Threshold for stability of weak saturation

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Abstract

We study the weak K_s-saturation number of the Erdős–Rényi random graph $\mathbb{G}(n,p)$, denoted by $\mathrm{wsat}(\mathbb{G}(n,p),K_s)$, where K_s is the complete graph on s vertices. In 2017, Korándi and Sudakov proved that the weak K_s-saturation number of K_n is stable, in the sense that it remains the same after removing edges with constant probability. In this paper, we prove that there exists a threshold for this stability property and give upper and lower bounds on the threshold. This generalizes the result of Korándi and Sudakov. A general upper bound on $\mathrm{wsat}(\mathbb{G}(n,p),K_s)$ is also provided.

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1. Introduction

Given a graph F, an *F-bootstrap percolation process* is a sequence of graphs $H_0 \subset H_1 \subset \cdots \subset H_m$ such that, for $i = 1, \ldots, m$, H_i is obtained from H_{i-1} by adding an edge e that belongs to a copy of F in H_i. As a customary term, it is said that the edge e is activated during the process. The F-bootstrap percolation process was introduced by Bollobás more than fifty years ago [6] and can be seen as a special case of the 'cellular automata' introduced by von Neumann [15]. The F-bootstrap percolation is also similar to r-neighborhood bootstrap percolation model having applications in physics; see, for example, [1], [8], and [14].

Given two graphs G and F, a spanning subgraph H of G is said to be a *weakly F-saturated subgraph of G* if H contains no subgraph isomorphic to F and there exists an F-bootstrap percolation process $H = H_0 \subset H_1 \subset \cdots \subset H_m = G$. The minimum number of edges in a weakly F-saturated subgraph of G is called the *weak F-saturation number* of G and is denoted by $\mathrm{wsat}(G, F)$.

We denote by $\mathbb{G}(n,p)$ the Erdős–Rényi random graph on vertex set $[\![n]\!] = \{1,\dots,n\}$ constructed by adding every edge $e \in \{xy \mid x, y \in [\![n]\!] \text{ and } x \neq y\}$ with probability p independently of all the others. Korándi and Sudakov [12] initiated the study of weak saturation numbers of random graphs. They proved that, for every fixed real number $p \in (0,1)$ and integer $s \geqslant 3$, $\mathrm{wsat}(\mathbb{G}(n,p), K_s) = \mathrm{wsat}(K_n, K_s)$ with high probability. Recall that the notion 'with high probability', which is written as '*whp*' for brevity, is used whenever an event occurs in $\mathbb{G}(n,p)$ with a probability approaching 1 as $n \to \infty$. It was already known that

$$\mathrm{wsat}(K_n, K_s) = (s-2)n - \binom{s-1}{2}$$

by a classic result proved by Lovász [13]. Several proofs for the above equality have been given by others. Korándi and Sudakov [12] also noticed that $\mathrm{wsat}(\mathbb{G}(n,p), K_s) = \mathrm{wsat}(K_n, K_s)$ *whp* when $p \geqslant n^{-\varepsilon}$ for small enough $\varepsilon > 0$ and asked about smaller p and about possible threshold for the property of having the weak K_s-saturation number of $\mathbb{G}(n,p)$ exactly $(s-2)n - \binom{s-1}{2}$. We denote this property by $\mathscr{A}_s$. In this paper, we prove that this threshold exists and present upper and lower bounds on that. Recall that a function $\breve{p}$ is a *threshold* for a sequence $\mathscr{X}_n$ of events in $\mathbb{G}(n,p)$ if either

$$\lim_{n\to\infty} \mathbb{P}\big[\mathscr{X}_n\big] = \begin{cases} 0 & \text{if } p \ll \breve{p}, \\ 1 & \text{if } p \gg \breve{p} \end{cases}$$

or

$$\lim_{n\to\infty} \mathbb{P}\big[\mathscr{X}_n\big] = \begin{cases} 1 & \text{if } p \ll \breve{p}, \\ 0 & \text{if } p \gg \breve{p}. \end{cases}$$

The existence of a threshold for the property $\mathscr{A}_s$ and its lower bound are proven in Section 3. It is done by approximating this property using an auxiliary increasing property. We also establish an upper bound on the aforementioned threshold in Section 4 by introducing a weakly K_s-saturated subgraph of $\mathbb{G}(n,p)$. The following theorem summarizes our results in Sections 3 and 4. Before stating it, we fix some notation. For any integer $s \geqslant 3$, let

$$a_s = \left(2\left(1 - \frac{1}{s+1}\right)(s-2)!\right)^{\frac{2}{(s-2)(s+1)}} \tag{1}$$

and

$$\sigma_s = \begin{cases} 2 & \text{if } s = 3, \\ 1 & \text{if } s \geqslant 4. \end{cases} \tag{2}$$

Theorem 1.1. *For any fixed integer $s \geqslant 3$, the property $\mathscr{A}_s$ in $\mathbb{G}(n,p)$ has a threshold. Moreover, there is a threshold $\tilde{p}$ for the property $\mathscr{A}_s$ in $\mathbb{G}(n,p)$ with $a_s n^{-2/(s+1)}(\log n)^{2/(s-2)(s+1)} \leqslant \tilde{p} \leqslant n^{-1/(2s-3)}(\log n)^{(s+\sigma_s-3)/(2s-3)}$ for all sufficiently large n.*

Furthermore, we establish in Section 5 a universal upper bound on $\mathrm{wsat}(\mathbb{G}(n,p), K_s)$ which is presented in the following theorem. It can be remarkable when p is between the provided upper and lower bounds on the threshold in Theorem 1.1.

Theorem 1.2. *Let $s \geqslant 3$ be a fixed integer and let $w(n)$ be a real-valued function such that $w(n) \to \infty$ as $n \to \infty$. Then, whp*

$$\operatorname{wsat}\Big(\mathbb{G}(n,p), K_s\Big) \leqslant (s-2)n + \frac{(\log n)^{2(s+\sigma_s-3)} w(n)}{p^{2s-3}}.$$

For the sake of clarity, we present in Section 2 the notation and terminology used in the paper and state the known results that we have referred to.

2. Notation and preliminaries

In this section, we introduce notation and recall the probabilistic facts that we use in the rest of the paper.

For a graph G, we denote the vertex set and the edge set of G by $V(G)$ and $E(G)$, respectively. The *size* of G is defined as $|E(G)|$ and is denoted by $e(G)$. For a vertex u of G, let $N_G(u) = \{x \in V(G) \,|\, u \text{ is adjacent to } x\}$, that is, the set of neighbors of u in G. For a subset U of $V(G)$, set $N_G(U) = \bigcap_{u \in U} N_G(u)$ and $N_G[U] = U \cup N_G(U)$. For the sake of convenience, $N_G(u_1, \ldots, u_k)$ is written instead of $N_G(\{u_1, \ldots, u_k\})$. Further, for a subset U of $V(G)$, we denote the induced subgraph of G on U by $G[U]$.

We also use the standard asymptotic notation in the rest of the paper. For two real-valued functions $f(n)$ and $g(n)$, we write $f(n) = O(g(n))$ if there exists a constant $c > 0$ such that $|f(n)| \leqslant cg(n)$ for every large enough integer n. Also, we use the notation $f(n) = o(g(n))$ if the same holds for any constant $c > 0$. We sometimes write $f(n) \ll g(n)$ or $g(n) \gg f(n)$ instead of $f(n) = o(g(n))$. Finally, we use the notation $f(n) = \Theta(g(n))$ if both $f(n) = O(g(n))$ and $g(n) = O(f(n))$ hold.

In what follows, we formulate the probabilistic inequalities that we make use of all in the next sections. We also recall some properties of random graphs.

Theorem 2.1 (Markov's inequality; Inequality (1.3) in [10])**.** *Let X be a nonnegative random variable. Then, for all $t > 0$,*

$$\mathbb{P}\Big[X \geqslant t\Big] \leqslant \frac{\mathbb{E}[X]}{t}.$$

Corollary 2.2. *Let $X_1, X_2, \ldots$ be a sequence of nonnegative integer-valued random variables. If $\mathbb{E}[X_n] = o(1)$, then $X_n = 0$ whp.*

Theorem 2.3 (Chebyshev's inequality; Inequality (1.2) in [10])**.** *Let X be a random variable with the expected value $\mathbb{E}[X]$ and the variance* $\operatorname{Var}[X]$*. Then, for all $t > 0$,*

$$\mathbb{P}\Big[\big|X - \mathbb{E}[X]\big| \geqslant t\Big] \leqslant \frac{\operatorname{Var}[X]}{t^2}.$$

Corollary 2.4. *Let $X_1, X_2, \ldots$ be a sequence of random variables with nonzero expected values. If* $\operatorname{Var}[X_n] = o(\mathbb{E}[X_n]^2)$*, then $X_n \neq 0$ whp.*

Theorem 2.5 (Chernoff's inequality; Theorem 2.1 in [10])**.** *Let $X \sim$* $\operatorname{Bin}(n,p)$ *be a binomial random variable with parameters n and p. Then, for any $t \geqslant 0$,*

$$\mathbb{P}\Big[X \leqslant np - t\Big] \leqslant \exp\left(-\frac{t^2}{2np}\right).$$

The following consequence of the Fortuin–Kasteleyn–Ginibre inequality appears in Page 31 of [10].

