proof of Lemma 5.1.13. $\square$

The following is now immediate.

Lemma 5.1.16. *The forcing* $(\mathbb{P}, \leq)$ *preserves cofinalities.*

Proof. By Lemma 5.1.13, $\mathbb{P}$ preserves cofinalities $\leq \omega_1$. On the other hand by a $\Delta-$system argument, $\mathbb{P}$ satisfies the ω_2-c.c and hence it preserves cofinalities $\geq \omega_2$. $\square$

Lemma 5.1.17. *Let G be $(\mathbb{P}, \leq)-$generic over V. Then $V[G] \models GCH$.*

Proof. By Lemma 5.1.13, $V[G] \models CH$. Now let $\kappa \geq \omega_1$. Then

$$(2^{\kappa})^{V[G]} \leq ((|\mathbb{P}|^{\omega_1})^{\kappa})^V \leq (2^{\kappa})^V = \kappa^+.$$

The result follows. □

Now we return to the proof of Theorem 5.1.1. Suppose that G is $(\mathbb{P}, \leq)-$generic over V, and let $V_1 = V[G]$. Then V_1 is a cofinality and GCH preserving generic extension of V. We show that adding a Cohen real over V_1 produces $\aleph_1-$many Cohen reals over V. Thus force to add a Cohen real over V_1. Split it into ω Cohen reals over V_1. Denote them by $\langle r_{n,m} : n, m < \omega \rangle$. Also let $\langle f_i : i < \omega_1 \rangle \in V$ be a sequence of almost disjoint functions from ω into ω. First we define a sequence $\langle s_{n,i} : i < \omega_1 \rangle$ of reals by

$$\forall k < \omega,\ s_{n,i}(k) = r_{n, f_i(k)}(0).$$

Let $\langle I_n : n < \omega \rangle$ be the partition of ω_1 produced by G. For $\alpha < \omega_1$ let

- *$n(\alpha) =$ that $n < \omega$ such that $\alpha \in I_n$,*
- *$i(\alpha) =$ that $i < \omega_1$ such that α is the $i-$th element of $I_{n(\alpha)}$.*

We define a sequence $\langle t_\alpha : \alpha < \omega_1 \rangle$ of reals by $t_\alpha = s_{n(\alpha), i(\alpha)}$. The following lemma completes the proof of Theorem 5.1.1.

Lemma 5.1.18. *$\langle t_\alpha : \alpha < \omega_1 \rangle$ is a sequence of $\aleph_1-$many Cohen reals over V.*

Notation 5.1.19. *For each set I, let $\mathbb{C}(I)$ be the Cohen forcing notion for adding $I-$many Cohen reals. Thus $\mathbb{C}(I) = \{p : p$ is a finite partial function from $I \times \omega$ into $2\}$, ordered by reverse inclusion.*

Proof. First note that $\langle r_{n,m} : n, m < \omega \rangle$ is $\mathbb{C}(\omega \times \omega)-$generic over V_1. By c.c.c of $\mathbb{C}(\omega_1)$ it suffices to show that for every countable $I \subseteq \omega_1$, $I \in V$, $\langle t_\alpha : \alpha \in I \rangle$ is $\mathbb{C}(I)-$generic over V. Thus it suffices to prove the following

For every $\langle \langle p_0, \underset{\sim}{p}_1 \rangle, q \rangle \in \mathbb{P} * \mathbb{C}(\omega \times \omega)$ and every open dense subset

(*) $D \in V$ of $\mathbb{C}(I)$, there is $\langle \langle q_0, \underset{\sim}{q}_1 \rangle, r \rangle \leq \langle \langle p_0, \underset{\sim}{p}_1 \rangle, q \rangle$ such that $\langle \langle q_0, \underset{\sim}{q}_1 \rangle$

$, r\rangle \Vdash \ulcorner \langle \underset{\sim}{t}_\nu : \nu \in I\rangle$ extends some element of $D\urcorner$

