Some Infinite Families of Large Sets of t-Designs

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Abstract

A set of necessary conditions for existence of a large set of a t-design, $LS\left(\binom{v-t}{k-t}/n;t,k,v\right)$, is $n|\binom{v-i}{k-i}$ for $i=0,1,\cdots,t$. We show that these conditions are sufficient for $n=3, t=2,3$, or 4, and $k\leq 8$.

1. Introduction

Let v,k,t, and λ be integers with $v\geq k\geq t\geq 0$ and $\lambda\geq 1$. A t-(v,k,λ) design is a collection $\mathcal{B}$ of k-subsets of a v-set X such that every t-subset of X occurs exactly λ times in $\mathcal{B}$.

A large set of disjoint t-(v,k,λ) designs, denoted by $LS(\lambda;t,k,v)$, is a partition of the k-subsets of a v-set into t-(v,k,λ) designs. The number of designs in partition equals to $n=\binom{v-t}{k-t}/\lambda$. We simply write $LS(1/n;t,k,v)$ for $LS\left(\binom{v-t}{k-t}/n;t,k,v\right)$.

A set of necessary conditions for the existence of a $LS(1/n;t,k,v)$ is $n|\binom{v-i}{k-i}$ for $i=0,1,\cdots,t$. In Table 1 some important cases with small k's in which the necessary conditions are sufficient, are shown.

A conjecture of A. Hartman[7] states that these conditions are sufficient for $n=2$. Concerning this conjecture, Ajoodani-Namini in [1], besides of establishing the truth of the conjecture for $t=2$(with some other partial results), proves some theorems for any t. In this paper, we generalize those results for any prime power number $n=p^{\alpha}$. As a

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consequence, we show that the necessary conditions are sufficient for $n = 3$, $t = 2, 3$, or 4, and $k \leq 8$.

Table 1.

t	k	n	v	Ref.
1	$*$	$*$[1]	$*$	[5]
2	3	$*$	$\neq 7$	[12,13,14]
2	$*$	2	$*$	[1,3]
2	4	11	$\equiv 14 \pmod{11}$	[9]
3	4	$*$	$\equiv 0 \pmod{3}$	[15]
3	5	3	$\equiv 4 \pmod{9}$	[17]
3	4	5	$\equiv 3 \pmod{10}$	[9]
4	6	3	$\equiv 5 \pmod{9}$	[9]
6	7	2	$*$	[8,11]

1. $*$ means all feasible values.

2. Preliminaries

Notation. Let n, t, and k be given. The set of all v's for which $LS(1/n; t, k, v)$ exist is denoted by $A(n; t, k)$. The set of all v's which satisfy the necessary conditions is denoted by $B(n; t, k)$.

The following two lemmas, which have recursive nature, provide some important machinary for constructing families of large setes from small cases and and have been utilized by different authors[3,9,10,17].

Lemma 1 [2]. If $v \in \bigcap_{i=0}^{l} A(n; t, k+i)$, then $v + l \in A(n; t, k+l)$.

Lemma 2 [2]. If $v \in \bigcap_{i=t+1}^{k} A(n; t, i)$ and $u \in A(n; t, k)$, then $u + l(v - t) \in A(n; t, k)$.

Let p be a prime, and let m, l be two integers such that $m > l$. Let $(m, l)_p$ denote the largest power of p which divides $\binom{m}{l}$.

Lemma 3.

$$(m,l)_p = \sum_{i\geq 1} \left[\frac{m}{p^i}\right] - \left[\frac{l}{p^i}\right] - \left[\frac{m-l}{p^i}\right].$$

Here $[a]$ denotes the largest integer smaller than or equal to a.

Proof. The largest power of p that divides $r!$ equals to $\left[\frac{r}{p}\right] + \left[\frac{r}{p^2}\right] + \cdots$. Using $\binom{m}{l} = \frac{m!}{l!(m-l)!}$, the assertion follows. □

Let l_p be the smallest power of p such that $l < p^{l_p}$. Now we have the following important lemma.

