On the classification of skew Hadamard matrices of order 36 and their codes

Makoto Araya[1] Masaaki Harada[2] Hadi Kharaghani[3]
Ali Mohammadian[4] Behruz Tayfeh-Rezaie[4]

[1]Department of Computer Science, Shizuoka University,
Hamamatsu 432–8011, Japan
[2]Research Center for Pure and Applied Mathematics,
Graduate School of Information Sciences, Tohoku University,
Sendai 980–8579, Japan
[3]Department of Mathematics and Computer Science, University of Lethbridge,
Lethbridge, Alberta, T1K 3M4, Canada
[4]School of Mathematics, Institute for Research in Fundamental Sciences (IPM),
P.O. Box 19395-5746, Tehran, Iran

araya@inf.shizuoka.ac.jp mharada@tohoku.ac.jp kharaghani@uleth.ca
ali_m@ipm.ir tayfeh-r@ipm.ir

Abstract

Two skew Hadamard matrices are considered SH-equivalent if they are similar by a signed permutation matrix. This paper determines the number of SH-inequivalent skew Hadamard matrices of order 36 for some types. We also study ternary self-dual codes and association schemes constructed from the skew Hadamard matrices of order 36.

Keywords: Classification, Skew Hadamard Matrix, Ternary Self-Dual Code, Association Scheme

AMS Mathematics Subject Classification: 05B20, 15B34, 94B05.

1. Introduction

A *Hadamard matrix* of order n is an $n \times n$ matrix H with entries in $\{-1, 1\}$ such that $HH^{\mathsf{T}} = nI$, where H^{T} is the transpose of H and I is the identity matrix. It is well known that the order of a Hadamard matrix is 1, 2, or a multiple of 4 [22]. It is a longstanding folklore conjecture that Hadamard matrices of order n exist for any n divisible by 4. The smallest order for which no known Hadamard matrix is 668 [11].

A Hadamard matrix H is said to be *skew Hadamard* if $H + H^{\mathsf{T}} = 2I$. Skew Hadamard matrices are equivalent to doubly regular tournaments [3]. They are used to construct several combinatorial objects, such as association schemes, self-dual codes, strongly regular graphs, and more. It is conjectured that skew Hadamard matrices of order n exist for any n divisible by 4 [24]. The smallest unknown order of a skew Hadamard matrix is 356 [5].

A *permutation matrix* (respectively, *signed permutation matrix*) is a matrix with entries in $\{0, 1\}$ (respectively, $\{-1, 0, 1\}$) which has exactly one nonzero entry in each row and each column. Two matrices X and Y with entries in $\{-1, 1\}$ are said to be H-*equivalent* if there exist two signed permutation matrices P and Q such that $Y = PXQ$, otherwise, they are called H-*inequivalent*. Two matrices X and Y with entries in $\{-1, 1\}$ are said to be SH-*equivalent* if there exists a signed permutation matrix P such that $Y = P^{\mathsf{T}}XP$, otherwise, they are called SH-*inequivalent*.

All H-inequivalent Hadamard matrices of orders up to 32 have been classified [13]. Also, all H-inequivalent and SH-inequivalent skew Hadamard matrices of orders up to 32 have been classified [8]. The resulting classification is shown in Table 1. The complete classification of Hadamard matrices of order 36 seems to be difficult and perhaps inaccessible.

order	1	2	4	8	12	16	20	24	28	32
# H-inequivalent Hadamard matrices	1	1	1	1	1	5	3	60	487	13710027
# H-inequivalent skew Hadamard matrices	1	1	1	1	1	2	2	16	54	6662
# SH-inequivalent skew Hadamard matrices	1	1	1	1	1	2	2	16	65	7227

Table 1. The number of H-inequivalent Hadamard matrices, H-inequivalent skew Hadamard matrices, and SH-inequivalent skew Hadamard matrices of orders up to 32.

This paper determines the number of SH-inequivalent skew Hadamard matrices of order 36 for some types. We find 157132 SH-inequivalent skew Hadamard matrices of order 36. As an application, we study ternary self-dual codes and association schemes constructed from the skew Hadamard matrices of order 36.

2. Preliminaries

In this section, we fix our notation and present some preliminary results. Throughout the paper, we denote all one vector of length r by $\mathbb{1}_r$. Let $H = [h_{uv}]$ be a Hadamard matrix of order n. We know from [4] that, by a sequence of column negations and column permutations, every four distinct rows i, j, k, ℓ of H may be transformed to the form

$$\begin{array}{cccccccc}
\mathbb{1}_s & \mathbb{1}_t & \mathbb{1}_t & \mathbb{1}_s & \mathbb{1}_t & \mathbb{1}_s & \mathbb{1}_s & \mathbb{1}_t \\
\mathbb{1}_s & \mathbb{1}_t & \mathbb{1}_t & \mathbb{1}_s & -\mathbb{1}_t & -\mathbb{1}_s & -\mathbb{1}_s & -\mathbb{1}_t \\
\mathbb{1}_s & \mathbb{1}_t & -\mathbb{1}_t & -\mathbb{1}_s & \mathbb{1}_t & \mathbb{1}_s & -\mathbb{1}_s & -\mathbb{1}_t \\
\mathbb{1}_s & -\mathbb{1}_t & \mathbb{1}_t & -\mathbb{1}_s & \mathbb{1}_t & -\mathbb{1}_s & \mathbb{1}_s & -\mathbb{1}_t
\end{array} \tag{1}$$

for some uniquely determined s, t with $s + t = \frac{n}{4}$. By negation of the last row in (1) and then a suitable column permutation if necessary, we may assume that $s \geqslant t$ and so $0 \leqslant t \leqslant \lfloor \frac{n}{8} \rfloor$. Following [15], we define the *type* of the four rows i, j, k, ℓ as $T_{ijk\ell} = t$. It is straightforward to check that $T_{ijk\ell} = \frac{n-|P_{ijk\ell}|}{8}$, where

$$P_{ijk\ell} = \sum_{r=1}^{n} h_{ir}h_{jr}h_{kr}h_{\ell r}.$$

This shows that 'type' is an equivalence invariant, meaning that any negation of rows and columns and any permutation of columns leaves the type unchanged. As H^{T} is a Hadamard matrix, we may define 'type' for any quadruple of columns of H in a similar way. A Hadamard matrix is of *type* t if it has a quadruple of rows of type t and no quadruple of rows of type less than t. We refer to [20] for more information about types of Hadamard matrices.