Theorem 2.6 ([9])**.** *Let $\mathcal{S}$ be a family of subgraphs of K_n and assume that the random variable X counts the number of graphs in $\mathcal{S}$ that appear in $\mathbb{G}(n,p)$. Then,*

$$\mathbb{P}\Big[X=0\Big] \geqslant \prod_{H\in\mathcal{S}} \Big(1-\mathbb{P}\Big[H\subset \mathbb{G}(n,p)\Big]\Big).$$

Theorem 2.7 (Janson's inequality; Theorem 2.18 in [10])**.** *Let $\mathcal{S}$ be a family of subgraphs of K_n. Assume that the random variable X counts the number of graphs in $\mathcal{S}$ that appear in $\mathbb{G}(n,p)$. For every $H_1,H_2\in\mathcal{S}$, let $H_1\sim H_2$ indicate that $H_1\neq H_2$ and H_1,H_2 share at least one edge. Define*

$$\varDelta_X = \sum_{\substack{H_1,H_2\in\mathcal{S}\\ H_1\sim H_2}} \mathbb{P}\Big[H_1,H_2\subset \mathbb{G}(n,p)\Big].$$

Then,

$$\mathbb{P}\Big[X=0\Big] \leqslant \exp\left(-\mathbb{E}[X]+\frac{\varDelta_X}{2}\right).$$

A graph property $\mathscr{P}$ is called *increasing* if adding an edge to a graph satisfying $\mathscr{P}$ does not destroy the property. A graph property $\mathscr{P}$ is called *decreasing* if removing an edge from a graph satisfying $\mathscr{P}$ does not destroy the property. A graph property which is either increasing or decreasing is called *monotone*. The assertions of the following theorem are also proved in Theorems 1.10 and 1.24 in [10].

Theorem 2.8 ([5, 7])**.** *For every $p_1\leqslant p_2$, the following two statements hold.*

(i) *If $\mathscr{P}$ is an increasing graph property, then $\mathbb{P}[\mathbb{G}(n,p_1)\in\mathscr{P}]\leqslant \mathbb{P}[\mathbb{G}(n,p_2)\in\mathscr{P}]$.*

(ii) *If $\mathscr{P}$ is a decreasing graph property, then $\mathbb{P}[\mathbb{G}(n,p_1)\in\mathscr{P}]\geqslant \mathbb{P}[\mathbb{G}(n,p_2)\in\mathscr{P}]$.*

Moreover, every monotone graph property has a threshold.

For a graph G, let $d(G)=\frac{|E(G)|}{|V(G)|}$ and let $m(G)=\max\{d(H)\,|\,H \text{ is a subgraph of } G\}$. A graph G is called *strictly balanced* if $d(H)<d(G)$ for every proper subgraph H of G. The following theorem also appears in [10] as Theorem 3.4.

Theorem 2.9 ([3])**.** *Let G be a fixed graph with at least one edge. Then, $n^{-1/m(G)}$ is a threshold for the property that $\mathbb{G}(n,p)$ contains a copy of G as a subgraph.*

The following theorem also appears in [10] as Theorem 3.19.

Theorem 2.10 ([4])**.** *Let G be a fixed strictly balanced graph. Assume that the random variable X counts the number of copies of G that can be found in $\mathbb{G}(n,p)$, where $np^{m(G)}\to c$ as $n\to\infty$ for a positive constant c. Then, X converges weakly to a Poisson random variable with expectation $c^{|V(G)|}/|Aut(G)|$ as $n\to\infty$, where $Aut(G)$ denotes the automorphism group of G.*

The following result is a part of Theorem 1 of [2].

Theorem 2.11 ([2])**.** *Let $s\geqslant 3$ be a fixed integer and let $p\gg n^{-2(s-2)/(s^2-s-4)}\log n$. Then, there exists an F-bootstrap percolation process $\mathbb{G}(n,p)=H_0\subset H_1\subset\cdots\subset H_m=K_n$ whp.*

3. The existence of the threshold

In this section, we prove the existence of the threshold for the property $\mathscr{A}_s$ and present a lower bound on it. We first prove the following lemma.

Lemma 3.1. *Let $s \geqslant 3$ be a fixed integer. If $p \ll n^{-2/(s+1)}$, then the property $\mathscr{A}_s$ does not hold in $\mathbb{G}(n,p)$ whp.*

Proof. First, assume that $p \leqslant 1/(n \log n)$. Since $p \ll n^{-2/(s-1)}$, we obtain from Theorem 2.9 that $\mathbb{G}(n,p)$ does not contain K_s as a subgraph. This forces that $\mathrm{wsat}(\mathbb{G}(n,p), K_s) = e(\mathbb{G}(n,p))$ *whp*. Furthermore, it follows from Theorem 2.1 that $e(\mathbb{G}(n,p)) \leqslant n/\sqrt{\log n}$ *whp*. This implies that $\mathrm{wsat}(\mathbb{G}(n,p), K_s) \neq \mathrm{wsat}(K_n, K_s)$ *whp*, as desired. Next, assume that $p \geqslant 1/(n \log n)$. Let $\varepsilon \in (0,1)$ be a small constant and denote by X_s the random variable that counts the number of K_s in $\mathbb{G}(n,p)$. As $p \ll n^{-2/(s+1)}$, we may consider a function $f \gg 1$ such that $(np^{(s+1)/2})^{s-2} f(n) = o(1)$. It follows from $\mathbb{E}[X_s] = \binom{n}{s} p^{s(s-1)/2}$ and Theorem 2.1 that

$$\mathbb{P}\left[X_s \geqslant \frac{n^2 p}{f(n)}\right] \leqslant \frac{\mathbb{E}[X_s]}{\frac{n^2 p}{f(n)}} \leqslant \frac{n^s p^{\frac{s(s-1)}{2}} f(n)}{n^2 p} = \left(np^{\frac{s+1}{2}}\right)^{s-2} f(n) = o(1),$$

implying that $X_s = o(n^2 p)$ *whp*. Since $e(\mathbb{G}(n,p)) - X_s \leqslant \mathrm{wsat}(\mathbb{G}(n,p), K_s) \leqslant e(\mathbb{G}(n,p))$, we get that *whp*

$$\mathrm{wsat}\Big(\mathbb{G}(n,p), K_s\Big) = e\Big(\mathbb{G}(n,p)\Big) + o\Big(n^2 p\Big). \tag{3}$$

Since $p \geqslant 1/(n \log n)$ and $e(\mathbb{G}(n,p))$ follows the binomial distribution with parameters $\binom{n}{2}$ and p, we obtain from Theorem 2.3 that $n^2 p/(2+2\varepsilon) \leqslant e(\mathbb{G}(n,p)) \leqslant n^2 p/(2-2\varepsilon)$ *whp*. Thus, we deduce from (3) that $\mathrm{wsat}(\mathbb{G}(n,p), K_s) \neq \mathrm{wsat}(K_n, K_s)$ *whp* when $p \ll n^{-2/(s+1)}$ and $p \notin I_s$ for all sufficiently large n, where

$$I_s = \left(\frac{2(1-\varepsilon)(s-2)}{n}, \frac{2(1+\varepsilon)(s-2)}{n}\right).$$

If $s \geqslant 4$ and $p \in I_s$, then Theorem 2.9 along with $m(K_s) = (s-1)/2$ yields that $X_s = 0$ *whp*. Hence, the weak saturation number is exactly $e(\mathbb{G}(n,p))$ that is not concentrated on a single value.

If $s = 3$ and $p \in I_3$ for all sufficiently large n, then $\mathrm{wsat}(\mathbb{G}(n,p), K_3) = e(\mathbb{G}(n,p)) - X_3$ *whp*. This is because all the triangles in $\mathbb{G}(n,p)$ are disjoint *whp*, since there are no subgraphs with at most 5 vertices and at least two cycles *whp* using Theorem 2.9. Note that the random variable $e(\mathbb{G}(n,p)) - X_3$ is also not concentrated in a unit set. This is because the number of edges has the binomial distribution with parameters $\binom{n}{2}$ and $p = \Theta(1/n)$, so it is outside any interval of length $o(\sqrt{n})$ *whp*, while the number of triangles is bounded from above by an asymptotically Poisson random variable. Indeed, for a constant nonnegative integer L, the property of having at most L triangles is decreasing, implying that $\mathbb{P}[X_3 \leqslant L]$ is minimum when $p = 2(1+\varepsilon)/n$ by Theorem 2.8. At the same time, the number of triangles in $\mathbb{G}(n, 2(1+\varepsilon)/n)$ converges in distribution to a Poisson random variable with parameter $\frac{4}{3}(1+\varepsilon)^3$ by Theorem 2.10. Thus, for any slowly increasing function $g \gg 1$, we have $X_3 \leqslant g(n)$ *whp*. With a suitable choice of $g(n) = o(\sqrt{n})$, we may assume that $|e(\mathbb{G}(n,p)) - \mathrm{wsat}(K_n, K_3)| \geqslant 2g(n)$ *whp*. Therefore, *whp*

$$\Big|\mathrm{wsat}\Big(\mathbb{G}(n,p), K_3\Big) - \mathrm{wsat}(K_n, K_3)\Big| = \Big|\Big(e\Big(\mathbb{G}(n,p)\Big) - X_3\Big) - \mathrm{wsat}(K_n, K_3)\Big|$$

$$\geqslant \left| e\big(\mathbb{G}(n,p)\big) - \mathrm{wsat}(K_n, K_3)\right| - X_3$$
$$\geqslant g(n)$$

which implies that $\mathrm{wsat}(\mathbb{G}(n,p), K_3) \neq \mathrm{wsat}(K_n, K_3)$ *whp*. □

For any integer $s \geqslant 3$, let

$$q_s(n) = n^{-\frac{2}{s+1}} (\log n)^{\frac{2}{(s-2)(s+1)}}$$

and denote by $\mathscr{B}_s$ the property that any edge of $\mathbb{G}(n,p)$ belongs to some K_s. In order to proceed, we state the following technical lemma about the property $\mathscr{B}_s$. We include the proof of Lemma 3.2 in Appendix A.