Lemma 4. Let p^α be a prime power and let m_1, m_2, and l be integers such that $m_1, m_2 \geq l$. Let $m_1 \equiv m_2 \pmod{p^{l_p+\alpha-1}}$. Then $p^\alpha | \binom{m_1}{l}$ iff $p^\alpha | \binom{m_2}{l}$.

Proof. If $i > l_p$ and $\left[\frac{m}{p^i}\right] - \left[\frac{l}{p^i}\right] - \left[\frac{m-l}{p^i}\right] = 1$, then by $m[x] \leq [mx] \leq m[x] + m - 1$, we have

$$\begin{aligned} \left[\frac{m}{p^{i-1}}\right] - \left[\frac{l}{p^{i-1}}\right] - \left[\frac{m-l}{p^{i-1}}\right] &\geq p\left[\tfrac{m}{p^i}\right] - p\left[\tfrac{m-l}{p^i}\right] - (p-1) \\ &= 1. \end{aligned}$$

Now let $m_2 = m_1 + jp^{l_p+\alpha-1}$ and $p^\alpha | \binom{m_1}{l}$. We take the minimal chain $i_1 < i_2 < \cdots < i_\alpha$ such that

$$\left[\frac{m_1}{p^{i_s}}\right] - \left[\frac{l}{p^{i_s}}\right] - \left[\frac{m_1-l}{p^{i_s}}\right] = 1, \quad \text{for} \quad s = 1, \cdots, \alpha.$$

So $i_\alpha \leq p^{l_p+\alpha-1}$,and for $s = 1, \cdots, \alpha$ we have

$$\begin{aligned} \left[\frac{m_2}{p^{i_s}}\right] - \left[\frac{l}{p^{i_s}}\right] - \left[\frac{m_2-l}{p^{i_s}}\right] &= \left[\tfrac{m_1}{p^{i_s}}\right] - \left[\tfrac{l}{p^{i_s}}\right] - \left[\tfrac{m_1-l}{p^{i_s}}\right] \\ &= 1. \end{aligned}$$

Therefore, $p^\alpha | \binom{m_2}{l}$. □

3. More on the Necessary Conditions

In this section we state the necessary conditions in terms of some congruency relations.

Theorem 1. Let p^α be a prime power. Then $v \in B(p^\alpha;t,k)$ iff one of the followings hold:

$$i) \qquad v \equiv t, \cdots, k-1 \pmod{p^{k_p+\alpha-1}}.$$

$$ii) \qquad v \equiv v_0 \pmod{p^{k_p+\alpha-1}},\ k < v_0 < p^{k_p+\alpha-1} \text{ and } v_0 \in B(p^\alpha;t,k).$$

Proof. Let $v \equiv v_0 \pmod{p^{k_p+\alpha-1}}$ such that $k \le v_0 \le p^{k_p+\alpha-1}+k-1$. By Lemma 4, $v \in B(p^\alpha;t,k)$ iff $v_0 \in B(p^\alpha;t,k)$.

First let $v_0 = p^{k_p+\alpha-1}+s$,where $t \le s \le k-1$. For $j = 0, \cdots, t$, we have

$$\begin{aligned}(v_0-j, k-j)_p &\ge \textstyle\sum_{i=0}^{\alpha-1} \left[\frac{v_0-j}{p^{k_p+i}}\right] - \left[\frac{k-j}{p^{k_p+i}}\right] - \left[\frac{v_0-k}{p^{k_p+i}}\right] \\ &= \textstyle\sum_{i=0}^{\alpha-1} \left[\frac{s-j}{p^{k_p+i}}\right] - \left[\frac{k-j}{p^{k_p+i}}\right] - \left[\frac{s-k}{p^{k_p+i}}\right] \\ &= \alpha.\end{aligned}$$

Hence $p^\alpha | \binom{v_0-j}{k-j}$ and we have $v_0 \in B(p^\alpha;t,k)$.

