A Note on 3-Factorizations of K_{10}*

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Abstract

In this note, we enumerate all nonisomorphic 3-factorizations of K_{10}.

1. An *m-factor* of the complete graph K_n is an m-regular spanning subgraph. An *m-factorization* of K_n is a collection $F = \{F_1, F_2, \ldots, F_k\}$ of m-factors which partition the edge set of K_n. F is *uniform* if $F_1 \cong F_2 \cong \ldots \cong F_k$. Two m-factorizations $F = \{F_1, \ldots, F_k\}$ and $F' = \{F'_1, \ldots, F'_k\}$ are said to be *isomorphic* if there exists a permutation σ on vertices (which is induced on edges) such that $\{\sigma F_1, \ldots, \sigma F_k\} = \{F'_1, \ldots, F'_k\}$. In the sequel, we focus only on 3-factorizations of K_{10}. It is well known that there are twenty one 3-regular graphs on 10 vertices which we denote by **a** to **u** [2]. We denote the number of all nonisomorphic factorizations by R and the number of all nonisomorphic uniform factorizations by r. In [1], it is shown that uniform factorizations are possible for only fifteen of the graphs **a**,...,**u** and in [4] the number of nonisomorphic factorizations of this type has been found to be $r = 122$. The number of nonisomorphic factorizations with exactly two isomorphic factors has also been established to be 2194 [5]. In this note, we

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determine all possible nonisomorphic factorizations, so the problem 2 in [3] is completely answered. Our results also agree with the results of [1], [4], and [5].

2. A factorization $F = \{F_1, F_2, F_3\}$ is said to be of *type* (for example) $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ if by renaming its factors we have $F_1 \cong \mathbf{a}, F_2 \cong \mathbf{b}$, and $F_3 \cong \mathbf{c}$. Similarly, a uniform factorization with isomorphic factors to (for example) $\mathbf{a}$ is said to be of type $(\mathbf{a}, \mathbf{a}, \mathbf{a})$. Clearly, factorizations of different types are nonisomorphic. To generate nonisomorphic factorizations of a certain type, we define a relation $\sim$ on $S_{10} \times S_{10} \times S_{10}$. We first typically consider the type $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ to describe $\sim$. Let $F = \{p_1\mathbf{a}, p_2\mathbf{b}, p_3\mathbf{c}\}$ and $F' = \{q_1\mathbf{a}, q_2\mathbf{b}, q_3\mathbf{c}\}$ be two factorizations of type $(\mathbf{a}, \mathbf{b}, \mathbf{c})$. Then, $F \cong F'$ if and only if there exists a permutation $\sigma \in S_{10}$ such that $\sigma p_1\mathbf{a} = q_1\mathbf{a}, \sigma p_2\mathbf{b} = q_2\mathbf{b}$, and $\sigma p_3\mathbf{c} = q_3\mathbf{c}$. Define $\sim$ as follows. $(p_1, p_2, p_3) \sim (q_1, q_2, q_3)$ if and only if there exists a permutation $\sigma \in S_{10}$ such that $q_1^{-1}\sigma p_1 \in \mathrm{Aut}(\mathbf{a}), q_2^{-1}\sigma p_2 \in \mathrm{Aut}(\mathbf{b})$, and $q_3^{-1}\sigma p_3 \in \mathrm{Aut}(\mathbf{c})$. The definition of $\sim$ for a type such as $(\mathbf{a}, \mathbf{a}, \mathbf{b})$ is somewhat different: $(p_1, p_2, p_3) \sim (q_1, q_2, q_3)$ if and only if there exist permutations $\sigma \in S_{10}$ and $\tau \in S_2$ such that $q_{\tau(1)}^{-1}\sigma p_1, q_{\tau(2)}^{-1}\sigma p_2 \in \mathrm{Aut}(\mathbf{a})$, and $q_3^{-1}\sigma p_3 \in \mathrm{Aut}(\mathbf{b})$. Finally, for a type like $(\mathbf{a}, \mathbf{a}, \mathbf{a})$, we define $\sim$ as follows: $(p_1, p_2, p_3) \sim (q_1, q_2, q_3)$ if and only if there exist permutations $\sigma \in S_{10}$ and $\tau \in S_3$ such that $q_{\tau(1)}^{-1}\sigma p_1, q_{\tau(2)}^{-1}\sigma p_2, q_{\tau(3)}^{-1}\sigma p_3 \in \mathrm{Aut}(\mathbf{a})$. It can be easily seen that $\sim$ is an equivalence relation. We now present an algorithm to generate nonisomorphic factorizations of a given type $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$. (i) Partition $S_{10} \times S_{10} \times S_{10}$ into equivalence classes of $\sim$. (ii) For each equivalence class, choose a representative (p_1, p_2, p_3) and determine whether $\{p_1\mathbf{x}_1, p_2\mathbf{x}_2, p_3\mathbf{x}_3\}$ is a factorization. In a representative (p_1, p_2, p_3), we can always assume that p_1 is identity. Moreover, note that if one fixes p_1 and p_2, then p_3 is determined uniquely.