Lemma 3.2. *For any fixed integer $s \geqslant 3$ and any fixed positive number $\varepsilon \leqslant 1 - 2^{-2/(s^2-s-4)}$, let $w(n)$ be a real-valued function such that*

$$h(n) = 1 + \frac{2\log\log n}{s(s-2)^2(s+1)\log n} + \frac{w(n)}{\log n} \geqslant n^{-\varepsilon}$$

and $p(n) = a_s q_s(n) h(n) \leqslant 1$ for all sufficiently large n, where a_s is defined in (1). Then, the following two statements hold.

(i) *If $w(n) \to \infty$ as $n \to \infty$, then $\mathbb{G}(n,p)$ satisfies the property $\mathscr{B}_s$ whp.*

(ii) *If $w(n) \to -\infty$ as $n \to \infty$, then $\mathbb{G}(n,p)$ does not have the property $\mathscr{B}_s$ whp.*

For any integer $s \geqslant 3$, define the event $\mathscr{C}_s$ in $\mathbb{G}(n,p)$ as follows. Let $\mathscr{C}_3$ be the event that for every two distinct vertices u and v there exists a path u, u', v', v of length 3 in $\mathbb{G}(n,p)$. For $s \geqslant 4$, let $\mathscr{C}_s$ be the event that for every two distinct vertices u and v there exist two distinct nonadjacent vertices $u', v' \in N_{\mathbb{G}(n,p)}(u,v)$ as well as two disjoint cliques $\varOmega \subseteq N_{\mathbb{G}(n,p)}(u,v,u',v')$ and $\varOmega' \subseteq N_{\mathbb{G}(n,p)}(u',v')$ of sizes $s-4$ and $s-2$, respectively. The following result is a special case of Theorem 2 of [18].

Lemma 3.3. *Let $s \geqslant 3$ be a positive fixed integer. Then, there exists a constant c_s such that for any*

$$p \geqslant c_s n^{-\frac{2(s-2)}{s^2-s-3}} (\log n)^{\frac{1}{s^2-s-3}},$$

the property $\mathscr{C}_s$ holds in $\mathbb{G}(n,p)$ whp.

We are now ready to prove the lower bound on the threshold for the property $\mathscr{A}_s$.

Theorem 3.4. *For any fixed integer $s \geqslant 3$ and any $p \leqslant a_s q_s$, the property $\mathscr{A}_s$ does not hold in $\mathbb{G}(n,p)$ whp.*

Proof. In view of Lemma 3.1, we may assume that $p \geqslant c_s n^{-2(s-2)/(s^2-s-3)} (\log n)^{1/(s^2-s-3)}$, where c_s comes from Lemma 3.3. As $p \leqslant a_s q_s$, Part (ii) of Lemma 3.2 implies that there exists an edge e in $\mathbb{G}(n,p)$ that is not contained in any K_s *whp* which immediately yields that e should belong to all weakly K_s-saturated subgraphs of $\mathbb{G}(n,p)$. Suppose by way of contradiction that there exists a weakly K_s-saturated subgraph H of $\mathbb{G}(n,p)$ of size $(s-2)n - \binom{s-1}{2}$. Then, $H - e$ is a weakly K_s-saturated subgraph of $\mathbb{G}(n,p) - e$ of size $(s-2)n - \binom{s-1}{2} - 1$. But, this contradicts $\mathrm{wsat}(K_n, K_s) = (s-2)n - \binom{s-1}{2}$, since $\mathbb{G}(n,p) - e$ is a weakly K_s-saturated subgraph of K_n *whp*. To see this, note that the edge e can be activated by going through two steps *whp* by Lemma 3.3 and moreover $\mathbb{G}(n,p)$ is a weakly K_s-saturated subgraph of K_n *whp* by Theorem 2.11. □

For every positive integers r and s, we say a graph Γ has the property $\mathscr{D}_{r,s}$ if for any subset $X \subseteq V(\Gamma)$ of size r, $N_\Gamma(X)$ contains a clique of size s. The following result is a special case of Theorem 2 of [18].

Lemma 3.5. *Let r and s be positive fixed integers. Then, there exists a constant $d_{r,s}$ such that for any*

$$p \geqslant d_{r,s} n^{-\frac{2}{2r+s-1}} (\log n)^{\frac{2}{s(2r+s-1)}},$$

the property $\mathscr{D}_{r,s}$ holds in $\mathbb{G}(n,p)$ whp.

Note that the lower bound given in Lemma 3.5 for the property $\mathscr{D}_{2,s-2}$ is equal to $q_s(n)$ up to a constant. We will make use the following lemma twice in the next theorem.

Lemma 3.6. *Let $s \geqslant 3$ be a fixed integer and let $p = a_s q_s(1 + w(n))$ for some function $w = O(1)$. For given vertices u and v of $\mathbb{G}(n,p)$, let the random variable X count the number of cliques of size $s-2$ in $N_{\mathbb{G}(n,p)}(u,v)$. Then,*

$$\mathbb{E}[X] = \frac{2s}{s+1}\log n + s(s-2)w(n)\log n + O\left(\log n\left(w(n)^2 + \frac{1}{n}\right)\right)$$

and $\Delta_X = \mathbb{E}[X]o(n^{-1/(s+1)}\log n)$, where Δ_X is introduced in Theorem 2.7.

Proof. For the sake of simplification, let $\lambda = \mathbb{E}[X]$ and $\Delta = \Delta_X$. We have

$$\begin{aligned}
\lambda &= \binom{n-2}{s-2} p^{\frac{(s-2)(s+1)}{2}} \\
&= \frac{n^{s-2}}{(s-2)!}\left(1 + O\left(\frac{1}{n}\right)\right)\frac{2s}{s+1}(s-2)!n^{-(s-2)}\log n\big(1+w(n)\big)^{\frac{(s-2)(s+1)}{2}} \\
&= \frac{2s}{s+1}\log n\left(1 + \frac{(s-2)(s+1)}{2}w(n) + O\big(w(n)^2\big)\right) + O\left(\frac{\log n}{n}\right) \\
&= \frac{2s}{s+1}\log n + s(s-2)w(n)\log n + O\left(\log n\left(w(n)^2 + \frac{1}{n}\right)\right),
\end{aligned}$$

as desired. If $s = 3$, then $\Delta = 0$, there is nothing to prove. For $s \geqslant 4$, we have

$$\begin{aligned}
\Delta &= \sum_{\ell=1}^{s-3}\binom{n-2}{s-2}\binom{s-2}{\ell}\binom{n-s}{s-2-\ell}p^{(s-2)(s+1)-\frac{\ell(\ell+3)}{2}} \\
&= \lambda\sum_{\ell=1}^{s-3}\binom{s-2}{\ell}\binom{n-s}{s-2-\ell}p^{\frac{(s-2)(s+1)-\ell(\ell+3)}{2}} \\
&= \lambda\sum_{\ell=1}^{s-3}\frac{(s-2)!}{\ell!(s-\ell-2)!^2}n^{s-\ell-2}\left(1 + O\left(\frac{1}{n}\right)\right)(a_s q_s)^{\frac{(s-2)(s+1)-\ell(\ell+3)}{2}}\Big(1 + O\big(w(n)\big)\Big) \\
&= \lambda\sum_{\ell=1}^{s-3} n^{-\frac{\ell(s-\ell-2)}{s+1}}\, o\left(\log n\right) \\
&= \lambda o\left(n^{-\frac{1}{s+1}}\log n\right),
\end{aligned}$$

as required. ∎

Theorem 3.7. *For any fixed integer $s \geqslant 3$, there is a threshold for the property $\mathscr{A}_s$ in $\mathbb{G}(n,p)$.*

Proof. Clearly, the property $\mathscr{A}_s \cap \mathscr{D}_{2,s-2}$ is increasing and so it has a threshold r_s by Theorem 2.8. We prove that r_s is a threshold for the property $\mathscr{A}_s$.

If $p \gg r_s$, then the definition of r_s shows that $\mathbb{G}(n,p)$ has the property $\mathscr{A}_s$ *whp*. It remains to prove the opposite for $p \ll r_s$. For the sake of convenience, we let $p' = a_s q_s$ and $p'' = d_{2,s-2} q_s$. Also, consider an auxiliary function

$$q(n) = p'(n)\left(1 + \frac{1}{\sqrt{\log n}\log\log n}\right).$$

For given vertices u and v, denote by X_s the random variable that counts the number of cliques of size $s-2$ in $N_{\mathbb{G}(n,p)}(u,v)$. From Theorem 2.7, we have

$$\mathbb{P}\Big[X_s = 0\Big] \leqslant \exp\left(-\mathbb{E}\Big[X_s\Big] + \frac{\varDelta_{X_s}}{2}\right). \tag{4}$$

We distinguish the following four cases.

