On halvings of the 2-$(10,3,8)$ design[†]

Z. Eslami, G. B. Khosrovshahi*, and B. Tayfeh-Rezaie
Department of Mathematics, University of Tehran
and
Institutes for Studies in Theoretical Physics and Mathematics (IPM)
Tehran, Iran

Abstract

In this paper, we classify all non-isomorphic $LS[2](2,3,10)$ with non-trivial automorphism group. Rigid large sets are also enumerated. Consequently, all simple 2-$(10,3,4)$ designs are classified and enumerated in the same sense.

1. Introduction

For given v, k, and t, let $X = \{1, 2, \ldots, v\}$ and let $P_k(X)$ denote the set of all k-subsets of X. The elements of X and $P_k(X)$ are called points and blocks, respectively.

A t-(v,k) *trade* $T = \{T_1, T_2\}$ consists of two disjoint collections of blocks T_1 and T_2 such that for every $A \in P_t(X)$, the number of blocks containing A is the same in both T_1 and T_2. T is called *simple* if there are no repeated blocks in T_1 (T_2). Here, we are concerned only with simple trades.

The *foundation* of a trade T, denoted by $found(T)$, is the set of all elements covered by T_1 and T_2. The number of blocks in T_1 (T_2) is called the *volume* of T and is denoted by $vol(T)$.

Two trades $T = \{T_1, T_2\}$ and $T' = \{T'_1, T'_2\}$ are called *isomorphic* if there exists a bijection $\sigma : \mathrm{found}(T) \to \mathrm{found}(T')$ such that $\sigma(T) = \{\sigma(T_1), \sigma(T_2)\} = \{T'_1, T'_2\} = T'$. An isomorphism σ such that $\sigma(T) = T$ is called an *automorphism* of T. Clearly, the set of all automorphisms of T forms a group. T is called *rigid* if its automorphism group is trivial.

For each point $x \in \mathrm{found}(T)$, we consider the set of all blocks containing it. By omitting x from these blocks, we obtain a $(t-1)$-$(v-1, k-1)$ trade and we call it the *derived* trade with respect to x.

[†]This research was partially supported by a grant from IPM.

*Corresponding author; Mailing address: IPM, P.O. Box 19395-5746, Tehran, Iran; Email: rezagbk@karun.ipm.ac.ir.

A t-(v,k,λ) *design* is a collection $\mathcal{B}$ of blocks of X such that every t-subset of X occurs exactly λ times in $\mathcal{B}$. Again, we are concerned only with *simple* designs. $P_k(X)$ is called the *complete* design. A *large set* of t-(v,k,λ) designs, denoted by $LS[N](t,k,v)$, is a partition of the complete design into N disjoint t-(v,k,λ) designs, where $N=\binom{v-t}{k-t}/\lambda$. An $LS[2](t,k,v)$ is called a *halving* of the complete design. A t-(v,k) trade $T=\{T_1,T_2\}$ of volume $\binom{v}{k}/2$ is exactly an $LS[2](t,k,v)$. Clearly, T_1 and T_2 are simple t-$(v,k,\binom{v-t}{k-t}/2)$ designs.

In [5], a lower bound of 961 is given for the number of non-isomorphic 2-$(10,3,4)$ designs (some with repeated blocks). In [2], a linear algebraic approach based on the *standard basis* of trades is presented to classify 2-$(8,3)$ trades. In Section 2, a description of this basis, quoted from [2], is provided. In this paper, we employ, with slight modifications, the same approach to classify all non-rigid 2-$(10,3)$ trades of volume 60, i.e. $LS[2](2,3,10)$. Rigid large sets are also enumerated. Consequently, all simple 2-$(10,3,4)$ designs are classified and enumerated in the same sense.

2. The standard basis for trades

Let $1\leq t<k<v$, and let X be a v-set. Let $P^v_{t,k}=[p_{A,B}]$ be the $\binom{v}{t}\times\binom{v}{k}$ *inclusion matrix*, where for a t-subset A of X and a block B, $p_{A,B}=1$ if $A\subseteq B$ and 0 otherwise. For $t<k<v-t$, it is known that the rank of $P^v_{t,k}$ is $\binom{v}{t}$ and hence its kernel, denoted by $N^v_{t,k}$, is a Z-module of dimension $\binom{v}{k}-\binom{v}{t}$. The trade $T=\{T_1,T_2\}$ corresponds to the $\binom{v}{k}$-integral vector F which is a solution of the equation $P^v_{t,k}F=0$. That is, the set of all t-(v,k) trades is the kernel of $P^v_{t,k}$.