3. To be confident about computational results of the algorithm above, we pursued and implemented a different approach. A description of this approach is as follows. Let G be a 3-regular graph on 10 vertices and let P be the incidence matrix of vertices vs. edges of $K_{10} \setminus G$. Any vector with entries ± 1 in the null space of P obviously implies a factorization which involves factor G. Therefore, by letting G run over all 3-regular graphs on 10 vertices, we can obtain all distinct factorizations. The set of nonisomorphic factorizations generated by the algorithm of paragraph 2 can also be used to find all distinct factorizations. The results of these two methods have been

checked to be the same for each type of graphs.

4. The twenty one 3-regular graphs on 10 vertices $\mathbf{a}, \ldots, \mathbf{u}$ are shown in Figure 1. The number of nonisomorphic factorizations for each type is given in Table I. We observe that out of $\binom{23}{3} = 1771$ possible types, there exist factorizations of exactly 1000 types. The number of all distinct factorizations is equal to 31,350,309,760. Finally, we obtain $R = \mathbf{8999}$.

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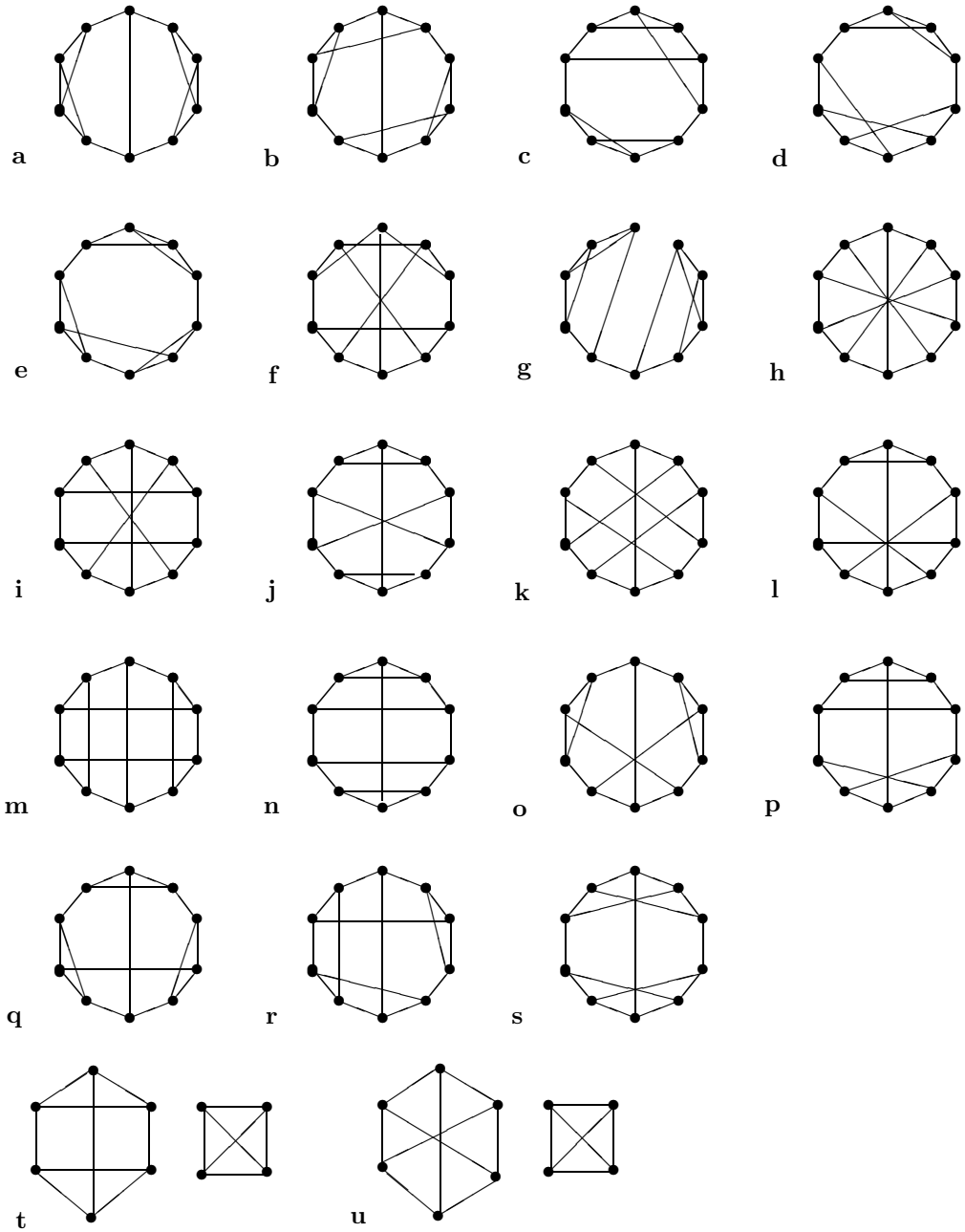