Case 3.8. *Assume that the set N_1 consisting of all positive integers n with $p(n) \leqslant p'(n)$ is infinite.*

Proof. It follows from Theorem 3.4 that $\mathbb{P}[\mathbb{G}(n,p) \in \mathscr{A}_s] \to 0$ when n runs over N_1. $\diamondsuit$

Case 3.9. *Assume that the set N_2 consisting of all positive integers n with $p(n) \geqslant p''(n)$ is infinite.*

Proof. It follows from $p \ll r_s$ and Lemma 3.5 that

$$\mathbb{P}\Big[\mathbb{G}(n,p) \in \mathscr{A}_s\Big] \leqslant \mathbb{P}\Big[\mathbb{G}(n,p) \in \mathscr{A}_s \cap \mathscr{D}_{2,s-2}\Big] + \mathbb{P}\Big[\mathbb{G}(n,p) \notin \mathscr{D}_{2,s-2}\Big] \longrightarrow 0$$

when n runs over N_2. $\diamondsuit$

Case 3.10. *Assume that the set N_3 consisting of all positive integers n with $q(n) \leqslant p(n) < p''(n)$ is infinite.*

Proof. In the following argument, we assume that n comes from N_3. As $p < p''$, by the coupling technique, $\mathbb{G}(n,p'')$ can be obtained by superimposing $\mathbb{G}(n,p)$ and $\mathbb{G}(n,(p''-p)/(1-p))$ and replacing eventual double edges by a single one. Formally, it is written as $\mathbb{G}(n,p'') = \mathbb{G}(n,p) \cup \mathbb{G}(n,(p''-p)/(1-p))$. Denote by $\mathscr{E}_s$ the property that there exists an edge xy in $\mathbb{G}(n,p'') \setminus \mathbb{G}(n,p)$ such that $N_{\mathbb{G}(n,p)}(x,y)$ does not contain a clique of size $s-2$. Using Lemma 3.6, $\mathbb{E}[X_s] \geqslant \mathbb{E}[Y_s] = \frac{2s}{s+1}\log n + s(s-2)\sqrt{\log n}/\log\log n + o(1)$ and $\varDelta_{X_s} = o(1)$, where the random variable Y_s counts the number of cliques of size $s-2$ in $N_{\mathbb{G}(n,q)}(u,v)$ for given vertices u and v. By (4) and the union bound, we get

$$\begin{aligned}
\mathbb{P}\Big[\mathscr{E}_s\Big] &\leqslant \binom{n}{2}\frac{p''-p}{1-p}\exp\left(-\mathbb{E}\Big[X_s\Big] + \frac{\varDelta_{X_s}}{2}\right)\\
&\leqslant n^2\Big(p''-p'\Big)\exp\left(-\frac{2s}{s+1}\log n - s(s-2)\frac{\sqrt{\log n}}{\log\log n} + o(1)\right)\\
&= n^2 O\Big(q_s\Big)\exp\left(-\frac{2s}{s+1}\log n - s(s-2)\frac{\sqrt{\log n}}{\log\log n} + o(1)\right)\\
&= O\left((\log n)^{\frac{2}{(s-2)(s+1)}}\right)\exp\left(-s(s-2)\frac{\sqrt{\log n}}{\log\log n}\right)\Big(1+o(1)\Big)
\end{aligned}$$

$$= o(1).$$

Since $p \ll r_s$ and p', p'' differ by a constant factor, we find that $p'' \ll r_s$. So, it follows from Lemma 3.5 that

$$\mathbb{P}\Big[\mathbb{G}(n,p'') \in \mathscr{A}_s\Big] \leqslant \mathbb{P}\Big[\mathbb{G}(n,p'') \in \mathscr{A}_s \cap \mathscr{D}_{2,s-2}\Big] + \mathbb{P}\Big[\mathbb{G}(n,p'') \notin \mathscr{D}_{2,s-2}\Big] \longrightarrow 0$$

when n runs over N_3. Hence, $\mathbb{P}[\mathbb{G}(n,p) \in \mathscr{A}_s] \leqslant \mathbb{P}[\mathbb{G}(n,p'') \in \mathscr{A}_s] + \mathbb{P}[\mathscr{E}_s] \to 0$ when n runs over N_3. $\diamondsuit$

Case 3.11. *Assume that the set N_4 consisting of all positive integers n with $p'(n) < p(n) < q(n)$ is infinite.*

Proof. In the following argument, we assume that n comes from N_4. As $p < q$, by the coupling technique, $\mathbb{G}(n,q)$ can be obtained by superimposing $\mathbb{G}(n,p)$ and $\mathbb{G}(n,(q-p)/(1-p))$ and replacing eventual double edges by a single one. Formally, it is written as $\mathbb{G}(n,q) = \mathbb{G}(n,p) \cup \mathbb{G}(n,(q-p)/(1-p))$. Let $\mathscr{E}_s$ be the property that there exists an edge xy in $\mathbb{G}(n,q) \setminus \mathbb{G}(n,p)$ such that $N_{\mathbb{G}(n,p)}(x,y)$ does not contain a clique of size $s-2$. Using Lemma 3.6, $\mathbb{E}[X_s] \geqslant \mathbb{E}[Y_s] = \frac{2s}{s+1}\log n + o(1)$ and $\Delta_{X_s} = o(1)$, where the random variable Y_s counts the number of cliques of size $s-2$ in $N_{\mathbb{G}(n,p')}(u,v)$ for given vertices u and v. By (4) and the union bound, we get

$$\begin{aligned}
\mathbb{P}\Big[\mathscr{E}_s\Big] &\leqslant \binom{n}{2}\frac{q-p}{1-p}\exp\Big(-\mathbb{E}\big[X_s\big] + \frac{\Delta_{X_s}}{2}\Big)\\
&\leqslant n^2\big(q-p'\big)\exp\Big(-\frac{2s}{s+1}\log n + o(1)\Big)\\
&= \frac{n^2 p'}{\sqrt{\log n}\log\log n}\exp\Big(-\frac{2s}{s+1}\log n + o(1)\Big)\\
&= \frac{a_s}{\log\log n}(\log n)^{-\frac{(s-3)(s+2)}{2(s-2)(s+1)}}\big(1+o(1)\big)\\
&= o(1).
\end{aligned}$$

We have $q \ll r_s$ and $\mathbb{P}[\mathbb{G}(n,p) \in \mathscr{A}_s] \leqslant \mathbb{P}[\mathbb{G}(n,q) \in \mathscr{A}_s] + \mathbb{P}[\mathscr{E}_s]$. Thus, if $\mathbb{P}[\mathbb{G}(n,p) \in \mathscr{A}_s] \not\longrightarrow 0$ when n runs over N_4, then $\mathbb{P}[\mathbb{G}(n,q) \in \mathscr{A}_s] \not\longrightarrow 0$ when n runs over N_4, contradicting Case 3.10. This shows that $\mathbb{P}[\mathbb{G}(n,p) \in \mathscr{A}_s] \to 0$ when n runs over N_4. $\diamondsuit$

From Cases 3.8–3.11, we deduce that $\mathbb{P}[\mathbb{G}(n,p) \in \mathscr{A}_s] \to 0$ as $n \to 0$, completing the proof. $\square$

4. An upper bound on the threshold

In this section, we present an upper bound on the threshold for the property $\mathscr{A}_s$. Let us first recall the following definition. For any positive integer k, the *k-th power* of a graph Γ, denoted by Γ^k, is the graph with vertex set $V(\Gamma)$ such that two distinct vertices x, y are adjacent in Γ^k if and only if the distance between x, y in Γ is at most k. We need the following result on the threshold of the appearance of the k-th power of a Hamiltonian cycle.

Theorem 4.1 ([11, 16, 17]). *There exists a constant $c > 0$ such that, if $p \geqslant \frac{c\log n}{n}$, then $\mathbb{G}(n,p)$ contains a Hamiltonian cycle whp. For every integer $k \geqslant 2$, if $p \gg n^{-1/k}$, then $\mathbb{G}(n,p)$ contains the k-th power of a Hamiltonian cycle whp.*

As a consequence of Theorem 4.1, we have the following.

Corollary 4.2. *For every positive fixed integers k and r, if $p \gg n^{-1/k}(\log n)^{\sigma_{k+2}}$, then $\mathbb{G}(n,p)$ contains the k-th power of a Hamiltonian path with probability at least $1-\frac{1}{n^r}$ for all sufficiently large n, where σ_{k+2} can be determined from (2).*

Proof. By Theorem 4.1, if $\breve{p} \gg n^{-1/k}(\log n)^{\sigma_{k+2}-1}$, then $\mathbb{G}(n,\breve{p})$ contains the k-th power of a Hamiltonian path with probability at least $1-\frac{1}{e}$ for all sufficiently large n. To boost this probability to $1-\frac{1}{n^r}$, it suffices to apply the coupling technique by taking the union of $\lceil r\log n\rceil$ independent copies of $\mathbb{G}(n,\breve{p})$ on vertex set $[\![n]\!]$ and letting $p=\lceil r\log n\rceil\breve{p}$. □

We say a graph Γ has the property $\mathscr{F}_s$ if for any subset $S\subseteq V(\Gamma)$ of size $s-1$, $\Gamma[N_\Gamma(S)]$ has at least $s-1$ vertices and contains the $(s-2)$-th power of a Hamiltonian path.