There are different bases for $N^v_{t,k}$ in the literature. For a brief description the reader is referred to [4], where the authors also introduce a new basis which is called the *standard basis*. Here, we show how this basis can be used to classify t-(v,k) trades of volume $\binom{v}{k}/2$. The $\binom{v}{k}-\binom{v}{t}$ trades of the standard basis constitute the columns of a matrix $M^v_{t,k}$ which has the following block structure:

$$M^v_{t,k}=\begin{bmatrix} I \\ \bar{M}^v_{t,k}\end{bmatrix} \tag{1}$$

The rows corresponding to I are indexed by the so-called *starting blocks* and the remaining rows by the *non-starting blocks* [3]. By (1), the following observation is clear.

Lemma. Let T be a trade. Then $T \neq 0$ if and only if T contains at least one starting block.

The starting blocks corresponding to the triple (v,k,t) on the point set $\{1,\ldots,v\}$ have the following property. If we choose from among these starting blocks the ones containing i (for $i=1,\ldots,v-k-t$) and omit i from them, the resulting blocks are the starting blocks for the triple $(v-1,k-1,t-1)$ on the point set $\{1,\ldots,v\}\backslash\{i\}$. Let i=1. Then we have the following block structure for $M^v_{t,k}$:

$$\begin{bmatrix} I & 0 \\ 0 & I \\ K & L \\ Q & R \end{bmatrix} \tag{2}$$

The indices corresponding to the first and the third rows of this block structure are the starting and non-starting blocks for the triple $(v-1,k-1,t-1)$, respectively. By the lemma, we have $L=0$ and therefore $K=\bar{M}^{v-1}_{t-1,k-1}$. Clearly $R=\bar{M}^{v-1}_{t,k}$. Hence by permuting the rows of $M^v_{t,k}$, we obtain

$$M^v_{t,k} = \begin{bmatrix} M^{v-1}_{t-1,k-1} & 0 \\ N & M^{v-1}_{t,k} \end{bmatrix}$$

We now focus only on trades with volume $\binom{v}{k}/2$. Hereafter, by "trade" we mean such a trade. A direct way to produce and classify all t-(v,k) trades is to compute linear combinations of the columns of $M^v_{t,k}$ with coefficients 1 and -1, and then to decide whether the result is a simple trade. However, this is practical only for small values of the parameters. Hence we make the following improvements to this crude procedure so that it will not be necessary to deal with all linear combinations of the columns of $M^v_{t,k}$.

Suppose $(t-1)$-$(v-1,k-1)$ trades have been classified so that we have one representative for each isomorphism class. Let T be a t-(v,k) trade and T' its derived trade with respect to the point 1. T' is clearly isomorphic to one of the representative $(t-1)$-$(v-1,k-1)$ trades, say T''. So, there exists a permutation π such that $T''=\pi T'$. Therefore, πT (an isomorphic copy of T) will be the extension of T''. Hence, to classify t-(v,k) trades, up to isomorphism, it suffices to extend only the representatives of the isomorphism classes of $(t-1)$-$(v-1,k-1)$ trades. The recursive structure of $M^v_{t,k}$ helps us in determining t-(v,k) trades by extending $(t-1)$-$(v-1,k-1)$ trades. Let T' be a $(t-1)$-$(v-1,k-1)$ trade. Then the coefficients of the first $\binom{v-1}{k-1}-\binom{v-1}{t-1}$ columns of $M^v_{t,k}$

are specified by the blocks of T'. To extend T', it suffices to determine the coefficients of the remaining columns in such a way that the result would be a simple trade. Finally, we check for isomorphism among all extensions.

3. Classification of non-rigid $LS[2](2,3,10)$

The approach described in Section 2 is employed to classify 2-$(10,3)$ trades of volume 60, i.e. $LS[2](2,3,10)$. In what follows, we only deal with such trades. First, 1-$(9,2)$ trades of volume 36 are classified. Up to isomorphism, there exist exactly 10 non-isomorphic 1-$(9,2)$ trades, $S_1,\dots,S_{10}$, which are given in Table I of the Appendix. The direct extensions of these derived trades result in over 200,000,000 solutions for which isomorphism testing would be clearly hard to carry out. To overcome this difficulty, we focus our attention only on non-rigid trades.