Figure 1 3-regular graphs on 10 vertices.

Table I The numbers of nonisomorphic 3-factorizations of K_{10}[1].

aab2	aai1	aak1	aal1	aan4	aap1	abb12	abc17	abd3	abe10	abf1	abg3
abh3	abi17	abj10	abk9	abl28	abm4	abn15	abo11	abp13	abq18	abr14	abs1
abt1	acc2	acd2	ace1	acf1	aci6	acj3	ack4	acl17	acm1	acn6	aco3
acp7	acq4	acr6	adg3	adh1	adi2	adk1	adl2	adn1	adp6	ads3	aee1
aei4	aej2	aek2	ael6	aem2	aen3	aeo2	aep4	aeq2	aer3	afi1	afp1
agi2	agj1	agk1	agl1	agp6	agr1	ags1	ahi2	ahk1	ahl1	ahm2	ahp2
ahr1	aii3	aij3	aik5	ail15	aim4	ain13	aio2	aip5	aiq4	air6	ais2
ajk1	ajl10	ajn6	ajo2	ajp3	ajq5	ajr1	akk2	akl8	akm2	akn7	ako1
akp5	akq3	akr3	aks2	all16	alm3	aln19	alo5	alp14	alq9	alr9	amn2
ann2	ano2	anp6	anq3	anr5	aop3	app9	apq2	apr6	aps2	aqq2	bbb39
bbc51	bbd14	bbe25	bbf2	bbg2	bbh14	bbi60	bbj30	bbk31	bbl108	bbm7	bbn54
bbo21	bbp48	bbq40	bbr35	bbs5	bbt5	bbu1	bcc30	bcd10	bce26	bcf2	bcg5
bch9	bci59	bcj33	bck26	bcl125	bcm11	bcn53	bco15	bcp58	bcq36	bcr38	bcs5
bct1	bdd1	bde7	bdg2	bdh4	bdi18	bdj6	bdk9	bdl29	bdm7	bdn14	bdo4
bdp15	bdq5	bdr18	bds1	bee6	bef1	beg2	beh5	bei31	bej16	bek16	bel75
bem4	ben27	beo12	bep25	beq23	ber20	bes2	bet1	bfi2	bfk1	bfl3	bfn1
bfo3	bfp2	bgh4	bgi9	bgj2	bgk3	bgl19	bgm3	bgn10	bgo1	bgp4	bgq1
bgr6	bgs1	bhh1	bhi15	bhj5	bhk8	bhl28	bhm5	bhn18	bho1	bhp19	bhq6
bhr12	bhs3	bii24	bij28	bik31	bil103	bim15	bin60	bio14	bip55	biq32	bir38
bis9	bit1	biu1	bjj6	bjk11	bjl61	bjm8	bjn29	bjo10	bjp23	bjq28	bjr22
bjs3	bjt1	bkk9	bkl54	bkm4	bkn30	bko10	bkp20	bkq20	bkr14	bks4	bkt2
bll104	blm28	bln107	blo42	blp105	blq82	blr76	bls8	blt1	bmm1	bmn17	bmo3
bmp14	bmq5	bmr6	bms1	bnn32	bno15	bnp54	bnq35	bnr40	bns7	boo5	bop13
boq16	bor8	bos1	bpp25	bpq34	bpr38	bps6	bpt1	bqq16	bqr30	bqs1	brr13
brs6	ccc7	ccd4	cce6	ccg2	cch2	cci16	ccj4	cck8	ccl26	ccm5	ccn19
cco3	ccp15	ccq6	ccr12	ccs1	ccu1	cdd4	cde2	cdg3	cdh2	cdi8	cdj3
cdk2	cdl15	cdm2	cdn11	cdp20	cdq2	cdr6	cds3	cee3	ceg1	ceh1	cei21
cej6	cek12	cel25	cem1	cen13	ceo4	cep13	ceq6	cer4	cfh1	cfi1	cfj1
cfk1	cfl2	cfn1	cfq2	cgg1	cgh3	cgi6	cgj1	cgk1	cgl5	cgm1	cgn3
cgp13	cgr1	cgs1	chh2	chi7	chj2	chk5	chl11	chm1	chn3	cho2	chp8
chq1	chr4	cii14	cij13	cik12	cil55	cim6	cin33	cio11	cip21	ciq17	cir22
cis2	cit1	cjj3	cjk7	cjl29	cjm2	cjn16	cjo7	cjp13	cjq11	cjr10	cjs1
ckk4	ckl25	ckn13	cko6	ckp17	ckq13	ckr13	cks2	cll64	clm9	cln55	clo19