Lemma 4.3. *Let $s\geqslant 3$ and $n\geqslant s-2$. Assume that both properties $\mathscr{D}_{s,s-2}$ and $\mathscr{F}_s$ hold for a graph G on n vertices. Then, $\mathrm{wsat}(G,K_s)\leqslant (s-2)n-\binom{s-1}{2}$.*

Proof. If $n\in\{s-2,s-1\}$, then the result is clearly valid. Let $n\geqslant s$ and let Ω be a clique of size $s-2$ in G. We define a spanning subgraph H of G as follows. The graph H contains all edges of G with both endpoints in Ω and also all edges of G with endpoints in both Ω and $N_G(\Omega)$. We still have to add to H some other edges going outside $N_G[\Omega]$. For every $v\in V(G)\setminus N_G[\Omega]$, we add $s-2$ edges of G adjacent to v described below. Since G satisfies $\mathscr{F}_s$, the graph $H_v=G[N_G(\{v\}\cup\Omega)]$ has at least $s-1$ vertices and contains the $(s-2)$-th power of a Hamiltonian path. Beginning from a starting vertex, denote the vertices of H_v going in the natural order induced by the Hamiltonian path by $x^v_1,\ldots,x^v_{h_v}$, where $h_v=|V(H_v)|$. Note that $h_v\geqslant s-1$. We add the edges $vx^v_1,\ldots,vx^v_{s-2}$ to H for any $v\in V(G)\setminus N_G[\Omega]$. It is easy to see that H is of size $(s-2)n-\binom{s-1}{2}$, so it suffices to prove that H is a weakly K_s-saturated subgraph of G.

First, all edges with both endpoints in $N_G(\Omega)$ can be activated, since they belong to a K_s containing Ω. Next, for each $v\in V(G)\setminus N_G[\Omega]$, we may activate the edges $vx^v_{s-1},\ldots,vx^v_{h_v}$ one by one, since every such edge belongs to a K_s containing the previous $s-2$ vertices of the $(s-2)$-th power of the Hamiltonian path. Finally, each edge xy with $x,y\in V(G)\setminus N_G[\Omega]$ can be activated. To see this, note that $N_G(\{x,y\}\cup\Omega)$ contains a clique of size $s-2$, say Ω_{xy}, since G satisfies $\mathscr{D}_{s,s-2}$. It follows from $\Omega_{xy}\subseteq N_G(\Omega)$ that the edges with both endpoints in Ω_{xy} are already activated and so xy is the last edge of the $\{x,y\}\cup\Omega_{xy}$ of size s and can be activated as well. □

We define here an event in $\mathbb{G}(n,p)$ to use in later proofs. For each subset U of vertices, let $\mathscr{G}_U$ be the event that $\mathbb{G}(n,p)[U]$ does not contain the $(s-2)$-th power of a Hamiltonian path.

Lemma 4.4. *Let $s\geqslant 3$ be a fixed integer and let $p\gg n^{-1/(2s-3)}(\log n)^{(s+\sigma_s-3)/(2s-3)}$. Then,*

$$\mathbb{P}\Big[\mathbb{G}(n,p)\text{ has both properties }\mathscr{D}_{s,s-2}\text{ and }\mathscr{F}_s\Big]\longrightarrow 1$$

as $n\to\infty$.

Proof. By Lemma 3.5, the property $\mathscr{D}_{s,s-2}$ holds in $\mathbb{G}(n,p)$ *whp* when

$$p\gg n^{-\frac{2}{3(s-1)}}(\log n)^{\frac{2}{3(s-2)(s-1)}}.$$

Therefore, by the assumption on p, we deduce that $\mathbb{P}[\mathbb{G}(n,p)$ has the property $\mathscr{D}_{s,s-2}]\to 1$ as $n\to\infty$. Below, we explore the behavior of $\mathbb{P}[\mathbb{G}(n,p)$ has the property $\mathscr{F}_s]$.

Fix $W \subseteq [\![n]\!]$ of size $s-1$ and set $m = np^{s-1}/2$. By Theorem 2.5 and by applying Corollary 4.2 after substituting k, r, n with $s-2$, $3s$, m, respectively, we derive that

$$\begin{aligned}
\mathbb{P}\left[\mathscr{G}_{N_{\mathbb{G}(n,p)}(W)}\right] &\leqslant \mathbb{P}\Big[\Big|N_{\mathbb{G}(n,p)}(W)\Big| \leqslant m\Big] + \sum_{\substack{U \subseteq [\![n]\!] \\ |U| \geqslant m}} \mathbb{P}\Big[U = N_{\mathbb{G}(n,p)}(W)\Big]\mathbb{P}\Big[\mathscr{G}_U\Big] \\
&\leqslant \exp\left(-\frac{m}{4}\right) + m^{-3s} \sum_{\substack{U \subseteq [\![n]\!] \\ |U| \geqslant m}} \mathbb{P}\Big[U = N_{\mathbb{G}(n,p)}(W)\Big] \\
&= \exp\left(-\frac{m}{4}\right) + m^{-3s}\mathbb{P}\Big[\Big|N_{\mathbb{G}(n,p)}(W)\Big| \geqslant m\Big] \\
&\leqslant \exp\left(-\frac{m}{4}\right) + m^{-3s}.
\end{aligned}$$

As $m \geqslant n^{1/3}$ for all sufficiently large n, it follows from the union bound that the probability that $\mathbb{G}(n,p)$ does not have the property $\mathscr{F}_s$ is at most

$$\sum_{|W|=s-1} \left(\exp\left(-\frac{m}{4}\right) + m^{-3s}\right) \leqslant \exp\left((s-1)\log n - \frac{n^{\frac{1}{3}}}{4}\right) + \frac{1}{n} = o(1).$$

This means that $\mathbb{P}[\mathbb{G}(n,p)$ has the property $\mathscr{F}_s] \to 1$ as $n \to \infty$, completing the proof. $\square$

Korándi and Sudakov [12] proved, if p is a constant probability and $s \geqslant 3$ is a fixed integer, then $\mathrm{wsat}(\mathbb{G}(n,p), K_s) = (s-2)n - \binom{s-1}{2}$ *whp*. The following theorem generalizes their result.

Theorem 4.5. *Let $s \geqslant 3$ be a fixed integer and let $p \gg n^{-1/(2s-3)}(\log n)^{(s+\sigma_s-3)/(2s-3)}$. Then, $\mathbb{G}(n,p)$ has the property $\mathscr{A}_s$ whp.*

Proof. By Theorem 2.11, $\mathbb{G}(n,p)$ is a weakly K_s-saturated subgraph of K_n *whp* and thus we get $\mathrm{wsat}(\mathbb{G}(n,p), K_s) \geqslant \mathrm{wsat}(K_n, K_s) = (s-2)n - \binom{s-1}{2}$ *whp*. It remains to prove that there exists a weakly K_s-saturated subgraph of $\mathbb{G}(n,p)$ of size at most $(s-2)n - \binom{s-1}{2}$ *whp*. So, the result immediately follows from Lemmas 4.3 and 4.4. $\square$

We point out here that Theorem 1.1 is concluded from Theorems 3.4, 3.7, and 4.5.

5. An upper bound on $\mathrm{wsat}(\mathbb{G}(n,p), K_s)$

By considering what we did in the first paragraph of the proof of Lemma 3.1, one obtains that $\mathrm{wsat}(\mathbb{G}(n,p), K_s) = e(\mathbb{G}(n,p))(1+o(1))$ *whp* when $p \ll n^{-2/(s+1)}$. Also, it follows from Theorem 4.5 that $\mathrm{wsat}(\mathbb{G}(n,p), K_s) = (s-2)n - \binom{s-1}{2}$ *whp* when $p \gg n^{-1/(2s-3)}(\log n)^{(s+\sigma_s-3)/(2s-3)}$. In this section, we prove Theorem 1.2 which gives an upper bound on $\mathrm{wsat}(\mathbb{G}(n,p), K_s)$ for all the remaining values of p. It is worth mentioning that $\mathrm{wsat}(\mathbb{G}(n,p), K_s) \geqslant (s-2)n - \binom{s-1}{2}$ *whp*, since $\mathbb{G}(n,p)$ is a weakly K_s-saturated subgraph of K_n *whp* for those values of p by Theorem 2.11.

Lemma 5.1. *Let $s \geqslant 3$, G be a graph, and W be a subset of $V(G)$ having the following two properties.*

(i) *For any vertex $u \in V(G) \setminus W$, $G[N_G(u) \cap W]$ has at least $s-1$ vertices and contains the $(s-2)$-th power of a Hamiltonian path.*

(ii) *For every two distinct vertices $u, v \in V(G) \setminus W$, $N_G(u,v) \cap W$ contains a clique of size $s-2$.*

Then, $\mathrm{wsat}(G, K_s) \leqslant e(G[W]) + (s-2)|V(G) \setminus W|$.