Below, we find a nice form for a quadruple of rows of a skew Hadamard matrix. It is not hard to verify that any 4×4 skew-symmetric matrix with entries in $\{-1, 1\}$ and with constant diagonal is SH-equivalent to one of the matrices

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{bmatrix} \quad \text{or} \quad B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \end{bmatrix}. \tag{2}$$

Therefore, by a sequence of row negations and row permutations, and simultaneously, column negations and column permutations, every quadruple of rows of a skew Hadamard matrix may be transformed to one of the forms

$$\begin{array}{cccccccccccc}
1 & 1 & 1 & 1 & \mathbb{1}_{s-1} & \mathbb{1}_{t-1} & \mathbb{1}_t & \mathbb{1}_{s-1} & \mathbb{1}_t & \mathbb{1}_s & \mathbb{1}_s & \mathbb{1}_{t-1} \\
-1 & 1 & 1 & 1 & \mathbb{1}_{s-1} & \mathbb{1}_{t-1} & \mathbb{1}_t & \mathbb{1}_{s-1} & -\mathbb{1}_t & -\mathbb{1}_s & -\mathbb{1}_s & -\mathbb{1}_{t-1} \\
-1 & -1 & 1 & 1 & \mathbb{1}_{s-1} & \mathbb{1}_{t-1} & -\mathbb{1}_t & -\mathbb{1}_{s-1} & \mathbb{1}_t & \mathbb{1}_s & -\mathbb{1}_s & -\mathbb{1}_{t-1} \\
-1 & -1 & -1 & 1 & \mathbb{1}_{s-1} & -\mathbb{1}_{t-1} & \mathbb{1}_t & -\mathbb{1}_{s-1} & \mathbb{1}_t & -\mathbb{1}_s & \mathbb{1}_s & -\mathbb{1}_{t-1}
\end{array} \tag{3}$$

or

$$\begin{array}{cccccccccccc}
1 & 1 & 1 & 1 & \mathbb{1}_s & \mathbb{1}_{t-1} & \mathbb{1}_{t-1} & \mathbb{1}_s & \mathbb{1}_{t-1} & \mathbb{1}_s & \mathbb{1}_s & \mathbb{1}_{t-1} \\
-1 & 1 & 1 & -1 & \mathbb{1}_s & \mathbb{1}_{t-1} & \mathbb{1}_{t-1} & \mathbb{1}_s & -\mathbb{1}_{t-1} & -\mathbb{1}_s & -\mathbb{1}_s & -\mathbb{1}_{t-1} \\
-1 & -1 & 1 & 1 & \mathbb{1}_s & \mathbb{1}_{t-1} & -\mathbb{1}_{t-1} & -\mathbb{1}_s & \mathbb{1}_{t-1} & \mathbb{1}_s & -\mathbb{1}_s & -\mathbb{1}_{t-1} \\
-1 & 1 & -1 & 1 & \mathbb{1}_s & -\mathbb{1}_{t-1} & \mathbb{1}_{t-1} & -\mathbb{1}_s & \mathbb{1}_{t-1} & -\mathbb{1}_s & \mathbb{1}_s & -\mathbb{1}_{t-1}
\end{array} \tag{4}$$

for some s, t with $s + t = \frac{n}{4}$. The quadruples of rows given in (3) and (4) are of type $\min\{s, t\}$. Letting

$$U = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and $V = \operatorname{diag}(-I_{s-1}, I_{t-1}, -I_t, I_{s-1}, -I_t, I_s, -I_s, I_{t-1})$ and multiplying the quadruple of rows given in (3) by U from the left and by $U^{\mathsf{T}} \oplus V$ from the right, we may assume that $s \geqslant t$. This means the quadruple of rows given in (3) is of type t. We define the *skew type* of the quadruple of rows given in (3) as $(t, 1)$. Also, we define the *skew type* of the quadruple of rows given in (4) as $(s, 2)$ if $s \leqslant t$ and as $(t, 0)$ if $s > t$. Note that skew types $(0, 0)$ and $(0, 1)$ are not admissible. A skew Hadamard matrix is of *skew type* (t, ε) if it is a Hadamard matrix of type t containing a quadruple of rows of skew type (t, ε) and no quadruples of rows of skew type (t, ε') with $\varepsilon' < \varepsilon$.

For a quadruple $\{i, j, k, \ell\}$ of rows of a skew Hadamard matrix $H = [h_{ij}]$, define

$$Q_{ijk\ell} = \sum_{r \in \{i,j,k,\ell\}} h_{ir} h_{jr} h_{kr} h_{\ell r}.$$

Note that $Q_{ijk\ell} = 0$ if and only if the 4×4 principal submatrix corresponding to the quadruple $\{i, j, k, \ell\}$ is SH-equivalent to A, given in (2). Moreover, the parameter $P_{ijk\ell} Q_{ijk\ell}$ is invariant under negation and permutation of rows, and simultaneously, negation and permutation of columns. The following proposition describes the skew type of a quadruple of rows of a skew Hadamard matrix. It is easily proved using the forms (3) and (4) and noting that $P_{ijk\ell} Q_{ijk\ell}$ is an invariant parameter.

Proposition 2.1. *Let H be a skew Hadamard matrix of order n. Then, the following statements hold for every quadruple $\{i, j, k, \ell\}$ of rows of H of type t.*

(i) *The skew type of $\{i, j, k, \ell\}$ is $(t, 0)$ if and only of $P_{ijk\ell} Q_{ijk\ell} < 0$.*

(ii) *The skew type of $\{i, j, k, \ell\}$ is $(t, 1)$ if and only of $Q_{ijk\ell} = 0$.*

(iii) *The skew type of $\{i, j, k, \ell\}$ is $(t, 2)$ if and only of $Q_{ijk\ell} \neq 0$ and $P_{ijk\ell} Q_{ijk\ell} \geqslant 0$.*

The following lemma shows the relation between the skew type of a quadruple of rows in a skew Hadamard matrix H and the skew type of their corresponding rows in H^{T}.

Lemma 2.2. *Let H be a skew Hadamard matrix of order n. Then, the following statements hold.*

(i) *If the quadruple $\{i, j, k, \ell\}$ of rows of H is of skew type $(t, 0)$, then the quadruple $\{i, j, k, \ell\}$ of rows of H^{T} is of skew type $(t - 1, 2)$.*

(ii) *If the quadruple $\{i, j, k, \ell\}$ of rows of H is of skew type $(t, 1)$, then the quadruple $\{i, j, k, \ell\}$ of rows of H^{T} is also of skew type $(t, 1)$.*

(iii) *If the quadruple $\{i,j,k,\ell\}$ of rows of H is of skew type $(s,2)$, then the quadruple $\{i,j,k,\ell\}$ of rows of $H^\intercal$ is of skew type*

$$\begin{cases} (s-1,2) & \textit{if } s=\frac{n}{8}, \\ (s,2) & \textit{if } s=\frac{n-4}{8}, \\ (s+1,2) & \textit{if } s=\frac{n}{8}-1, \\ (s+1,0) & \textit{otherwise.} \end{cases}$$

Proof. To prove (i), let $\{i,j,k,\ell\}$ be a quadruple of rows of H of skew type $(t,0)$ and let $s=\frac{n}{4}-t$. By (4), we have $T_{ijk\ell}=\frac{n-|P_{ijk\ell}|}{8}=\frac{4(s+t)-4|s-t+2|}{8}=t-1$ in $H^\intercal$, by noting that $s>t$. Since $P_{ijk\ell}Q_{ijk\ell}>0$ in $H^\intercal$, it follows from Proposition 2.1(iii) that the quadruple $\{i,j,k,\ell\}$ of rows of $H^\intercal$ is of skew type $(t-1,2)$.