Let $\langle\langle p_0, \underset{\sim}{p}_1\rangle, q\rangle$ and D be as above. Let $\alpha = supI$. We may suppose that $lh(\underset{\sim}{p}_1) \geq \alpha$. Let $J = \{n : \exists m, k, \langle n, m, k\rangle \in domq\}$. We apply (2b) to $\langle p_0, \underset{\sim}{p}_1\rangle, I, p_0$ and J to find a finite set $a \subseteq I$ such that:

(**) For every finite $b \subseteq I \backslash a$ there is $p_0' \leq^* p_0$ such that $p_0' \Vdash \ulcorner (\forall \beta \in b, \forall k \in J, \underset{\sim}{p}_1(\beta) \neq k) \& (\forall \beta_1 \neq \beta_2 \in b, \underset{\sim}{p}_1(\beta_1) \neq \underset{\sim}{p}_1(\beta_2))\urcorner$.

Let

$$S = \{\langle \nu, k, j\rangle : \nu \in a, k < \omega, j < 2, \langle n(\nu), f_{i(\nu)}(k), 0, j\rangle \in q\}.$$

Then $S \in \mathbb{C}(\omega_1)$. Pick $k_0 < \omega$ such that for all $\nu_1 \neq \nu_2 \in a$, and $k \geq k_0$, $f_{i(\nu_1)}(k) \neq f_{i(\nu_2)}(k)$. Let

$$S^* = S \cup \{\langle \nu, k, 0\rangle : \nu \in a, k < \kappa_0,\ \langle \nu, k, 1\rangle \notin S\}.$$

The reason for defining S^* is to avoid possible collisions. Then $S^* \in \mathbb{C}(\omega_1)$. Pick $S^{**} \in D$ such that $S^{**} \leq S^*$. Let $b = \{\nu : \exists k, j, \langle \nu, k, j\rangle \in S^{**}\}\backslash q$. By $(**)$ there is $p_0' \leq^* p_0$ such that

$$p_0' \Vdash \ulcorner (\forall \nu \in b, \forall k \in J, \underset{\sim}{p}_1(\nu) \neq k) \& (\forall \nu_1 \neq \nu_2 \in b, \underset{\sim}{p}_1(\nu_1) \neq \underset{\sim}{p}_1(\nu_2))\urcorner.$$

Let $p_0'' \leq p_0'$ be such that $\langle p_0'', \underset{\sim}{p}_1\rangle$ decides all the colors of elements of $a \cup b$. Let

$$q^* = q \cup \{\langle n(\nu), f_{i(\nu)}(k), 0, S^{**}(\nu, k)\rangle : (\nu, k) \in domS^{**}\}.$$

Then q^* is well defined and $q^* \in C(\omega \times \omega)$. Now $q^* \leq q$, $\langle\langle p_0'', \underset{\sim}{p}_1\rangle, q^*\rangle \leq \langle\langle p_0, \underset{\sim}{p}_1\rangle, q\rangle$ and for $\langle \nu, k\rangle \in domS^{**}$

$$\langle\langle p_0'', \underset{\sim}{p}_1\rangle, q^*\rangle \Vdash \ulcorner S^{**}(\nu, k) = q^*(n(\nu), f_{i(\nu)}(k), 0) = \underset{\sim}{r}_{n(\nu), f_{i(\nu)}(k)}(0) = \underset{\sim}{t}_\nu(k)\urcorner.$$

It follows that

$$\langle\langle p_0'', \underset{\sim}{p}_1\rangle, q^*\rangle \Vdash \ulcorner \langle \underset{\sim}{t}_\nu : \nu \in I\rangle \text{ extends } S^{**}\urcorner.$$

$(*)$ and hence Lemma 5.1.18 follows. □

This completes the proof of Theorem 5.1.1. □

5.2 An impossibility result

In this section we prove the following result.

Theorem 5.2.1. *([9]) Suppose that $V_1 \supseteq V$ are such that V_1 and V have the same cardinals and reals. Suppose $\aleph_\delta <$ the first fixed point of the $\aleph-$function. Then adding $\aleph_\delta-$many Cohen reals over V_1 can not produce $\aleph_{\delta+1}-$many Cohen reals over V.*

The above Theorem answers an open question from [6]. The proof follows from the next two lemmas.