Proof. According to (i), for each $u \in V(G)\setminus W$, we may fix a Hamiltonian path P_u of $G[N_G(u)\cap W]$ such that $P_u^{s-2} \subset G[N_G(u) \cap W]$. Let H be a spanning subgraph of G containing all edges with both endpoints in W and also all $(s-2)|V(G) \setminus W|$ edges in

$$\left\{ uv \;\middle|\; \begin{array}{l} u \in V(G) \setminus W \text{ and } v \text{ is one of the } s-2 \text{ initial} \\ \text{vertices of } P_u \text{ beginning from a starting vertex} \end{array} \right\}.$$

Let us show that H is a weakly K_s-saturated subgraph of G. Note that all edges with both endpoints in W are initially activated. Let $u \in V(G) \setminus W$. Beginning from a starting vertex, denote the vertices of $G[N_G(u) \cap W]$ going in the natural order induced by P_u by $v_1, \ldots, v_{g_u}$, where $g_u = |N_G(u) \cap W|$. Property (i) ensures that $g_u \geqslant s-1$. The edges $uv_1, \ldots, uv_{s-2}$ are initially activated and so we may active the edges $uv_{s-1}, \ldots, uv_{g_v}$ one by one, since every such edge belongs to a K_s containing the previous $s-2$ vertices of P_u^{s-2}. Hence, all edges of G going out of W are now activated. Let $u, v \in V(G) \setminus W$ be adjacent in G. Property (ii) ensures that the edge uv belongs to a K_s whose other edges are already activated and so the edge uv can be activated as well. This shows that H is a weakly K_s-saturated subgraph of G which implies that

$$\mathrm{wsat}(G, K_s) \leqslant e\big(G[W]\big) + (s-2)\big|V(G) \setminus W\big|. \qquad \square$$

Proof of Theorem 1.2. For the purpose of simplification, set $f(n) = (\log n)^{s+\sigma_s-3}\sqrt{w(n)}$. If $p \leqslant 1/(n \log n)$, then Theorem 2.1 yields that $\mathrm{wsat}(\mathbb{G}(n,p), K_s) \leqslant e(\mathbb{G}(n,p)) \leqslant n/\sqrt{\log n}$ *whp* which results in the assertion. Also, if $1/(n \log n) \leqslant p \leqslant (f/n)^{1/(s-1)}$, then $n^2 p \to \infty$ as $n \to \infty$ and therefore, it follows from Theorem 2.3 that $\mathrm{wsat}(\mathbb{G}(n,p), K_s) \leqslant e(\mathbb{G}(n,p)) \leqslant n^2 p \leqslant f^2/p^{2s-3}$ *whp* which results in the assertion. Furthermore, the result follows from Theorem 4.5 when $p \gg n^{-1/(2s-3)}(\log n)^{(s+\sigma_s-3)/(2s-3)}$. So, we may assume that

$$\left(\frac{f}{n}\right)^{\frac{1}{s-1}} \leqslant p \leqslant n^{-\frac{1}{2s-3}} (\log n)^{\frac{s+\sigma_s-2}{2s-3}}. \tag{5}$$

Now, let $G \sim \mathbb{G}(n,p)$ and $W = [\![m]\!]$, where $m = \lfloor f/p^{s-1} \rfloor$. Note that, it follows from (5) that $m \leqslant n$ and moreover, $m^2 p \to \infty$ as $n \to \infty$. Hence, $e(G[W]) \leqslant m^2 p$ *whp* by Theorem 2.3 and so $e(G[W]) \leqslant f^2/p^{2s-3}$ *whp*. From the latter inequality and in view of Lemma 5.1, it remains to show that the properties (i) and (ii) in the statement of Lemma 5.1 hold *whp* for G and our choice of W.

Set $g = \lfloor mp/2 \rfloor$. For a given vertex $u \in V(G) \setminus W$, using Theorem 2.5 and (5), we may write

$$\mathbb{P}\Big[|N_G(u) \cap W| \leqslant g\Big] \leqslant \exp\left(-\frac{\big(mp-g\big)^2}{2mp}\right) \leqslant \exp\left(-\frac{f}{8p^{s-2}}\right) \leqslant \exp\left(-n^{\frac{s-2}{2s-3}}\right)$$

when n is sufficiently large. Noting that $w(n) \to \infty$ as $n \longrightarrow \infty$ and by applying Corollary 4.2 after substituting k, r, n with $s-2$, 4, g, respectively, we derive that

$$\mathbb{P}\Big[\mathscr{G}_{N_G(u)\cap W}\Big] \leqslant \mathbb{P}\Big[|N_G(u) \cap W| \leqslant g\Big] + \sum_{\substack{U \subseteq [\![n]\!] \\ |U| \geqslant g}} \mathbb{P}\Big[U = N_G(u) \cap W\Big]\mathbb{P}\Big[\mathscr{G}_U\Big]$$

$$\leqslant \exp\left(-n^{\frac{s-2}{2s-3}}\right) + g^{-4} \sum_{\substack{U \subseteq [\![n]\!] \\ |U| \geqslant g}} \mathbb{P}\Big[U = N_G(u) \cap W\Big]$$

$$= \exp\left(-n^{\frac{s-2}{2s-3}}\right) + g^{-4}\mathbb{P}\Big[|N_G(u) \cap W| \geqslant g\Big]$$

$$\leqslant \exp\left(-n^{\frac{s-2}{2s-3}}\right) + \frac{(\log n)^{2s}}{n^{\frac{4s-8}{2s-3}}}$$

for all large enough n. Hence, if n is large enough, then

$$\sum_{u \in V(G) \setminus W} \mathbb{P}\Big[\mathscr{G}_{N_G(u) \cap W}\Big] \leqslant n \exp\left(-n^{\frac{s-2}{2s-3}}\right) + \frac{(\log n)^{2s}}{n^{\frac{2s-5}{2s-3}}} = o(1).$$

Therefore, (i) holds in G *whp* using the union bound.

Finally, concerning (ii), let $\mathscr{H}_s$ be the event that there exist two distinct vertices $u, v \in V(G) \setminus W$ such that $N_G(u,v) \cap [\![h]\!]$ does not contain a clique of size $s-2$, where $h = \lfloor (\log n)^2 / p^{(s+1)/2} \rfloor$. Note that $h \leqslant m$ for all sufficiently large n. From Theorem 2.7, we have

$$\mathbb{P}\Big[X_s = 0\Big] \leqslant \exp\left(-\mathbb{E}[X_s] + \frac{\varDelta_{X_s}}{2}\right), \tag{6}$$

where the random variable X_s counts the number of cliques of size $s-2$ in $N_{\mathbb{G}(n,p)}(u,v) \cap [\![h]\!]$. In view of (5) and for some constants i_s and j_s, we may write

$$\begin{aligned}
\varDelta_{X_s} &= \sum_{\ell=1}^{s-3} \binom{h}{s-2} \binom{s-2}{\ell} \binom{h-s+2}{s-2-\ell} p^{(s-2)(s+1) - \frac{\ell(\ell+3)}{2}} \\
&= \binom{h}{s-2} p^{\frac{(s-2)(s+1)}{2}} \sum_{\ell=1}^{s-3} \binom{s-2}{\ell} \binom{h-s+2}{s-2-\ell} p^{\frac{(s-2)(s+1) - \ell(\ell+3)}{2}} \\
&= \mathbb{E}[X_s] \sum_{\ell=1}^{s-3} \frac{(s-2)!}{\ell!(s-\ell-2)!^2} h^{s-\ell-2} \left(1 + O\left(\frac{1}{h}\right)\right) p^{\frac{(s-2)(s+1) - \ell(\ell+3)}{2}} \\
&= \mathbb{E}[X_s] O\left((\log n)^{i_s}\right) \sum_{\ell=1}^{s-3} p^{\frac{\ell(s-\ell-2)}{2}} \\
&= \mathbb{E}[X_s] O\left((\log n)^{j_s}\right) n^{-\frac{s-3}{2(2s-3)}} \\
&= o\Big(\mathbb{E}[X_s]\Big).
\end{aligned}$$

Now, by (6) and the union bound, we get

$$\begin{aligned}
\mathbb{P}\Big[\mathscr{H}_s\Big] &\leqslant \binom{n}{2} \exp\left(-\binom{h}{s-2} p^{\binom{s}{2}-1} \Big(1 + o(1)\Big)\right) \\
&\leqslant \exp\left(2\log n - \frac{h^{s-2} p^{\binom{s}{2}-1}}{(s-2)!} \left(1 + O\left(\frac{1}{h}\right)\right) \Big(1 + o(1)\Big)\right) \\
&\leqslant \exp\left(2\log n - \frac{(\log n)^{2(s-2)}}{(s-2)!} \Big(1 + o(1)\Big)\right) \\
&= o(1).
\end{aligned}$$

This shows that (ii) occurs in G *whp* and completes the proof. $\square$

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References

[1] J. Adler and U. Lev, Bootstrap percolation: visualizations and applications, *Braz. J. Phys.* **33** (2003), 641–644.

[2] J. Balogh, B. Bollobás, and R. Morris, Graph bootstrap percolation, *Random Structures Algorithms* **41** (2012), 413–440.

[3] B. Bollobás, Random graphs, in: Combinatorics (Swansea, 1981), London Math. Soc. Lecture Note Ser., vol. **52**, Cambridge University Press, Cambridge-New York, 1981, pp. 80–102.

[4] B. Bollobás, Threshold functions for small subgraphs, Math. Proc. Camb. Phil. Soc. **90** (1981), 197–206.

[5] B. Bollobás, *Graph Theory — An Introductory Course*, Graduate Texts in Mathematics, **63**, Springer-Verlag, New York-Berlin, 1979.

[6] B. Bollobás, Weakly k-saturated graphs, Beiträge zur Graphentheorie (Kolloquium, Manebach, 1967), Teubner, Leipzig, 1968, pp. 25–31.