Let $T=\{T_1,T_2\}$ be a trade with a non-trivial automorphism, say π. As in [1], we can take π to be of the type $1^n a^m$, that is, π consists of n fixed points and m disjoint cycles of length a, where a is a prime and $n+am=10$. The case $a=7$ can be ruled out as follows. Let x_1, x_2, and x_3 be the fixed points of π and let $x_1x_2x_3, x_1x_2x_4, x_1x_2x_5, x_1x_2x_6 \in T_1$. Then x_4, x_5, and x_6 form a cycle of length 3 of π, a contradiction. Therefore $a \in \{2,3,5\}$. First suppose that $a \neq 2$. By a suitable relabeling of the points, we can assume that $\pi(1)=2$ and $\pi(2)=3$. Let D_i be the derived trade of T with respect to i. So we have $D_2=\pi D_1$ and $D_3=\pi D_2$. D_1, D_2, and D_3 constitute the first 61 columns of $M_{2,3}^{10}$. If $a=2$, then by similar arguments we have $D_2=\pi D_1$. Moreover, D_1 and D_2 together identify the coefficients of the first 47 columns of $M_{2,3}^{10}$.

The foregoing observations lead us to the following procedure. Let S_i be one of the 10 non-isomorphic 1-$(9,2)$ trades and let $x, y \in \mathrm{found}(S_i)$. Suppose γ is a permutation on $\{2,\dots,10\}$ such that $\gamma(x)=2$ and $\gamma(y)=3$. We take D_1 to be γS_i. We also choose π from one of the basic types $1^n a^m$, where $a \in \{3,5\}$ such that $\pi(1)=2$ and $\pi(2)=3$. Now, D_2 and D_3 are obtained by $D_2=\pi D_1$ and $D_3=\pi D_2$. We set the remaining 14 columns of $M_{2,3}^{10}$ in such a way that π is an automorphism of the resulting trade. We repeat this procedure for each S_i, $i=1,\dots,10$. We then consider the case $a=2$ and assume that every automorphism of T is of order 2. In this case the size of automorphism group is a power of 2. By similar arguments, we assume that $\pi(1)=2$ and $\pi(2)=1$, then $D_2=\pi D_1$, and this case is also treated as before. We then use McKay's **nauty** to determine the automorphism group of one half of the trades and obtain the following results:

$\vert\mathrm{Aut}\vert$	#trades
1	$168,514$
2	$72,526$
3	$4,670$
4	$3,457$
5	52
6	304
8	362
9	8
10	21
16	39
20	6
24	32
32	8
48	19
320	1
720	1

Isomorphism testing within each of these classes is done by applying only a fraction of 10! permutations. For example, the 43,322 trades in the largest subclass of the second class (those having an automorphism with 2 fixed points) can be tested for isomorphism with only 768 permutations. The final results show that there exist **19,945** non-rigid trades. In Table 1, the number of non-isomorphic trades with non-trivial automorphism group is given. The trades with at least 40 automorphisms are presented in Table II of the Appendix.

Each 2-$(10,3)$ trade $T = \{T_1, T_2\}$ consists of two 2-$(10,3,4)$ designs, that is T_1 and T_2. There exist 15,220 trades with exactly two automorphisms in which T_1 and T_2 are rigid and therefore isomorphic. This means that we have classified a total of 15,220 of the rigid 2-$(10,3,4)$ designs. On the other hand, there are 442 trades in which T_1 and T_2 are non-rigid but isomorphic. Therefore, the number of non-isomorphic simple 2-$(10,3,4)$ designs with non-trivial automorphism group is **9,008**. The number of these designs in each automorphism group size are given in Tabel 2.

We are not able to classify trades with trivial automorphism group. However, we enumerate in the next section the exact number of this class of trades and consequently we obtain the exact number of rigid 2-$(10, 3, 4)$ designs.

Table 1.

The number of non-rigid $LS[2](2, 3, 10)$.

|Aut|	#LS	|Aut|	#LS
2	19,180	20	1
3	214	24	2
4	426	32	2
5	2	40	2
6	23	48	1
8	70	64	1
9	2	640	1
10	10	1440	1
16	7		
total:	19,945		

Table 2.
The number of non-rigid 2-$(10, 3, 4)$ designs.