Lemma 5.2.2. *Suppose that $V_1 \supseteq V$ are such that V_1 and V have the same cardinals and reals. Suppose $\aleph_\delta <$ the first fixed point of the $\aleph-$function, $X \subseteq \aleph_\delta, X \in V_1$ and $|X| \geq \delta^+$ (in V_1). Then X has a countable subset which is in V.*

Proof. By induction on $\delta <$ the first fixed point of the $\aleph-$function.

Case 1. $\delta = 0$. Then $X \in V$ by the fact that V_1 and V have the same reals.

Case 2. $\delta = \delta^{'} + 1$. We have $\delta^{'} < \aleph_{\delta'}$, hence $\delta^+ < \aleph_\delta$, thus we may suppose that $|X| \leq \aleph_{\delta'}$. Let $\eta = sup(X) < \aleph_\delta$. Pick $f_\eta : \aleph_{\delta'} \leftrightarrow \eta, f_\eta \in V$. Set $Y = f_\eta^{-1''} X$. Then $Y \subseteq \aleph_{\delta'}, \delta^{'} < \aleph_{\delta'}$ and $|Y| \geq \delta^+ = \delta^{'+}$. Hence by induction there is a countable set $B \in V$ such that $B \subseteq Y$. Let $A = f_\eta^{''} B$. Then $A \in V$ is a countable subset of X.

Case 3. $limit(\delta)$. Let $\langle \delta_\xi : \xi < cf\delta \rangle$ be increasing and cofinal in δ. Pick $\xi < cf\delta$ such that $|X \cap \aleph_{\delta_\xi}| \geq \delta^+$. By induction there is a countable set $A \in V$ such that $A \subseteq X \cap \aleph_{\delta_\xi} \subseteq X$.

The lemma follows. □

Lemma 5.2.3. *Suppose that $V_1 \supseteq V$ are such that*

(a) *V_1 and V have the same cardinals and reals,*

(b) *$\kappa < \lambda$ are infinite cardinals of V_1 and $cf^{V_1}(\lambda) \neq cf^{V_1}(\kappa)$,*

(c) *there is no $C \in V_1$ such that $C \subseteq \lambda, |C| = \lambda$ and $|C \cap A| < \aleph_0$ for every countable set $A \in V$.*

Then adding $\kappa-$many Cohen reals over V_1 can not produce $\lambda-$many Cohen reals over V.

Proof. Suppose not. Let $\langle r_\alpha : \alpha < \lambda \rangle$ be a sequence of $\lambda-$many Cohen reals over V added after forcing with $\mathbb{C}(\kappa)$ over V_1. Let G be $\mathbb{C}(\kappa)-$generic over V_1. For each $p \in \mathbb{C}(\kappa)$ set

$$C_p = \{\alpha < \lambda : p \text{ decides } \underset{\sim}{r}_\alpha(0)\}.$$

Then by genericity $\lambda = \bigcup_{p \in G} C_p$. Hence as $cf^{V_1}(\lambda) \neq cf^{V_1}(\kappa)$ we can find $p \in G$ such that $|C_p| = \lambda$. Suppose for simplicity that $\forall \alpha \in C_p, p \Vdash \ulcorner \underset{\sim}{r}_\alpha(0) = 0 \urcorner$. By (c) there is a countable set $A \in V$ such that $A \subseteq C_p$. Let $q \in \mathbb{C}(\lambda)$ be such that

$$q \Vdash^V \ulcorner A \in V \text{ is countable and } \forall \alpha \in A, \underset{\sim}{r}_\alpha(0) = 0 \urcorner.$$

Pick $\langle 0, \alpha \rangle \in \omega \times A$ such that $\langle 0, \alpha \rangle \notin supp(q)$. Let $\bar{q} = q \cup \{\langle \langle 0, \alpha \rangle, 1 \rangle\}$. Then $\bar{q} \in \mathbb{C}(\lambda), \bar{q} \leq q$ and $\bar{q} \Vdash \ulcorner \underset{\sim}{r}_\alpha(0) = 1 \urcorner$ which is a contradiction. $\square$

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به

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