[7] B. Bollobás and A. Thomason, Threshold functions, *Combinatorica* **7** (1987), 35–38.

[8] L.R. Fontes, R.H. Schonmann, and V. Sidoravicius, Stretched exponential fixation in stochastic Ising models at zero temperature, *Comm. Math. Phys.* **228** (2002), 495–518.

[9] C.M. Fortuin, P.W. Kasteleyn, and J. Ginibre, Correlation inequalities on some partially ordered sets, *Comm. Math. Phys.* **22** (1971), 89–103.

[10] S. Janson, T. Łuczak, and A. Ruciński, *Random Graphs*, Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley-Interscience, New York, 2000.

[11] J. Kahn, B. Narayanan, and J. Park, The threshold for the square of a Hamilton cycle, *Proc. Amer. Math. Soc.* **149** (2021), 3201–3208.

[12] D. Korándi and B. Sudakov, Saturation in random graphs, *Random Structures Algorithms* **51** (2017), 169–181.

[13] L. Lovász, Flats in matroids and geometric graphs, in: Combinatorial surveys (Proc. Sixth British Combinatorial Conf., Royal Holloway Coll., Egham, 1977), Academic Press, London, 1977, pp. 45–86.

[14] R. Morris, Zero-temperature Glauber dynamics on $\mathbb{Z}^d$, *Probab. Theory Related Fields* **149** (2011), 417–434.

[15] J. von Neumann, *Theory of Self-Reproducing Automata*, Univ. Illinois Press, Urbana, 1966.

[16] L. Pósa, Hamiltonian circuits in random graphs, *Discrete Math.* **14** (1976), 359–364.

[17] O. Riordan, Spanning subgraphs of random graphs, *Combin. Probab. Comput.* **9** (2000), 125–148.

[18] J. Spencer, Threshold functions for extension statements, *J. Combin. Theory Ser. A* **53** (1990), 286–305.

Appendix A. Proof of Lemma 3.2

It could be that Lemma 3.2 is a known result in the literature. Since we could not find any reference, we include its proof here.

Proof of Lemma 3.2. Let X_s be the number of edges of $\mathbb{G}(n,p)$ that do not belong to any K_s. For every two distinct vertices $u,v \in [\![n]\!]$ and every subset $W \subseteq [\![n]\!] \setminus \{u,v\}$ of size $s-2$, consider the event $K[W]$ saying that W is a clique in $N_{\mathbb{G}(n,p)}(u,v)$. Let $\mu(u,v)$ count the number of subsets W as above such that $K[W]$ occurs. We have

$$\mathbb{E}[X_s] = \binom{n}{2} p \mathbb{P}\big[\mu(1,2) = 0\big]. \tag{7}$$

Let

$$\lambda = \sum_{W \in \binom{[\![n]\!] \setminus \{u,v\}}{s-2}} \mathbb{P}\big[K[W]\big] = \binom{n-2}{s-2} p^{\frac{(s-2)(s+1)}{2}} \tag{8}$$

and

$$\begin{aligned} \varDelta &= \sum_{\substack{W_1, W_2 \in \binom{[\![n]\!] \setminus \{u,v\}}{s-2} \\ W_1 \neq W_2 \\ W_1 \cap W_2 \neq \varnothing}} \mathbb{P}\big[K[W_1] \cap K[W_2]\big] \\ &= \sum_{\ell=1}^{s-3} \binom{n-2}{s-2} \binom{s-2}{\ell} \binom{n-s}{s-2-\ell} p^{(s-2)(s+1) - \frac{\ell(\ell+3)}{2}}. \end{aligned} \tag{9}$$

We have

$$\begin{aligned} \lambda &= \frac{n^{s-2}}{(s-2)!} \left(1 + O\left(\frac{1}{n}\right)\right) \frac{2s}{s+1} (s-2)! n^{-(s-2)} (\log n) h(n)^{\frac{(s-2)(s+1)}{2}} \\ &= \frac{2s \log n}{s+1} h(n)^{\frac{(s-2)(s+1)}{2}} + O\left(\frac{\log n}{n}\right) \end{aligned} \tag{10}$$

and thus

$$2\log n + \log p - \lambda = 2\log n + \left(-\frac{2\log n}{s+1} + \frac{2\log\log n}{(s-2)(s+1)} + \log\big(h(n)\big)\right)$$

$$
\begin{aligned}
&- \left(\frac{2s\log n}{s+1} h(n)^{\frac{(s-2)(s+1)}{2}}\right) + O(1) \\
&= \frac{2\log\log n}{(s-2)(s+1)} + \log\Big(h(n)\Big) - \frac{2s\log n}{s+1}\left(h(n)^{\frac{(s-2)(s+1)}{2}} - 1\right) + O(1). \qquad (11)
\end{aligned}
$$

By applying Lemma 3.5, we may assume that $h(n) = O(1)$. Since $\lambda = O(\log n)$ by (10) and $\varDelta = \lambda o(n^{-1/(s+1)}\log n)$ by Lemma 3.6, we find from Theorem 2.7 that

$$
\begin{aligned}
\mathbb{E}[X_s] &\leqslant \binom{n}{2} p \exp\left(-\lambda + \frac{\varDelta}{2}\right) \qquad (12)\\
&= \exp\Big(2\log n + \log p - \lambda + o(1)\Big). \qquad (13)
\end{aligned}
$$

Let $w(n) \to \infty$ as $n \to \infty$. As $h(n) = O(1)$, we obtain from (11) that

$$
\begin{aligned}
2\log n + \log p - \lambda &\leqslant \frac{2\log\log n}{(s-2)(s+1)} - \frac{2s\log n}{s+1}\left(\frac{\log\log n}{s(s-2)\log n} + \frac{(s-2)(s+1)w(n)}{2\log n}\right) + O(1) \\
&= -s(s-2)w(n) + O(1) \longrightarrow -\infty \qquad (14)
\end{aligned}
$$

as $n \to \infty$. Hence, it follows from (13) and (14) that $\mathbb{E}[X_s] = o(1)$ and so Part (i) of Lemma 3.2 follows from Corollary 2.2.

In order to prove Part (ii) of Lemma 3.2, let $w(n) \to -\infty$ as $n \to \infty$. It is easy to check that $1 - x \geqslant \exp(-x - 2x^2)$ for each real number $x \in [0, 1/4]$. Using this fact and since $\lambda = O(\log n)$, we find from (8) and Theorem 2.6 that

$$
\begin{aligned}
\mathbb{P}\Big[\mu(1,2) = 0\Big] &\geqslant \prod_{W \in \binom{[\![n]\!]\setminus\{u,v\}}{s-2}} \Big(1 - \mathbb{P}\big[K[W]\big]\Big) \\
&= \left(1 - p^{\frac{(s-2)(s+1)}{2}}\right)^{\binom{n-2}{s-2}} \\
&\geqslant \exp\left(\binom{n-2}{s-2}\left(-p^{\frac{(s-2)(s+1)}{2}} - 2p^{(s-2)(s+1)}\right)\right) \\
&= \exp\left(-\lambda - 2\lambda p^{\frac{(s-2)(s+1)}{2}}\right) \qquad (15)\\
&= \exp(-\lambda)\exp\left(o\left(n^{-\frac{1}{s+1}}(\ln n)^2\right)\right) \\
&= \exp(-\lambda)\left(1 + o\left(n^{-\frac{1}{s+1}}(\ln n)^2\right)\right) \qquad (16)
\end{aligned}
$$

for all large enough n. Also, since $\lambda = O(\log n)$, it follows from (7) and (15) that

$$
\mathbb{E}[X_s] \geqslant \binom{n}{2} p \exp\left(-\lambda - 2\lambda p^{\frac{(s-2)(s+1)}{2}}\right) = \exp\Big(2\log n + \log p - \lambda + o(1)\Big). \qquad (17)
$$

Below, we apply (17) to show that

$$
\mathbb{E}[X_s] = o\Big(\mathbb{E}[X_s]^2\Big). \qquad (18)
$$

Before proving (18), we need to establish some inequalities. It is straightforward to verify that $(1-x)^m \leqslant 1 - \frac{mx}{2}$ for every integer $m \geqslant 2$ and positive real number $x \leqslant 1 - 2^{-1/(m-1)}$. Using this fact and letting $h_0 = 1 - 2^{-2/(s^2-s-4)}$, we find that

$$
h(n)^{\frac{(s-2)(s+1)}{2}} - 1 \leqslant (1 - h_0)^{\frac{(s-2)(s+1)}{2}} - 1 \leqslant -\frac{(s-2)(s+1)}{4} h_0 \qquad (19)
$$

if $0 < h(n) \leqslant 1 - h_0$, and moreover,

$$h(n)^{\frac{(s-2)(s+1)}{2}} - 1 = \Big(1 - \big(1 - h(n)\big)\Big)^{\frac{(s-2)(s+1)}{2}} - 1 \leqslant -\frac{(s-2)(s+1)}{4}\big(1 - h(n)\big) \tag{20}$$

if $1 - h_0 \leqslant h(n) < 1$.

To prove (18), it suffices to consider the following three cases.