$\lvert\text{Aut}\rvert$	#designs	$\lvert\text{Aut}\rvert$	#designs
2	8,285	16	4
3	428	20	2
4	179	24	4
5	10	32	1
6	46	48	2
8	32	320	1
9	4	720	1
10	9		
total:	9,008		

4. Enumeration of rigid $LS[2](2, 3, 10)$

In this section, rigid 2-$(10, 3)$ trades of volume 60, i.e. $LS[2](2, 3, 10)$, are enumerated. Let, up to isomorphism, R be the number of such trades and let S be the total number of all distinct trades. Recall that $S_1, \ldots, S_{10}$ are the 10 non-isomorphic 1-$(9, 2)$ trades. Let d_i be the size of automorphism group of S_i and let r_i be the number of its extensions. Suppose that $T_1, \ldots, T_{19945}$ are the non-isomorphic trades with non-trivial automorphism group and $t_1, \ldots, t_{19945}$ are the sizes of their corresponding automorphism group.Therefore, we have

$$S = \sum_{i=1}^{10} \frac{r_i \times 9!}{d_i}.$$

$$R = \frac{S - \sum_{i=1}^{19945} \frac{10!}{t_i}}{10!}. \tag{3}$$

Using $M_{2,3}^{10}$, we extend $S_1, \ldots, S_{10}$ and determine the numbers r_i. The results are as follows:

i	d_i	r_i
1	2	$21,263,595$
2	4	$21,287,629$
3	4	$21,279,485$
4	8	$21,274,056$
5	12	$21,301,446$
6	16	$21,242,020$
7	16	$21,325,376$
8	18	$21,277,437$
9	64	$21,306,583$
10	144	$21,288,480$

Consequently, by (3) we obtain $R =$**2,993,342**. In Section 3, we classified a total of 15,220 of the rigid 2-$(10,3,4)$ designs. Therefore, the exact number of rigid 2-$(10,3,4)$ designs is **6,001,904**.

5. Appendix

Table I. The 10 **non-isomorphic** $LS[2](1,2,9)$.

$\lvert\mathrm{Aut}\rvert$	2	4	4	8	12	16	16	18	64	144
	23	23	23	23	23	23	23	23	23	23
	24	24	24	24	24	24	24	24	24	24
	25	25	25	25	25	25	25	25	25	25
	26	26	26	26	26	26	26	26	26	26
	38	38	38	38	38	38	38	38	38	36
	39	39	39	39	39	39	39	39	39	39
	47	47	48	47	47	46	48	47	48	45
	49	49	49	49	49	47	49	49	49	48
D_1	56	56	56	57	56	56	57	56	56	57
	58	58	57	58	59	57	59	58	57	59
	67	67	67	67	67	59	67	67	67	67
	69	68	69	68	68	68	68	69	78	68
	78	78	78	69	78	79	78	78	79	78
	79	79	30	30	30	30	79	30	30	79
	30	30	40	40	40	40	30	40	40	30
	40	40	50	50	50	70	40	50	50	40
	50	50	70	70	70	89	50	70	60	80
	80	90	89	89	89	81	60	89	89	90
	27	27	27	27	27	27	27	27	27	27
	28	28	28	28	28	28	28	28	28	28
	29	29	29	29	29	29	29	29	29	29
	34	34	34	34	34	34	34	34	34	34
	35	35	35	35	35	35	35	35	35	35
	36	36	36	36	36	36	36	36	36	37
	37	37	37	37	37	37	37	37	37	38
	45	45	45	45	45	45	45	45	45	46
D_2	46	46	46	46	46	48	46	46	46	47
	48	48	47	48	48	49	47	48	47	49
	57	57	58	56	57	58	56	57	58	56
	59	59	59	59	58	67	58	59	59	58
	68	69	68	78	69	69	69	68	68	69
	20	20	79	79	79	78	20	79	69	20
	60	60	20	20	20	20	70	20	20	50
	70	70	60	60	60	50	89	60	70	60
	89	89	80	80	80	60	80	80	80	70
	90	80	90	90	90	90	90	90	90	89

Table II. The $LS[2](2,3,10)$ with at least 40 automorphisms.