For (ii), let $\{i,j,k,\ell\}$ be a quadruple of rows of H of skew type $(t,1)$ and let $s=\frac{n}{4}-t$. By (3), we have $T_{ijk\ell}=\frac{n-|P_{ijk\ell}|}{8}=\frac{4(s+t)-4|s-t|}{8}=t$ in $H^\intercal$, by noting that $s\geqslant t$. Since $Q_{ijk\ell}=0$ in $H^\intercal$, it follows from Proposition 2.1(ii) that the quadruple $\{i,j,k,\ell\}$ of rows of $H^\intercal$ is of skew type $(t,1)$.

Finally, to establish (iii), let $\{i,j,k,\ell\}$ be a quadruple of rows of H of skew type $(s,2)$ and let $t=\frac{n}{4}-s$. As $s\leqslant t$, it follows from (4) that

$$T_{ijk\ell}=\frac{n-|P_{ijk\ell}|}{8}=\frac{4(s+t)-4|s-t+2|}{8}=\begin{cases} s-1 & \text{if } s=\frac{n}{8}, \\ s & \text{if } s=\frac{n-4}{8}, \\ s+1 & \text{otherwise,} \end{cases}$$

in $H^\intercal$. If $s\in\{\frac{n}{8},\frac{n-4}{8},\frac{n}{8}-1\}$, then $P_{ijk\ell}Q_{ijk\ell}\geqslant 0$ in $H^\intercal$, and otherwise, $P_{ijk\ell}Q_{ijk\ell}<0$ in $H^\intercal$. As $Q_{ijk\ell}\neq 0$ in $H^\intercal$, the result follows from Proposition 2.1. □

We recall the following two known lemmas on the types of a Hadamard matrix and its transpose.

Lemma 2.3 ([12, 15, 20])**.** *There is no Hadamard matrix of order n and type $\lfloor\frac{n}{8}\rfloor$ if $n\notin\{4,12\}$. There is no Hadamard matrix of order n and type 0 if $n\equiv 4 \pmod 8$.*

Lemma 2.4 ([12])**.** *Let H be a Hadamard matrix of order n having a quadruple of rows of type 1. Then, the type of $H^\intercal$ is 0 if $n\equiv 0 \pmod 8$ and 1 if $n\equiv 4 \pmod 8$.*

Theorem 2.5. *The following statements hold.*

(i) *There is no skew Hadamard matrix of skew type $(2,0)$.*

(ii) *There is no skew Hadamard matrix of order n and skew type $(1,0)$ if $n\equiv 4 \pmod 8$.*

(iii) *There is no skew Hadamard matrix of order n and skew type $(1,1)$ or $(1,2)$ if $n\equiv 0 \pmod 8$.*

Proof. Let H be a skew Hadamard matrix of order n. Suppose the skew type of H is $(2,0)$. Then, Lemma 2.2(i) implies that $H^\intercal$ has a quadruple of rows of skew type $(1,2)$ and so $H^\intercal$ is of type 0 or 1. If $H^\intercal$ is of type 1, then, by applying Lemma 2.4 for $H^\intercal$, we deduce that H is of type 0 or 1 which is a contradiction, since the skew type of H is $(2,0)$. Therefore, $H^\intercal$ is of type 0 and hence $n \equiv 0 \pmod 8$ by Lemma 2.3. Since the skew type of $H^\intercal$ is $(0,2)$, Lemma 2.2(iii) implies that H has a quadruple of rows of type 1, which is again a contradiction, as the skew type of H is $(2,0)$.

Now, let $n \equiv 4 \pmod 8$. If H is of skew type $(1,0)$, then Lemma 2.2(i) implies that $H^\intercal$ has a quadruple of rows of skew type $(0,2)$ and so $H^\intercal$ is of type 0, contradicting Lemma 2.3.

Finally, suppose that $n \equiv 0 \pmod 8$ and H is of skew type $(1,1)$ or $(1,2)$. It follows from Lemma 2.3 that $n \neq 8$. Also, Lemma 2.4 yields that the type of $H^\intercal$ is 0, so $H^\intercal$ has a quadruple of skew type $(0,2)$ rows. So, it follows from Lemma 2.2(iii) that H has a quadruple of rows of skew type $(1,0)$. As H is of type 1, we conclude that the skew type of H is $(1,0)$, a contradiction. $\square$

To proceed, we must recall the following result on the type of quadruples of rows of a Hadamard matrix.

Lemma 2.6 ([20])**.** *Let H be a Hadamard matrix of order $4m$. Fix three rows of H and let κ_t be the number of other rows of type t with these three rows. Then,*

$$\sum_{t=0}^{\lfloor \frac{m}{2} \rfloor} \kappa_t (m-2t)^2 = m^2.$$

Lemma 2.7. *Let H be a skew Hadamard matrix of order n. Then, the number of all 4×4 principal submatrices of H which are* SH*-equivalent to A is $n(n-1)(n-2)(n-4)/32$, where A is introduced in* (2)*.*

Proof. For simplicity, let $n = 4m$. It is easy to see that, by a sequence of row negations and row permutations, and simultaneously, column negations and column permutations, every triple of rows of H may be transformed to the form

$$\begin{array}{ccccccc} 1 & 1 & 1 & \mathbb{1}_{m-1} & \mathbb{1}_{m-1} & \mathbb{1}_{m-1} & \mathbb{1}_{m} \\ -1 & 1 & 1 & \mathbb{1}_{m-1} & \mathbb{1}_{m-1} & -\mathbb{1}_{m-1} & -\mathbb{1}_{m} \\ -1 & -1 & 1 & \mathbb{1}_{m-1} & -\mathbb{1}_{m-1} & -\mathbb{1}_{m-1} & \mathbb{1}_{m} \end{array}.$$

One may straightforwardly check that $Q_{123\ell} = 0$ for the quadruple of rows $\{1,2,3,\ell\}$ if and only if $\ell \in \{4,\ldots,3m\}$. This means that the quadruple of rows $\{1,2,3,\ell\}$ forms a 4×4 principal submatrix of H which is SH-equivalent to A if and only if $\ell \in \{4,\ldots,3m\}$. Therefore, every triple of rows of H is contained in exactly $3m-3$ quadruples of rows whose corresponding 4×4 principal submatrix is SH-equivalent to A. By double counting, we derive that the number of all quadruples of rows of H whose corresponding 4×4 principal submatrices are SH-equivalent to A is equal to $(3m-3)\binom{n}{3}/4 = n(n-1)(n-2)(n-4)/32$, as desired. $\square$

Theorem 2.8. *There is no skew Hadamard matrix of order* 36 *and skew type* $(3, 2)$.