Case A.1. *Assume that the set N_1 consisting of all positive integers n with $h(n) \geqslant 1$ is infinite.*

Proof. Since $w(n) \to -\infty$ as $n \to \infty$, we deduce that $w(n) = o(\log n)$. From (11), we get

$$\begin{aligned}
2\log n + \log p - \lambda &= \frac{2\log\log n}{(s-2)(s+1)} - \frac{2s\log n}{s+1}\left(\frac{\log\log n}{s(s-2)\log n} + \frac{(s-2)(s+1)w(n)}{2\log n}\right) \\
&\quad + O\left(\log n \sum_{k=2}^{\frac{(s-2)(s+1)}{2}} \sum_{i=0}^{k} \left(\frac{\log\log n}{\log n}\right)^{k-i}\left(\frac{w(n)}{\log n}\right)^{i}\right) \\
&= -s(s-2)w(n) + O\left(w(n) \sum_{k=2}^{\frac{(s-2)(s+1)}{2}} \sum_{i=1}^{k} \left(\frac{\log\log n}{\log n}\right)^{k-i}\left(\frac{w(n)}{\log n}\right)^{i-1} + o(1)\right) \\
&= -s(s-2)w(n)\big(1 + o(1)\big) \longrightarrow \infty
\end{aligned}$$

when n runs over N_1. Thus, it follows from (17) that $\mathbb{E}[X_s] \longrightarrow \infty$ when n runs over N_1. $\diamond$

Case A.2. *Assume that the set N_2 consisting of all positive integers n with $0 < h(n) \leqslant 1 - h_0$ is infinite.*

Proof. From (11), (19), and the assumptions of Lemma 3.2, we find that

$$2\log n + \log p - \lambda \geqslant \frac{2\log\log n}{(s-2)(s+1)} - \varepsilon\log n + \frac{s(s-2)}{2}h_0\log n \longrightarrow \infty$$

when n runs over N_2. Thus, it follows from (17) that $\mathbb{E}[X_s] \longrightarrow \infty$ when n runs over N_2. $\diamond$

Case A.3. *Assume that the set N_3 consisting of all positive integers n with $1 - h_0 \leqslant h(n) < 1$ is infinite.*

Proof. We obtain from (11) and (20) that

$$2\log n + \log p - \lambda \geqslant \frac{2\log\log n}{(s-2)(s+1)} + \log(1 - h_0) + \frac{s(s-2)}{2}\big(1 - h(n)\big)\log n \longrightarrow \infty$$

when n runs over N_3. Thus, it follows from (17) that $\mathbb{E}[X_s] \longrightarrow \infty$ when n runs over N_3. $\diamond$

From Cases A.1–A.3, we deduce that $\mathbb{E}[X_s] \longrightarrow \infty$ as $n \to 0$, proving (18).

To proceed, we estimate $\mathbb{P}[\mu(1,2) = \mu(3,4) = 0]$. For each subset $W \subseteq [\![n]\!] \setminus [\![4]\!]$ of size $s-2$, consider the event $K'[W]$ saying that W is a clique in either $N_{\mathbb{G}(n,p)}(1,2)$ or $N_{\mathbb{G}(n,p)}(3,4)$. Let μ' count the number of subsets W as above such that $K'[W]$ happens. We have

$$\mathbb{P}\big[\mu(1,2) = \mu(3,4) = 0\big] \leqslant \mathbb{P}\big[\mu' = 0\big]. \tag{21}$$

Let

$$\lambda' = \sum_{W' \in \binom{[\![n]\!] \setminus [\![4]\!]}{s-2}} \mathbb{P}\Big[K'[W]\Big] = \binom{n-4}{s-2} p^{\frac{(s-2)(s+1)}{2}} \left(2 - p^{2(s-2)}\right) \tag{22}$$

and

$$\begin{aligned} \Delta' &= \sum_{\substack{W_1, W_2 \in \binom{[\![n]\!] \setminus [\![4]\!]}{s-2} \\ W_1 \neq W_2 \\ W_1 \cap W_2 \neq \varnothing}} \mathbb{P}\Big[K'[W_1] \cap K'[W_2]\Big] \\ &= \sum_{\ell=1}^{s-3} \binom{n-4}{s-2} \binom{s-2}{\ell} \binom{n-s-2}{s-2-\ell} 2p^{(s-2)(s+1) - \frac{\ell(\ell+3)}{2}} \left(1 + p^{2\ell}\right). \end{aligned} \tag{23}$$

It follows from (8) and (22) that $\lambda' = 2\lambda(1 + O((\log n)/n))$. Also, it follows from (9) and (23) that $\Delta' = 2\Delta(1 + o(1))$. Since $\lambda = O(\log n)$ by (10) and $\Delta = \lambda o(n^{-1/(s+1)} \log n)$ by Lemma 3.6, we conclude from Theorem 2.7 that

$$\mathbb{P}[\mu' = 0] \leqslant \exp\left(-\lambda' + \frac{\Delta'}{2}\right) \tag{24}$$

$$\begin{aligned} &= \exp\left(-2\lambda \left(1 + O\left(\frac{\log n}{n}\right)\right) + \Delta\Big(1 + o(1)\Big)\right) \\ &= \exp(-2\lambda) \exp\left(o\left(n^{-\frac{1}{s+1}} (\log n)^2\right)\right) \\ &= \exp(-2\lambda) \left(1 + o\left(n^{-\frac{1}{s+1}} (\log n)^2\right)\right). \end{aligned} \tag{25}$$

In the same way, let us estimate $\mathbb{P}[\mu(1,2) = \mu(2,3) = 0]$. For each subset $W \subseteq [\![n]\!] \setminus [\![3]\!]$ of size $s-2$, consider the event $K''[W]$ saying that W is a clique in either $N_{\mathbb{G}(n,p)}(1,2)$ or $N_{\mathbb{G}(n,p)}(2,3)$. Let μ'' count the number of subsets W as above such that $K''[W]$ happens. We have

$$\mathbb{P}\Big[\mu(1,2) = \mu(2,3) = 0\Big] \leqslant \mathbb{P}\Big[\mu'' = 0\Big]. \tag{26}$$

Let

$$\lambda'' = \sum_{W' \in \binom{[\![n]\!] \setminus [\![3]\!]}{s-2}} \mathbb{P}\Big[K''[W]\Big] = \binom{n-3}{s-2} p^{\frac{(s-2)(s+1)}{2}} \left(2 - p^{s-2}\right) \tag{27}$$

and

$$\begin{aligned} \Delta'' &= \sum_{\substack{W_1, W_2 \in \binom{[\![n]\!] \setminus [\![3]\!]}{s-2} \\ W_1 \neq W_2 \\ W_1 \cap W_2 \neq \varnothing}} \mathbb{P}\Big[K''[W_1] \cap K''[W_2]\Big] \\ &= \sum_{\ell=1}^{s-3} \binom{n-3}{s-2} \binom{s-2}{\ell} \binom{n-s-1}{s-2-\ell} 2p^{(s-2)(s+1) - \frac{\ell(\ell+3)}{2}} \left(1 + p^{\ell}\right). \end{aligned} \tag{28}$$

It follows from (8) and (27) that $\lambda'' = 2\lambda(1 + O(\sqrt{(\log n)/n}))$. Also, it follows from (9) and (28) that $\Delta'' = 2\Delta(1 + o(1))$. Since $\lambda = O(\log n)$ by (10) and $\Delta = \lambda o(n^{-1/(s+1)} \log n)$ by Lemma 3.6, we derive from Theorem 2.7 that

$$\mathbb{P}[\mu'' = 0] \leqslant \exp\left(-\lambda'' + \frac{\Delta''}{2}\right) \tag{29}$$

$$
\begin{aligned}
&= \exp\left(-2\lambda\left(1+O\left(\sqrt{\frac{\log n}{n}}\right)\right)+\varDelta\big(1+o(1)\big)\right)\\
&= \exp(-2\lambda)\exp\left(o\left(n^{-\frac{1}{s+1}}(\log n)^2\right)\right)\\
&= \exp(-2\lambda)\left(1+o\left(n^{-\frac{1}{s+1}}(\log n)^2\right)\right). \qquad (30)
\end{aligned}
$$

Finally, by combining the relations (7), (16), (18), (21), (25), (26), and (30), we derive for each $s \geqslant 3$ that

$$
\begin{aligned}
\mathrm{Var}[X_s] &= \mathbb{E}\Big[X_s^2\Big] - \mathbb{E}[X_s]^2\\
&= \mathbb{E}[X_s] + \binom{n}{2}\binom{n-2}{2}p^2\mathbb{P}\big[\mu(1,2)=\mu(3,4)=0\big]\\
&+ n(n-1)(n-2)p^2\mathbb{P}\big[\mu(1,2)=\mu(2,3)=0\big] - \left(\binom{n}{2}p\mathbb{P}\big[\mu(1,2)=0\big]\right)^2\\
&\leqslant \mathbb{E}[X_s] + \binom{n}{2}^2 p^2\exp(-2\lambda)\left(1+o\left(n^{-\frac{1}{s+1}}(\log n)^2\right)\right)\\
&+ n^3p^2\exp(-2\lambda)\left(1+o\left(n^{-\frac{1}{s+1}}(\log n)^2\right)\right) - \binom{n}{2}^2p^2\exp(-2\lambda)\left(1+o\left(n^{-\frac{1}{s+1}}(\log n)^2\right)\right)\\
&= o\Big(\mathbb{E}[X_s]^2\Big).
\end{aligned}
$$

Thus, Part (ii) of Lemma 3.2 follows from Corollary 2.4. $\square$