$\lvert\mathrm{Aut}\rvert$	40	40	48	64	640	1440
	123 457	123 378	123 456	123 456	123 379	123 378
	124 459	124 457	124 459	124 457	124 456	124 456
	125 467	125 458	125 467	125 458	125 458	125 467
	126 469	126 459	126 469	126 459	126 459	126 479
	138 568	138 467	136 568	138 467	138 468	136 568
	139 569	139 468	139 569	139 468	139 469	139 569
	146 578	147 568	145 579	148 469	148 478	145 579
	147 130	149 569	148 130	149 567	149 479	148 130
	156 140	157 579	157 140	156 130	156 567	157 140
	157 170	158 130	159 180	157 140	157 130	159 180
	159 189	167 140	167 190	167 150	167 140	167 190
	168 180	168 150	168 250	178 160	178 150	168 250
	179 250	169 170	178 260	179 189	179 160	178 260
	234 260	234 189	179 289	234 240	234 189	179 280
D_1	237 280	235 230	234 280	235 270	237 230	236 290
	239 290	246 270	236 290	236 280	247 240	237 350
	246 350	249 289	237 350	247 290	257 289	238 370
	248 360	256 280	245 370	258 340	258 280	245 389
	257 370	258 290	247 389	259 350	259 290	247 390
	258 389	267 350	258 390	268 360	267 340	249 460
	269 450	278 360	269 460	269 389	268 370	258 470
	278 489	279 450	278 470	278 470	269 470	269 489
	279 480	347 460	279 489	279 589	345 570	278 480
	345 490	348 480	347 480	348 580	346 580	279 570
	347 590	349 560	348 570	349 590	356 590	345 589
	348 678	357 690	349 580	356 689	358 670	346 580
	356 670	359 789	356 678	357 680	359 680	348 678
	358 680	367 780	357 670	367 690	368 690	349 670
	367 789	368 790	358 690	378 780	369 789	356 689
	369 790	369 890	368 789	379 790	378 890	357 690

Table II. *Continued.*

	127	379	127	456	127	379	127	369	127	457	127	457
	128	456	128	469	128	457	128	478	128	467	128	458
	129	458	129	478	129	458	129	479	129	568	129	459
	134	468	134	479	134	468	134	568	134	569	134	468
	135	478	135	567	135	478	135	569	135	578	135	469
	136	479	136	578	137	479	136	578	136	579	137	478
	137	567	137	120	138	567	137	579	137	120	138	567
	145	579	145	160	146	578	145	120	145	170	146	578
	148	120	146	180	147	120	146	170	146	180	147	120
	149	150	148	190	149	150	147	180	147	190	149	150
	158	160	156	240	156	160	158	190	158	250	156	160
	167	190	159	250	158	170	159	230	159	260	158	170
	169	230	178	260	169	189	168	250	168	270	169	189
D_2	178	240	179	340	235	230	169	260	169	350	234	230
	235	270	236	370	238	240	237	289	235	360	235	240
	236	289	237	389	239	270	238	370	236	389	239	270
	238	340	238	380	246	340	239	380	238	380	246	289
	245	380	239	390	248	360	245	390	239	390	248	340
	247	390	245	470	249	380	246	450	245	450	256	360
	249	460	247	489	256	450	248	460	246	460	257	380
	256	470	248	490	257	490	249	489	248	489	259	450
	259	560	257	570	259	560	256	480	249	480	267	490
	267	570	259	589	267	589	257	490	256	490	268	560
	268	589	268	580	268	590	267	560	278	560	347	590
	346	580	269	590	345	679	345	570	279	589	358	679
	349	679	345	678	346	689	346	678	347	678	359	680
	357	689	346	679	359	680	347	679	348	679	367	789
	359	690	356	670	367	780	358	670	349	689	368	780
	368	780	358	689	369	790	359	789	357	780	369	790
	378	890	379	680	378	890	368	890	367	790	379	890

References

1. C. J. Colbourn, S. S. Magliveras, and D. R. Stinson, *Steiner triple systems of order 19 with nontrivial automorphism group*, Math. Comp. **199** (1992), 283-295.

2. Z. Eslami, G. B. Khosrovshahi, and B. Tayfeh-Rezaie, *On classification of* 2-$(8, 3)$ *and* 2-$(9, 3)$ *trades*, submitted.

3. G. B. Khosrovshahi and S. Ajoodani-Namini, *A new basis for trades*, SIAM J. Discrete Math. **3** (1990), 364-372.

4. G. B. Khosrovshahi and CH. Maysoori, *On the bases for trades*, Linear Algebra Appl. **226-228** (1990), 731-748.

5. R. Mathon and A. Rosa, *On* 2-(v, k, λ) *designs of small order*, in: The CRC Handbook of Combinatorial Designs, (C. J. Colbourn and J. H. Dinitz, eds.), CRC Press, New York, 1996, pp. 3-41.