Proof. Suppose by way of contradiction that H is a skew Hadamard matrix of order 36 and skew type $(3, 2)$. So, every quadruple of rows of H has type 3 or 4. Using the notation of Lemma 2.6, we have $9\kappa_3^2 + \kappa_4 = 81$. Since $\kappa_3 + \kappa_4 = 33$, we obtain that $\kappa_3 = 6$ and $\kappa_4 = 27$. By double counting, we derive that the number of all quadruples of rows of H of type 3 is $6\binom{36}{3}/4 = 10710$ and therefore, the number of all quadruples of rows of H of type 4 is $\binom{36}{4} - 10710 = 48195$.

The possible skew types for any quadruple of rows of H are $(3, 2)$, $(4, 0)$, $(4, 1)$, $(4, 2)$. Using Lemma 2.2, the possible skew types for any quadruple of rows of $H^\intercal$ are $(3, 2)$, $(4, 0)$, $(4, 1)$, $(4, 2)$ and therefore, $H^\intercal$ is of skew type $(3, 2)$. Since the number of all quadruples of rows of H of type 3 is 10710, the number of all quadruples of rows of H of skew type $(3, 2)$ is 10710. Also, Lemma 2.2 implies that the number of all quadruples of rows of H of skew type $(4, 0)$ is equal to the number of all quadruples of rows of $H^\intercal$ of skew type $(3, 2)$. Therefore, the number of all quadruples of rows of H of skew type $(4, 0)$ is 10710.

The number of all quadruples of rows of H of skew type $(4, 1)$ is equal to the number of all quadruples of rows of H whose corresponding 4×4 principal submatrices are SH-equivalent to A which equals 42840, by Lemma 2.7. Now, the number of all quadruples of rows of H of skew type $(4, 2)$ is $48195 - (10710 + 42840) < 0$, a contradiction. □

We end this section with the following lemma, which is crucial in our computational search.

Lemma 2.9. *Let H be a Hadamard matrix of order n whose first four rows are in the form* (1)*. Let $(h_1, \ldots, h_n)$ be a row of H other than the first four rows and define*

$$x_1 = \sum_{i=1}^{s} h_i, \qquad x_2 = \sum_{i=s+2t+1}^{2s+2t} h_i, \qquad x_3 = \sum_{i=2s+3t+1}^{3s+3t} h_i, \qquad x_4 = \sum_{i=3s+3t+1}^{4s+3t} h_i$$

and

$$y_1 = \sum_{i=s+1}^{s+t} h_i, \qquad y_2 = \sum_{i=s+t+1}^{s+2t} h_i, \qquad y_3 = \sum_{i=2s+2t+1}^{2s+3t} h_i, \qquad y_4 = \sum_{i=4s+3t+1}^{n} h_i.$$

Then, $x_1 + x_2 + x_3 - x_4 = 2s - \frac{n}{2} \pmod 4$ *and* $y_1 + y_2 + y_3 - y_4 = 2t - \frac{n}{2} \pmod 4$.

Proof. Since $(h_1, \ldots, h_n)$ is orthogonal to the four rows given in (1), we obtain that

$$x_1 + y_1 + y_2 + x_2 + y_3 + x_3 + x_4 + y_4 = 0, \tag{5}$$

$$x_1 + y_1 + y_2 + x_2 - y_3 - x_3 - x_4 - y_4 = 0, \tag{6}$$

$$x_1 + y_1 - y_2 - x_2 + y_3 + x_3 - x_4 - y_4 = 0, \tag{7}$$

$$x_1 - y_1 + y_2 - x_2 + y_3 - x_3 + x_4 - y_4 = 0. \tag{8}$$

By subtracting (8) from the equality obtained by taking the sum of (5)–(7), we conclude that $x_1 + x_2 + x_3 - x_4 = -2y_1$. Since $2y_1 = 2t \pmod 4$ and $s + t = \frac{n}{4}$, one deduces that $x_1 + x_2 + x_3 - x_4 = 2s - \frac{n}{2} \pmod 4$. Also, by summing (5)–(8), we get $y_1 + y_2 + y_3 - y_4 = -2x_1$. Since $2x_1 = 2s \pmod 4$ and $s + t = \frac{n}{4}$, we get $y_1 + y_2 + y_3 - y_4 = 2t - \frac{n}{2} \pmod 4$, the result follows. □

3. Search using skew types

In this section, we describe our search method for skew Hadamard matrices, which uses the notion of skew types. Let H be a skew Hadamard matrix of order n and skew type (t, ε). By Lemma 2.2, one can easily see that if $\varepsilon = 0$, then H^{T} is of skew type $(t-1, 2)$. Therefore, the classification of skew Hadamard matrices of skew type $(t, 0)$ can be extracted from the classification of skew Hadamard matrices of skew type $(t-1, 2)$. Hence, we may assume that $\varepsilon = 1, 2$. Using the forms (3), (4) and by a permutation on columns, we may assume that the first four rows of H are of the form

$$\begin{array}{cccccccccccc} 1 & 1 & 1 & 1 & \mathbb{1}_{t-1} & \mathbb{1}_t & \mathbb{1}_t & \mathbb{1}_{t-1} & \mathbb{1}_{s-1} & \mathbb{1}_{s-1} & \mathbb{1}_s & \mathbb{1}_s \\ -1 & 1 & 1 & 1 & \mathbb{1}_{t-1} & \mathbb{1}_t & -\mathbb{1}_t & -\mathbb{1}_{t-1} & \mathbb{1}_{s-1} & \mathbb{1}_{s-1} & -\mathbb{1}_s & -\mathbb{1}_s \\ -1 & -1 & 1 & 1 & \mathbb{1}_{t-1} & -\mathbb{1}_t & \mathbb{1}_t & -\mathbb{1}_{t-1} & \mathbb{1}_{s-1} & -\mathbb{1}_{s-1} & \mathbb{1}_s & -\mathbb{1}_s \\ -1 & -1 & -1 & 1 & -\mathbb{1}_{t-1} & \mathbb{1}_t & \mathbb{1}_t & -\mathbb{1}_{t-1} & \mathbb{1}_{s-1} & -\mathbb{1}_{s-1} & -\mathbb{1}_s & \mathbb{1}_s \end{array} \tag{9}$$

or

$$\begin{array}{cccccccccccc} 1 & 1 & 1 & 1 & \mathbb{1}_t & \mathbb{1}_t & \mathbb{1}_t & \mathbb{1}_t & \mathbb{1}_{s-1} & \mathbb{1}_{s-1} & \mathbb{1}_{s-1} & \mathbb{1}_{s-1} \\ -1 & 1 & 1 & -1 & \mathbb{1}_t & \mathbb{1}_t & -\mathbb{1}_t & -\mathbb{1}_t & \mathbb{1}_{s-1} & \mathbb{1}_{s-1} & -\mathbb{1}_{s-1} & -\mathbb{1}_{s-1} \\ -1 & -1 & 1 & 1 & \mathbb{1}_t & -\mathbb{1}_t & \mathbb{1}_t & -\mathbb{1}_t & \mathbb{1}_{s-1} & -\mathbb{1}_{s-1} & \mathbb{1}_{s-1} & -\mathbb{1}_{s-1} \\ -1 & 1 & -1 & 1 & \mathbb{1}_t & -\mathbb{1}_t & -\mathbb{1}_t & \mathbb{1}_t & -\mathbb{1}_{s-1} & \mathbb{1}_{s-1} & \mathbb{1}_{s-1} & -\mathbb{1}_{s-1} \end{array} \tag{10}$$

where $s = \frac{n}{4} - t \geqslant t$.