Now, for $0 \le j \le t$, assume that $v_0 = p^{k_p+\alpha-1}-1+j$. We show that $(v_0-j, k-j)_p < \alpha$. Taking $k-j=i$, we have $\binom{p^{k_p+\alpha-1}-1-j-j}{k-j} = \binom{p^{k_p+\alpha-1}-1}{i}$. If $r \le k_p$, then

$$\begin{aligned}&\left[\tfrac{p^{k_p+\alpha-1}-1}{p^r}\right] - \left[\tfrac{i}{p^r}\right] - \left[\tfrac{p^{k_p+\alpha-1}-1-i}{p^r}\right] \\ &= \left[\tfrac{-1}{p^r}\right] - \left(\left[\tfrac{i}{p^r}\right] + \left[\tfrac{-i-1}{p^r}\right]\right) \\ &= -1-(-1) \\ &= 0.\end{aligned}$$

Therefore, $(v_0-j, k-j)_p < \alpha$ and $p^\alpha \not| \binom{v_0-j}{k-j}$.

If $k \le v_0 \le k+t$, we let $v_0 = k+i$, for $0 \le i \le t$ and we have

$$\binom{v_0-j}{k-j} = \binom{k+i-j}{i}, \quad for\ j = 0, 1, \cdots, t.$$

By Lucas' lemma, one can easily see that $g.c.d.\{\binom{k+j}{i} |\ j = 0, 1, \cdots, i\} = 1$. Therefore, $v_0 \notin B(p^\alpha;t,k)$. This completes the proof. □

For $k = t+1$, we can completely characterize all the feasible v's. To do this, we have the following lemma.

Lemma 5. Let n be an integer with a prime factorization $\prod_1^s p_i^{\alpha_i}$. Then

$$B(n;t,t+1) = \{v|\ v \equiv t \pmod{n \prod_1^s p_i^{(t+1)_{p_i}-1}}\}.$$

Proof. We use the following result of Teirlinck[16]:

$$\lambda_{min} = \text{g.c.d.}\{v-t, \text{l.}c.m.\{1,2,\cdots,t+1\}\}.$$

If $v \in B(n;t,t+1)$, then $p_i^{\alpha_i} | \frac{v-t}{\lambda_{min}}$, for $i=1,\cdots,s$. Conversely, it is obvious that $p_i^{(t+1)_{p_i}-1} |\ l.c.m.\ \{1,\cdots,t+1\}$. Therefore, we must have $p_i^{\alpha_i+(t+1)_{p_i}-1} | v-t$ for $i=1,\cdots,s$. $\square$

4. Main Results

In this section we prove a theorem with recursive nature and then obtain some results on large sets for $n=3$.

Theorem 2. Let t and k be integers such that $k > t+1$ and let p^α be a prime power. Assume that the following conditions are satisfied:

$i)$ $A(p^\alpha;t,k_1) = B(p^\alpha;t,k_1)$ for $k_1 = t+1, t+2, \cdots, k-1$.

$ii)$ If $2k \le v_0 \le p^{k_p+\alpha-1}$ and $v_0 \in B(p^\alpha;t,k)$, then $v_0 \in A(p^\alpha;t,k)$.

$iii)$ If $k \le \lfloor \frac{p^{k_p+\alpha-1}}{2} \rfloor + \lceil \frac{t}{2} \rceil$, then $p^{k_p+\alpha-1}+t \in A(p^\alpha;t,k)$.

Then $A(p^\alpha;t,k) = B(p^\alpha;t,k)$.

Proof. If $k > \lfloor \frac{p^{k_p+\alpha-1}}{2} \rfloor + \lceil \frac{t}{2} \rceil$, then

$$\begin{aligned} p^{k_p+\alpha-1}+t-k &\le p^{k_p+\alpha-1}+t-\lfloor \tfrac{p^{k_p+\alpha-1}}{2} \rfloor - \lceil \tfrac{t}{2} \rceil - 1 \\ &\le \lfloor \tfrac{p^{k_p+\alpha-1}}{2} \rfloor + \lceil \tfrac{t}{2} \rceil. \end{aligned}$$

Therefore, by iii), we have $p^{k_p+\alpha-1}+t \in A(p^\alpha;t,p^{k_p+\alpha-1}+t-k)$ and hence, $p^{k_p+\alpha-1}+t \in A(p^\alpha;t,k)$.