clp55	clq36	clr41	cls5	clt1	cmn8	cmo2	cmp8	cmq2	cmr4	cnn15	cno7
cnp30	cnq13	cnr19	cns2	cnt1	coo4	cop5	coq12	cor5	cos1	cpp21	cpq14
cpr22	cps6	cpu1	cqq12	cqr8	cqs1	crr11	css1	ddd2	dde1	ddg2	ddl1
ddn2	ddp6	ddr1	ddt1	ddu2	deg3	dei4	dek2	del4	dem3	den3	deo1
dep10	der1	dfj1	dfq1	dgg3	dgn2	dgp3	dgs3	dhj1	dhl4	dhn1	dho1
dhp2	dii1	dij5	dik3	dil13	dim1	din9	dio3	dip8	diq10	dir4	djk5
djl11	djn2	djo2	djp2	djr1	djs2	dkl8	dkm1	dkn5	dko2	dkq7	dkr2
dll20	dln8	dlo8	dlp13	dlq4	dlr11	dls1	dmm1	dmn5	dmp2	dmq1	dnn6
dnp10	dnq4	dnr6	dns1	dor1	dpp6	dpq4	dpr4	dps2	dpt1	dqr1	drr2
eee1	eef1	eei3	eej2	eek4	eel9	eem1	een5	eeo1	eep4	eeq2	eer4
eet1	efl2	efq1	efr1	egh2	egi1	egk2	egl2	egm2	egn3	egp4	egr2
egs2	ehi4	ehj1	ehk2	ehl7	ehm1	ehn4	eho1	ehp4	ehq1	ehs2	eii8
eij5	eik11	eil30	eim8	ein23	eio2	eip22	eiq7	eir9	eis1	ejj3	ejk6
ejl13	ejm1	ejn4	ejo5	ejp8	ejq8	ejr2	ekk1	ekl15	ekm4	ekn9	eko3
ekp4	ekq5	ekr6	eks1	ekt1	ell21	elm6	eln31	elo8	elp22	elq30	elr15
els3	emm2	emn3	emo1	emp5	emq1	emr3	ems1	enn8	eno3	enp13	enq7
enr8	ens1	eoo1	eop2	eoq10	eor5	epp17	epq3	epr12	eps4	eqq5	eqr3
err3	ers2	ess1	ffh1	fgl1	fhi1	fii1	fij1	fin1	fip1	fjl1	fjn2
fjp1	fjq1	fkl1	fks1	fll2	fln1	flo1	flp5	flr1	fmq1	fmr1	fnn1
fnq1	fnr2	for1	fqr2	fqt1	ggg1	ggh1	ggs1	ggt1	ggu1	ghi1	ghk1
ghl2	ghm1	ghn3	ghp4	ghq1	ghs1	gii3	gij2	gik2	gil11	gim1	gin6
gio2	gip1	giq3	gir2	gis1	gjj1	gjk5	gjl5	gjm1	gjn3	gjo1	gjq1
gjr2	gjs1	gkk2	gkl6	gkn1	gkp1	gkq1	gkr2	gll6	glm2	gln4	glo5
glp6	glq8	glr1	gmn3	gmr1	gnn2	gnp3	gnq2	gnr1	gns1	gop1	gor1
gpp4	gpq2	gpr4	gps1	gpt1	gqr6	grr1	grs1	gss1	gst1	hhn1	hhq1
hii3	hij2	hil8	hin7	hio4	hip4	hiq2	hjj2	hjl4	hjm1	hjn3	hjp3
hjq2	hkk4	hkl4	hkm2	hkn3	hko2	hkp2	hkq1	hll13	hln6	hlo6	hlp7
hlq4	hlr8	hmm1	hmn3	hmq1	hnn5	hno2	hnp9	hnq3	hnr4	hns2	hor2
hpp3	hpq5	hpr6	hqq2	hqr2	hqs1	iii4	iij6	iik6	iil24	iim2	iin12
iio4	iip14	iiq15	iir10	iit2	ijj3	ijk6	ijl26	ijm1	ijn16	ijo2	ijp15
ijq8	ijr10	ijs1	ikk4	ikl27	ikm4	ikn12	iko5	ikp12	ikq9	ikr4	iks6
ill48	ilm6	iln51	ilo15	ilp56	ilq32	ilr31	ils2	ilt1	imm2	imn8	imo2
imp6	imq6	imr3	ims2	inn11	ino8	inp28	inq20	inr22	ins3	iop13	ioq6
ior5	ios1	ipp8	ipq27	ipr16	ips2	ipt1	iqq4	iqr14	iqs3	irr5	jjj4
jjk2	jjl3	jjn7	jjo2	jjp4	jjq2	jjr4	jjs1	jkk4	jkl10	jkm3	jkn11
jko3	jkp9	jkq2	jkr8	jks1	jku1	jll33	jlm10	jln25	jlo8	jlp20	jlq18