By applying Lemma 2.9, if the first four rows of H are of the form (9), then the column $4t+2$ of H is uniquely determined from the columns $1, 3, 5, \ldots, 4t+1$ of H and hence the row $4t+2$ of H is uniquely determined from the rows $1, 3, 5, \ldots, 4t+1$ of H. Also, using Lemma 2.9, if the first four rows of H are of the form (10), then the column $4t+4$ of H is uniquely determined from the columns $5, \ldots, 4t+3$ of H and thus the row $4t+4$ of H is uniquely determined from the rows $5, \ldots, 4t+3$ of H.

To classify skew Hadamard matrices of skew type (t, ε) for $\varepsilon = 1, 2$, we first fix the first four rows according to one of the forms (9) and (10). We then use a backtrack algorithm to construct the next rows one by one. By the above paragraph, when we reach the row $4t+2$ or $4t+4$, depending on the form of the first four rows, we have pruning criteria for the backtracking. We compute this row from the preceding rows. If there is no solution, we backtrack. Otherwise, we proceed to the next step.

During the backtracking algorithm, we perform some checks. Regarding the number of solutions permitted in the primary steps of backtracking, we do the SH-equivalence test on the obtained partial skew Hadamard matrices. Also, we check for the skew type of partial skew Hadamard matrices whenever it is computationally not time-consuming.

After we obtain all solutions, we need to check them for the SH-equivalence. It could take a long time because of the large number of solutions. To do this, we partition the set of solutions according to their skew profiles, which count the number of quadruples of rows of different skew types. Then, in each class, which is usually much smaller, we do the SH-equivalence test.

In order 36, in view of Lemma 2.3, there is no skew Hadamard matrix of skew type $(0, \varepsilon)$ and $(4, \varepsilon)$ for any ϵ. By Lemma 2.5, there is no such matrix of skew type $(1, 0)$ and $(2, 0)$.

We also have no matrix of type $(3, 2)$, using Lemma 2.8. So, we need to find solutions for skew types $(1, 1)$, $(1, 2)$, $(2, 1)$, $(2, 2)$, $(3, 0)$ and $(3, 1)$. We have run the program for skew types $(1, 1)$, $(1, 2)$ and $(2, 1)$. The two skew types $(1, 1)$ and $(1, 2)$ are fast enough to be run on a single PC for a few days. The computation time for the skew type $(2, 1)$ was about twenty days on a cluster of 48 cores. We also implemented some parts of the algorithm twice with different codes. The remaining cases $(2, 2)$, $(3, 0)$ and $(3, 1)$ need much more computational resources. We also run the program for all orders smaller than 36. The results, which are given in Table 2, confirm the numbers shown in Table 1. The obtained list of skew Hadamard matrices of order 36 is available electronically at [9]. We summarize our main result in the following theorem.

Theorem 3.1. *There are at least* 157132 SH*-inequivalent skew Hadamard matrices of order* 36.

Order \ Skew Type	(0, 2)	(1, 0)	(1, 1)	(1, 2)	(2, 1)	(2, 2)	(3, 0)	(3, 1)
4	1	0	0	0	0	0	0	0
8	1	0	0	0	0	0	0	0
12	0	0	1	0	0	0	0	0
16	2	0	0	0	0	0	0	0
20	0	0	1	1	0	0	0	0
24	14	1	0	0	1	0	0	0
28	0	0	43	21	0	1	0	0
32	6903	283	0	0	40	0	0	1
36	0	0	23260	123326	10546	?	?	?

Table 2. The number of SH-inequivalent skew Hadamard matrices of orders up to 36 for all possible skew types.

4. Search through the canonical method

This section uses the orderly generation method to classify skew Hadamard matrices of order 36. The technique was independently introduced in [7] and [23]. The method consists of two parallel subroutines. These are the construction of objects and the rejection of equivalence copies. The most natural and widely used method for the construction phase is backtracking, which has quite an old history. We use the notion of 'canonical form' to reject equivalence copies. A canonical form is a special representative for each equivalence class. The canonical forms are constructed step by step using backtracking, and the canonicity test is carried out at each step.

Here, we present the canonical form for skew Hadamard matrices. For a skew Hadamard matrix $H = [h_{ij}]$ of order n, assign a vector

$$v(H) = \big(h_{1,2}, h_{1,3}, h_{2,3}, h_{1,4}, h_{2,4}, h_{3,4}, \ldots, h_{n-1,n}\big).$$

We say that H is in the *canonical form* if $v(P^{\mathsf{T}}HP) \preccurlyeq v(H)$ for any signed permutation matrix P, where $\preccurlyeq$ denotes the lexicographical order.

We used an orderly algorithm with backtracking and the above canonical form to generate SH-equivalence classes of skew Hadamard matrices of order 36. Using this orderly algorithm, we first find all solutions for the 14×14 leading principal submatrices in canonical form. In total, there are 80122802 solutions for 14×14 matrices. For each solution, we find all 14×36 partial Hadamard matrices constituting the first 14 rows of desired skew Hadamard matrix of order 36 using an exhaustive search. Then, for each obtained 14×36 partial Hadamard matrix and $i = 15, \ldots, 36$, we complete every row i by solving a system of linear equations coming from the orthogonality row i to the first 14 rows. To speed up the computation, we solve the system of linear equations in modulo 2. Finally, for the complete solutions, we do the canonicity test. We have not been able to run the computation for all 80122802 solutions. It is done for about 10 million of them. The approximate computation time was three months on a cluster of 160 cores. We found 157132 skew Hadamard matrices of order 36 up to the SH-equivalence. The list matches the list obtained in the previous section.