To proceed, first we assume that $v_0 = p^{k_p+\alpha-1}+t+l$ in which $0 < l < k-t$. By i), we have $p^{k_p+\alpha-1}+t \in \bigcap_{i=0}^{l} A(p^\alpha;t,k-l+i)$. So by Lemma 1 we conclude that $v_0 \in A(p^\alpha;t,k)$.

Now we let $k + t < v_0 < 2k$ and $v_0 \in B(p^\alpha; t, k)$. Then $t < v_0 - k < k$ and therefore $v_0 \in A(p^\alpha; t, v_0 - k)$. Thus by i), $v_0 \in A(p^\alpha; t, k)$.

Let $v \in B(p^\alpha; t, k)$. Let $v = l \cdot p^{k_p+\alpha-1} + v_0$ where $k \leq v_0 \leq p^{k_p+\alpha-1} + k - 1$. By Theorem 1, $v \in B(p^\alpha; t, k)$ implies that $v_0 \in B(p^\alpha; t, k)$. From this and the above paragraphs, we conclude that $v_0 \in A(p^\alpha; t, k)$. Now, in Lemma 2, by substituting $p^{k_p+\alpha-1} + t$ for v, and v_0 for u, we obtain $v \in A(p^\alpha; t, k)$. This completes the proof. $\square$

Lemma 6. Let $2 \leq t < k \leq 8$. Then

$$B(3; t, k) = \{v |\ v \equiv t, \cdots, k - 1 \pmod 9\}.$$

Proof. By Theorem 1, the proof is straightforward. $\square$

Theorem 3. Let $t = 2, 3$, or 4 and $k \leq 8$. Then $A(3; t, k) = B(3; t, k)$.

Proof. By Theorem 2 and Lemmas 5 and 6 we need the following large sets:

$$\begin{gathered}
1)\ LS(1/3; 2, 3, 11), \quad 2)\ LS(1/3; 2, 4, 11), \quad 3)\ LS(1/3; 2, 5, 11), \\
4)\ LS(1/3; 3, 4, 12), \quad 5)\ LS(1/3; 3, 5, 12), \quad 6)\ LS(1/3; 3, 6, 12), \\
7)\ LS(1/3; 4, 5, 13), \quad 8)\ LS(1/3; 4, 6, 13).
\end{gathered}$$

7) exists by [9], and 1) and 4) are two derived large sets of 7). 2), 3), 5), and 6) are derived and residual large sets of 8) which exists by[6]. $\square$

Note: Let $2 \leq t < k \leq 8$. If a $LS(1/3; 5, 6, 14)$ and a $LS(1/3; 6, 7, 15)$ exist, then $A(3; t, k) = B(3; t, k)$. By a theorem of Alltop[4], the existence of a $LS(1/3; 4, 6, 13)$ and a $LS(1/3, 6, 7, 15)$ imply the existence of a $LS(1/3, 5, 7, 14)$ and a $LS(1/3; 7, 8, 16)$, respectively. Now, by utilizing Theorems 2 and 3, and Lemmas 5 and 6 the statement follows.

For larger values of k, we have Theorem 7. But first we need the following lemma and theorems[2].

Lemma 7. If a $LS(1/n; t, k, v)$ and a $LS(1/n; t, k + 1, v)$ exist, then a $LS(1/n; t, k + 1, v + 1)$ also exist.

Theorem 4[Ajoodani-Namini]. If a $LS(1/p; t, k, v-1)$ exists, then a $LS(1/p; t+1, pk+i, pv)$ exists for $1 \leq i \leq p-1$.

Theorem 5[Ajoodani-Namini]. If a $LS(1/p; t, k, v)$ exists, then a $LS(1/p; t, pk, pv)$ also exists.

We can say something further.

Theorem 6. If a $LS(1/p; t, k, v)$ exists, then a $LS(1/p; t, pk+i, pv+j)$ exists for $0 \leq j \leq 2p-2$ and $-p+j+1 \leq i \leq p-1$.