[1]By the notation $x_1x_2x_3n$ we mean that there exist n nonisomorphic factorizations of type $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$

Table I Continued.

jlr15	jls3	jlt1	jmm1	jmn2	jmo1	jmp3	jms2	jnn8	jno5	jnp10	jnq14
jnr7	joo1	jop7	joq1	jor5	jpp9	jpq5	jpr10	jqq5	jqr4	jqs2	jrr1
jrs1	kkk3	kkl2	kkn3	kko1	kkp4	kkr2	kkt1	kll29	klm5	kln29	klo5
klp26	klq19	klr11	kls2	klt1	kmm2	kmn4	kmo1	kmq5	kmr3	knn12	kno3
knp7	knq13	knr8	kns1	koo1	kop7	kor3	kos1	kpp8	kpq15	kpr10	kqq3
kqr10	krr6	lll33	llm12	lln64	llo19	llp53	llq33	llr42	lls5	llt2	llu2
lmn10	lmo4	lmp7	lmq2	lmr6	lnn25	lno27	lnp58	lnq46	lnr29	lns4	lnt2
loo2	lop27	loq10	lor10	los2	lpp24	lpq50	lpr33	lps2	lqq13	lqr24	lqs2
lrr10	lrs4	lrt1	mmn2	mmt1	mnn3	mno1	mnp8	mnq1	mnr6	mns3	mnt1
mop1	moq1	mor1	mpp7	mpq5	mpr3	mqr3	mqs1	mrr1	nnn9	nno2	nnp17
nnq4	nnr12	nns2	nnt1	nnu1	noo5	nop8	noq9	nor8	npp17	npq12	npr20
nps4	nqq10	nqr11	nqs3	nrr9	nrs2	nrt1	nss1	ooo1	oop1	oor1	oot1
opp2	opq4	opr5	ops1	oqq5	oqr8	oqt1	orr3	ppp6	ppq7	ppr11	pps2
ppt3	ppu1	pqq1	pqr12	pqs2	prr6	prs1	prt1	qqq5	qqr2	qqt2	qrr2
rrr6	rru2	sst1	uuu1								