Denote by J_k the skew-symmetric matrix of order k whose all entries on and above the main diagonal are 1. For example, the matrix A given in (2) equals J_4. The number of solutions for the 8×8 leading principal submatrices of skew Hadamard matrices of order 36 in the canonical form is 9. Our computation shows that only one of these 9 solutions, J_8, is extendable to skew Hadamard matrices. So, every skew Hadamard matrix of order 36 in canonical form contains J_8. Note that J_8 has 11 feasible extensions to 9×9 submatrices of skew Hadamard matrices of order 36. Let us show these solutions by $S_0 = J_9$, $S_1, \ldots, S_{10}$ from the largest to smallest in the canonical form. Our results show that $S_6, \ldots, S_{10}$ have no extensions. For S_0 and S_1, we found the complete list of extensions. However, the search is not complete for S_2. Also, we have not done anything for S_3, S_4 and S_5.

Inspired by the previous paragraph, we may ask for the maximum number $k(n)$ such that $J_{k(n)}$ appears as a principal submatrix of a skew Hadamard matrix of order n. This is related to a problem posed in [6]: What is the maximum number $t(n)$ such that every tournament of order n contains the transitive tournament of order $t(n)$ as a subtournament? The following theorem presents an upper bound on $k(n)$, which is especially tight for $n = 36$.

Theorem 4.1. *For any n,*

$$k(n) \leqslant \left\lfloor \frac{\pi}{2\cot^{-1}\left(\sqrt{n-1}\right)} \right\rfloor .$$

Proof. Let H be a skew Hadamard matrix of order n. It is easy to see that $A = \mathbb{i}(I - H)$ is a Hermitian matrix with $\frac{n}{2}$ eigenvalues $\sqrt{n-1}$ and $\frac{n}{2}$ eigenvalues $-\sqrt{n-1}$, where $\mathbb{i}$ is the unit imaginary complex number. Let $k = k(n)$ and $B = \mathbb{i}(I - J_k)$. As J_k is a principal submatrix of H, B is a principal submatrix of A. Since B is a negacyclic matrix, it straightforwardly follows from a result of [16] that the eigenvalues of B are $\cot(\frac{\pi\ell}{2k})$ for $\ell = 1, 3, \ldots, 2k-1$. By the interlacing theorem, the largest eigenvalue of B is less than or equal to the largest eigenvalue of A, that is, $\cot(\frac{\pi}{2k}) \leqslant \sqrt{n-1}$. This yields the desired inequality. $\square$

5. Their ternary codes

Self-dual codes are one of the most interesting classes of codes. This interest is justified by many combinatorial objects and algebraic objects related to self-dual codes (see, e.g., [21]). In this section, we classify ternary self-dual codes constructed from the 157132 skew Hadamard matrices of order 36 given in Table 2. We also study unimodular lattices and 1-designs built from the ternary near-extremal self-dual codes.

5.1. Ternary self-dual codes

Let $\mathbb{F}_3 = \{0,1,2\}$ denote the finite field of order 3. A *ternary* $[n,k]$ *code* C is a k-dimensional vector subspace of $\mathbb{F}_3^n$. The parameters n and k are called the *length* and *dimension* of C, respectively. A ternary *self-dual* code C of length n is a ternary $[n,n/2]$ code satisfying $C = C^{\perp}$, where $C^{\perp}$ is the dual code of C under the standard inner product. It was shown in [17] that the minimum weight d of a ternary self-dual code of length n is bounded by $d \leqslant 3\lfloor n/12 \rfloor + 3$. If $d = 3\lfloor n/12 \rfloor + 3$ and $d = 3\lfloor n/12 \rfloor$, then the code is called *extremal* and *near-extremal*, respectively. For length 36, the Pless symmetry code is a currently known extremal ternary self-dual code.

Two ternary codes C and C' are said to be *equivalent* if there exists a monomial matrix P over $\mathbb{F}_3$ with $C' = \{xP \mid x \in C\}$, otherwise, they are called *inequivalent*. Here, a monomial matrix is a matrix with entries in $\mathbb{F}_3$, with exactly one nonzero entry in each row and each column. Let H be a Hadamard matrix of order n. Throughout this section, let $C(H)$ denote the ternary code generated by the rows of H, where the entries of the matrix are regarded as elements of $\mathbb{F}_3$. It is trivial that $C(H)$ and $C(K)$ are equivalent if H and K are SH-equivalent skew Hadamard matrices.

Although the proof of the following lemma is somewhat trivial, we give it for the sake of completeness.

Lemma 5.1. *Let H be a skew Hadamard matrix of order n. If $n \equiv 0 \pmod{12}$, then $C(H)$ is self-dual.*

Proof. Since $n \equiv 0 \pmod{12}$, $HH^{\mathsf{T}} \equiv O \pmod 3$, where O denotes the $n \times n$ zero matrix. This implies $C(H) \subset C(H)^{\perp}$. On the other hand, since $H + H^{\mathsf{T}} = 2I$,

$$n = \mathrm{rank}_3(2I) = \mathrm{rank}_3(H + H^{\mathsf{T}}) \leqslant 2\,\mathrm{rank}_3(H),$$

where $\mathrm{rank}_3(A)$ denotes the 3-rank of a matrix A. Thus, we have $\frac{n}{2} \leqslant \mathrm{rank}_3(H)$. This means that the dimension of C is at least $\frac{n}{2}$. Therefore, $C(H)$ is self-dual. $\square$

We describe how to classify ternary self-dual codes of length 36 constructed from the 157132 skew Hadamard matrices of order 36 given in Table 2. By comparing pairs (A_6, A_9), the 157132 ternary self-dual codes are divided into 829 classes, where A_w denotes the number of codewords of weight w. For each class, to test the equivalence of ternary self-dual codes, we used the method given in [10, Section 7.3.3] as follows. For a ternary self-dual code C, define the vertex-colored graph $\Gamma(C)$ with vertex set $C_9 \cup (\{1, 2, \dots, 36\} \times (\mathbb{F}_3 - \{0\}))$ and edge set $\{\{c, (j, c_j)\} \mid c = (c_1, c_2, \dots, c_{36}) \in C_9, 1 \leqslant j \leqslant 36, c_j \neq 0\} \cup \{\{(j, y), (j, 2y)\} \mid 1 \leqslant j \leqslant 36,\ y \in \mathbb{F}_3 - \{0\}\}$, where C_9 denotes the set of codewords of weight 9 in C. Suppose that two ternary self-dual codes C and C' are generated by codewords of weight 9. Then, C and C' are equivalent if and only if $\Gamma(C)$ and $\Gamma(C')$ are isomorphic. We verified that the 157129 ternary self-dual codes of length 36 are generated by codewords of weight 9, and the remaining three codes are not generated by codewords of weight 9. For the three codes, we have that $(A_6, A_9) = (36, 464)$, $(96, 224)$ and $(96, 1520)$. Thus, each of the three codes is inequivalent to the other codes. We used NAUTY [18] for vertex-coloured graph isomorphism testing corresponding to the 157129 ternary self-dual codes. Then, we have the following classification of ternary self-dual codes of length 36 constructed from the 157132 skew Hadamard matrices of order 36.