Proof. From $LS(1/p; t, k, v)$, we have $LS(1/p; t-1, k, v-1)$ and so by Theorem 4 we have $LS(1/p; t, pk+i, pv)$ for $1 \leq j \leq p-1$. By Theorem 5, there exists a $LS(1/p; t, pk, pv)$. From $LS(1/p; t-1, k-1, v-1)$ and Theroem 4 we have $LS(1/p; t, pk-p+i, pv)$ for $1 \leq i \leq p-1$. Putting all these together, there exists a $LS(1/p; t, pk+i, pv)$ for $-p+1 \leq i \leq p-1$. Now by Lemma 7, there exist $LS(1/p; t, pk+i, pv+j)$ for $0 \leq j \leq 2p-2$ and $-p+j+1 \leq i \leq p-1$. $\square$

Utilizing Theorem 6, the following theorem is immediate.

Theorem 7. Let $t = 2, 3$, or 4, $t < k \leq 8$, and $v \equiv t, \cdots, k-1 \pmod 9$, then a $LS(1/3; t, 3^\alpha k+r, 3^\alpha v+s)$ exists for $\alpha \geq 0$ and $0 \leq r, s \leq 3^\alpha - 1$.

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References

1. S. Ajoodani-Namini, *All block designs with $b = \binom{v}{k}/2$ exist*, preprint.

2. S. Ajoodani-Namini, *Extending large sets of t-designs*, J. Combin. Theory Ser. A **76** (1996), 139-144.

3. S. Ajoodani-Namini and G.B. Khosrovshahi, *More on halving complete desings,* Discrete Math. **135** (1994), 29-37.

4. W. O. Alltop, *An infinite class of 5-designs,* J. Combin. Theory Ser. A **12** (1972), 390-395.

5. Z. Baranyai, *On the factorizations of the complete uniform hypergraph,* in: Finite and Infinite Sets, Colloq. Math. Soc., Janos Bolyai, Vol. 10, North-Holland, Amesterdam, 1975, pp. 91-108.

6. C. A. Cusack, R. Laue, and S. S. Magliveras, *New large sets of t-designs on up to 18 points*, in preparation.

7. A. Hartman, *Halving the complete design,* Annals of Discrete Math. **34** (1987), 207-224.

8. G.B. Khosrovshahi and S. Ajoodani-Namini, *Combining t-designs,* J. Combin. Theory Ser. A **58** (1991), 26-34.

9. E.S. Kramer, S.S. Magliveras, and E.A. O'Brien, *Some new large sets of t-designs*, Australas. J. of Combin. **7** (1993), 189-193.

10. E.S. Kramer, S.S. Magliveras, and D.R. Stinson, *Some small large sets of t-desings*, Australas. J. of Combin. **3** (1991), 191-205.

11. D.L. Kreher and S.P. Radsznowski, *The existence of simple* 6–(14,7,4) *designs*, J. Combin. Theory Ser. A **43** (1986), 237-243.

12. J.X. Lu, *On large sets of disjoint Stiener triple systems I, II, III,* J. Combin. Theory Ser. A **34** (1983), 140-146, 147-155, 156-182.

13. J.X. Lu, *On large sets of disjoint Stiener triple systems IV, V, VI,* J. Combin. Theory Ser. A **37** (1984), 136-163, 164-188, 189-192.

14. L. Teirlinck, *A completion of Lu's determination of the spectrum for large sets of Steiner triple ssystems,* J.Combin. Theory Ser. A **57** (1991), 302-305.

15. L. Teirlinck, *On large sets of disjoint quadruple systems,* Ars Combin. **17** (1984), 173-176.

16. L. Teirlinck, *Large sets of disjoint designs and related structures,* in: Contemporary design theory, A collection of surveys (J. H. Dinitz and D. R. Stinson, eds.), John Wiley & Sons, Inc., New York, 1992, pp. 561-592.

17. Qiu-rong Wu, *A note on extending t-desings,* Australas. J. of Combin. **4** (1991), 229-235.