Proposition 5.2. 153979 *inequivalent ternary self-dual codes of length* 36 *are constructed from the* 157132 *skew Hadamard matrices of order* 36 *given in Table 2.* 20848 *of them have minimum weight* 9 *and the others have minimum weight* 6.

The inequivalence of the above 153979 ternary self-dual codes was verified independently by MAGMA [2]. The obtained list of the ternary self-dual codes of length 36 is available electronically at [9].

5.2. Ternary near-extremal self-dual codes

Here, we concentrate on the 20848 ternary near-extremal self-dual codes given in Proposition 5.2. Recently, a restriction on the weight enumerators of ternary near-extremal self-dual codes of lengths divisible by 12 has been given in [1]. The possible weight enumerators of ternary near-extremal self-dual codes of length 36 are as follows:

$$\begin{aligned} W_\alpha = & 1 + \alpha y^9 + (42840 - 9\alpha) y^{12} + (1400256 + 36\alpha) y^{15} \\ & + (18452280 - 84\alpha) y^{18} + (90370368 + 126\alpha) y^{21} \\ & + (162663480 - 126\alpha) y^{24} + (97808480 + 84\alpha) y^{27} \\ & + (16210656 - 36\alpha) y^{30} + (471240 + 9\alpha) y^{33} + (888 - \alpha) y^{36}, \end{aligned}$$

where α is an integer with $\alpha \equiv 0 \pmod 8$ and $0 < \alpha \leqslant 888$. For the ternary near-extremal self-dual codes given in Proposition 5.2, α in their weight enumerators W_α are listed in Table 3. In addition, the numbers N_α of the ternary near-extremal self-dual codes having weight enumerator W_α are listed in Table 3.

A (Euclidean) lattice $L \subset \mathbb{R}^n$ in dimension n is *unimodular* if $L = L^*$, where L^* is the dual lattice of L under the standard inner product. The *minimum norm* of a unimodular lattice L is the smallest the norm among all nonzero vectors of L. The *kissing number* of L is the number of vectors of minimum norm in L. Two lattices L and L' are said to be *isomorphic* if there exists an orthogonal matrix A with $L' = \{xA \mid x \in L\}$, otherwise, they

(α, N_α)	(α, N_α)	(α, N_α)	(α, N_α)	(α, N_α)	(α, N_α)
$(72, 1)$	$(224, 25)$	$(288, 1105)$	$(352, 1352)$	$(416, 83)$	$(480, 2)$
$(144, 1)$	$(232, 41)$	$(296, 1451)$	$(360, 1123)$	$(424, 49)$	$(488, 2)$
$(168, 3)$	$(240, 105)$	$(304, 1580)$	$(368, 870)$	$(432, 34)$	$(496, 1)$
$(176, 1)$	$(248, 153)$	$(312, 1860)$	$(376, 679)$	$(440, 26)$	$(512, 1)$
$(184, 1)$	$(256, 263)$	$(320, 1750)$	$(384, 466)$	$(448, 9)$	$(544, 1)$
$(200, 3)$	$(264, 409)$	$(328, 1894)$	$(392, 342)$	$(456, 15)$	$(600, 1)$
$(208, 9)$	$(272, 577)$	$(336, 1762)$	$(400, 200)$	$(464, 8)$	
$(216, 13)$	$(280, 864)$	$(344, 1566)$	$(408, 142)$	$(472, 5)$	

Table 3. The number N_α of the ternary near-extremal self-dual codes.

are called *non-isomorphic*. Let C be a ternary self-dual code of length n. It is known that the following lattice,

$$A_3(C) = \frac{1}{\sqrt{3}}\{(x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n \mid (x_1 \pmod 3), x_2 \pmod 3), \ldots, x_n \pmod 3)) \in C\}$$

is an unimodular lattice in dimension n. This construction of lattices is well known as *Construction A*. Let C be a ternary near-extremal self-dual code of length 36 having weight enumerator W_α. Then, it follows that $A_3(C)$ has minimum norm 3 and kissing number $\alpha + 72$. We verified by Magma [2] that the 20848 unimodular lattices, which are constructed from the 20848 inequivalent ternary near-extremal self-dual codes given in Proposition 5.2, are non-isomorphic. This also verifies the inequivalence of the 20848 ternary near-extremal self-dual codes.

Let C be a ternary near-extremal self-dual code of length 36 having weight enumerator W_α. Then, the supports of codewords of weight 9 in C form a 1-$(36, 9, \frac{\alpha}{8})$ design [19]. Let $D_i = (X_i, \mathcal{B}_i)$ be a 1-design, where X_i is the set of points and $\mathcal{B}_i$ is the collection of blocks of D_i $(i = 1, 2)$. Two 1-designs D_1 and D_2 are said to be *isomorphic* if there exists a bijection $\phi : X_1 \to X_2$ such that if we rename every point x of X_1 by $\phi(x)$ then $\mathcal{B}_1$ is transformed into $\mathcal{B}_2$, otherwise, they are called *non-isomorphic*. We verified by Magma [2] that these 20848 1-designs, which are constructed from the 20848 inequivalent ternary near-extremal self-dual codes given in Proposition 5.2, are non-isomorphic. This also verifies the inequivalence of the 20848 ternary near-extremal self-dual codes.

6. Their association schemes

In this section, we construct association schemes of order 35 and class 2 from skew-Hadamard matrices of order 36. We define association schemes in matrix form. An *association scheme* of order n and class 2 is a set of nonzero $n \times n$ matrices A_0, A_1, A_2 with entries in $\{0, 1\}$ satisfying the following conditions:

(i) $A_0 = I$.

(ii) $A_0 + A_1 + A_2 = J$, where J denotes the all-one matrix.

(iii) For any $i, j \in \{0, 1, 2\}$, $A_i^{\intercal} \in \{A_0, A_1, A_2\}$ and A_iA_j is a linear combination of A_0, A_1, A_2.

An association scheme $\mathcal{S} = \{A_0, A_1, A_2\}$ is said to be *symmetric* if all matrices A_i are symmetric. Two association schemes $\mathcal{S}$ and $\mathcal{T}$ are said to be *isomorphic* if there exists a permutation matrix P such that $\mathcal{T} = P^{-1}\mathcal{S}P$, otherwise, they are called *non-isomorphic*.

There is a known correspondence between skew-Hadamard matrices of order n and nonsymmetric association schemes of order $n-1$ and class 2 for $n \equiv 0 \pmod 4$ (see [8]). Let H be a skew-Hadamard matrix H of order n. Let D_i be the diagonal matrix whose diagonal entries are the ith row of H. Let R_i be the $(n-1) \times (n-1)$ matrix obtained from $D_i^{-1}HD_i$ by deleting the ith row and column. Define the matrices

$$A_0 = I, A_1 = \frac{J - 2I + R_i}{2} \text{ and } A_2 = \frac{J - R_i}{2}.$$

Then, $\mathcal{A}(H, i) = \{A_0, A_1, A_2\}$ is an association scheme of order $n-1$ and class 2.

Let $H_1, H_2, \ldots, H_{157132}$ denote the 157132 skew Hadamard matrices of order 36 given in Table 2. Then, we have a set of nonsymmetric association schemes of order 35 and class 2:

$$\mathcal{A}(35) = \{\mathcal{A}(H_k, i) \mid k = 1, 2, \ldots, 157132 \text{ and } i = 1, 2, \ldots, 36\}.$$

We describe how to classify nonsymmetric association schemes $\mathcal{A}(H_k, i)$ constructed from $H_1, H_2, \ldots, H_{157132}$. Let $\mathcal{S} = \{A_0, A_1, A_2\}$ and $\mathcal{T} = \{B_0, B_1, B_2\}$ be nonsymmetric association schemes of order 35 and class 2. Let $dg(M)$ denote the digraph corresponding to a 35×35 matrix M with entries in $\{0, 1\}$. Then, $\mathcal{S}$ and $\mathcal{T}$ are isomorphic if and only if $dg(A_1) \cong dg(B_1)$ or $dg(A_1) \cong dg(B_2)$, where $dg(A) \cong dg(B)$ means that $dg(A)$ and $dg(B)$ are isomorphic [8]. Let $clg(A)$ denote the canonically labelled graph of $dg(A)$ (see [10] and [18] for the definition of canonically labelled graphs). It is known that $clg(A) = clg(B)$ if and only if $dg(A) \cong dg(B)$ (see [10] and [18]). Then, $\mathcal{S}$ and $\mathcal{T}$ are isomorphic if and only if $\{clg(A_1), clg(A_2)\} = \{clg(B_1), clg(B_2)\}$. Considering canonically labelled graphs significantly reduces the amount of our computation. In this way, by comparing canonically labelled graphs $\{clg(A_1), clg(A_2)\}$ in $\mathcal{S} = \{A_0, A_1, A_2\}$, we determined by MAGMA [2] that $\mathcal{A}(35)$ contains 2793032 non-isomorphic nonsymmetric association schemes. This calculation was also done by using NAUTY [18].

Proposition 6.1. *At least* 2793032 *non-isomorphic nonsymmetric association schemes of order* 35 *and class* 2 *exist.*

The obtained list of the non-isomorphic nonsymmetric association schemes of order 35 and class 2 is available electronically at [9].

7. Concluding remarks

Whether or not our list of skew Hadamard matrices of order 36 is complete is a question. Referring to Table 2, we strongly feel that no skew Hadamard matrices of order 36 exist with skew types $(2, 2)$, $(3, 0)$ and $(3, 1)$.

Acknowledgments

Masaaki Harada was supported by JSPS KAKENHI Grant Numbers 19H01802 and 23H01087. Hadi Kharaghani was supported by the Natural Sciences and Engineering Research Council of Canada (NSERC).

References

[1] M. Araya and M. Harada, Some restrictions on the weight enumerators of near-extremal ternary self-dual codes and quaternary Hermitian self-dual codes, *Des. Codes Cryptogr.* **91** (2023), 1813–1843.

[2] W. Bosma, J. Cannon and C. Playoust, The Magma algebra system I: The user language, *J. Symbolic Comput.* **24** (1997), 235–265.

[3] E. Brown and K.B. Reid, Doubly regular tournaments are equivalent to skew Hadamard matrices, *J. Combin. Theory Ser. A* **12** (1972), 332–338.

[4] J. Cooper, J. Milas and W.D. Wallis, Hadamard equivalence, *Combinatorial Mathematics*, in: Lecture Notes in Math., vol. 686, Springer, pp. 126–135, 1978.

[5] D.Ž. Đoković, Skew-Hadamard matrices of order 276, `https://arxiv.org/abs/2301.02751`.

[6] P. Erdős and L. Moser, On the representation of directed graphs as unions of orderings, *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **9** (1964), 125–132.

[7] I.A. Faradžev, Constructive enumeration of combinatorial objects, in: *Problèmes combinatoires et théorie des graphes* (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976), pp. 131–135, Colloq. Internat. CNRS, 260, CNRS, Paris, 1978.

[8] A. Hanaki, H. Kharaghani, A. Mohammadian and B. Tayfeh-Rezaie, Classification of skew-Hadamard matrices of order 32 and association schemes of order 31, *J. Combin. Des.* **28** (2020), 421–427.

[9] M. Harada, Codes and association schemes, available at: `https://www.math.is.tohoku.ac.jp/~mharada/F3-H36/`.

[10] P. Kaski and P.R.J. Östergård, *Classification Algorithms for Codes and Designs*, Springer, Berlin, 2006.

[11] H. Kharaghani and B. Tayfeh-Rezaie, A Hadamard matrix of order 428, *J. Combin. Des.* **13** (2005), 435–440.

[12] H. Kharaghani and B. Tayfeh-Rezaie, On the classification of Hadamard matrices of order 32, *J. Combin. Des.* **18** (2010), 328–336.

[13] H. Kharaghani and B. Tayfeh-Rezaie, Hadamard matrices of order 32, *J. Combin. Des.* **21** (2013), 212–221.

[14] J.-L. Kim and P. Solé, Skew Hadamard designs and their codes, *Des. Codes Cryptogr.* **49** (2008), 135–145.

[15] H. Kimura, Classification of Hadamard matrices of order 28, *Discrete Math.* **133** (1994), 171–180.

[16] E.G. Koçer, Circulant, negacyclic and semicirculant matrices with the modified Pell, Jacobsthal and Jacobsthal-Lucas numbers, *Hacet. J. Math. Stat.* **36** (2007), 133–142.

[17] C.L. Mallows and N.J.A. Sloane, An upper bound for self-dual codes, *Inform. Control* **22** (1973), 188–200.

[18] B.D. McKay and A. Piperno, Practical graph isomorphism, II, *J. Symbolic Comput.* **60** (2014), 94–112.

[19] T. Miezaki, A. Munemasa and H. Nakasora, A note on Assmus–Mattson type theorems, *Des. Codes Cryptogr.* **89** (2021), 843–858.

[20] A. Mohammadian and B. Tayfeh-Rezaie, Hadamard matrices with few distinct types, *Linear Multilinear Algebra* **67** (2019), 1596–1605.

[21] E. Rains and N.J.A. Sloane, *Self-dual codes*, in: Handbook of coding theory, Elsevier, pp. 177–294, 1998.

[22] R.E.A.C. Paley, On orthogonal matrices, *J. Math. Phys.* **12** (1933), 311–320.

[23] R.C. Read, Everyone a winner or how to avoid isomorphism search when cataloguing combinatorial configurations, *Ann. Discrete Math.* **2** (1978), 107–120.

[24] J. Wallis, Some $(1, -1)$ matrices, *J. Combin. Theory Ser. B* **10** (1971), 1–11.