Spectral characterization of graphs with index at most $\sqrt{2+\sqrt{5}}$

N. Ghareghani[b,a] G. R. Omidi[b,a] B. Tayfeh-Rezaie[a,1]

[a]*Institute for Studies in Theoretical Physics and Mathematics (IPM), P.O. Box 19395-5746, Tehran, Iran*
[b]*School of Mathematics, Statistics and Computer Science, University of Tehran, Tehran, Iran*
Emails: ghareghani@ipm.ir, omidi@ipm.ir, tayfeh-r@ipm.ir

Abstract

A graph is said to be determined by the adjacency spectrum (DS for short) if there is no other nonisomorphic graph with the same spectrum. All connected graphs with index at most $\sqrt{2+\sqrt{5}}$ are known. In this paper, we show that with few exceptions all of these graphs are DS.

AMS Subject Classification: 05C50.
Keywords: Spectral characterization, adjacency spectrum, DS graphs, index.

1 Introduction

In this paper, we are concerned only with undirected simple graphs (loops and multiple edges are not allowed). Let G be a graph with the adjacency matrix A. We denote $\det(\lambda I - A)$, the characteristic polynomial of G, by $P(G, \lambda)$. The multiset of eigenvalues

[1] Corresponding author.

of A is called the *adjacency spectrum*, or simply the *spectrum* of G. Since A is a symmetric matrix, the eigenvalues of G are real. The maximum eigenvalue of G is called the *index* of G. Two graphs with the same spectrum are called *cospectral*. We say that a graph is determined by the spectrum (*DS* for short) if there is no other nonisomorphic graph with the same spectrum. For a recent survey on DS and cospectral graphs, see [4]. It is conjectured that almost all graphs are DS [4]. However the set of graphs which are known to be DS is small and therefore it would be interesting to find more examples of DS graphs.

There have been some attempts to characterize graphs having index at most a given number. In [8] all graphs with index at most 2 were identified. Subsequently, graphs with index not exceeding $\sqrt{2+\sqrt{5}}$ were determined in [1, 2]. Most of connected graphs with index at most 2 are known to be DS [4, 7, 9]. In this paper, we consider connected graphs with index at most $\sqrt{2+\sqrt{5}}$ and identify the DS ones among them. The results are summarized in the following theorem (see below for notation).

Theorem 1 *All connected graphs with index at most* $\sqrt{2+\sqrt{5}}$ *are DS except for* $W_n(n \geq 2)$, $T(2,2,2)$ *and* $K_{1,4}$.

Notation The path and cycle with n vertices are denoted by P_n and C_n, respectively. For $a, b, c \geq 1$, we denote the graph shown in Figure 1 (left) by $T(a, b, c)$. In particular, Z_n $(n \geq 2)$ stands for $T(1, n-1, 1)$. For $a, c \geq 2$ and $b \geq 1$, we denote the graph shown in Figure 1 (right) by $Q(a, b, c)$. We also use W_n $(n \geq 2)$ to denote $Q(2, n-1, 2)$.

2 Graphs with index at most 2

In this section, we determine all connected DS graphs with index at most 2. Note that this characterization follows from the previously known results [3, 4, 7, 9].

Theorem 2 [8] (See also [3, page 78]) *The list of all connected graphs with index at most* 2 *includes precisely the following graphs:*
(i) $P_n, C_n, Z_n(n \geq 2), W_n(n \geq 2)$,

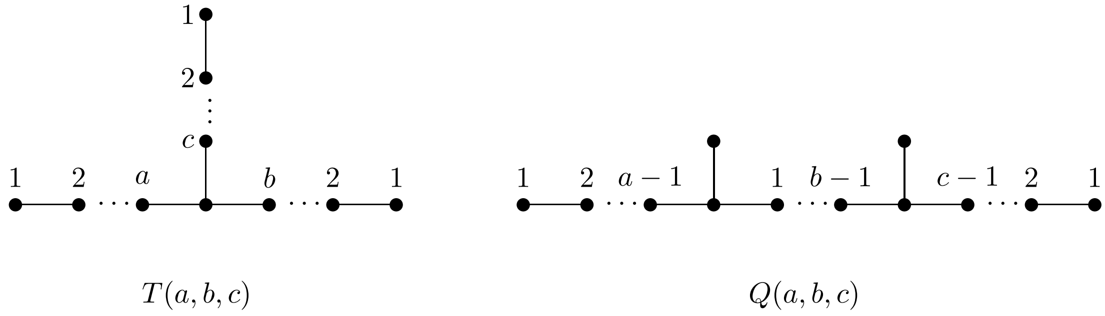

Figure 1: The graphs $T(a,b,c)$ and $Q(a,b,c)$

(ii) $T(a,b,c)$ *for* $(a,b,c) \in \{(1,2,2),(1,2,3),(1,2,4),(1,2,5),(1,3,3),(2,2,2)\}$,
(iii) $K_{1,4}$.

Theorem 3 *All connected graphs with index at most* 2 *are DS except for* $W_n (n \geq 2)$, $T(2,2,2)$ *and* $K_{1,4}$.

Proof. The following graphs are known to be DS: P_n, C_n [4], Z_n [7], $T(1,2,c)$ for $2 \leq c \leq 5$ and $T(1,3,3)$ [9]. The graph $W_n (n \geq 2)$ is cospectral with $C_4 + P_n$ [3, page 77]. Also it is well known that $K_{1,4}$ is cospectral with $C_4 + K_1$. The graph $T(2,2,2)$ is cospectral with $C_6 + K_1$, since both share the spectrum $\{\pm 2, \pm 1, \pm 1, 0\}$. $\square$

3 Graphs with index at most $\sqrt{2+\sqrt{5}}$

In this section, we consider graphs with index in the interval $(2, \sqrt{2+\sqrt{5}}]$. The following theorem characterizes these graphs.

Theorem 4 [1, 2] (See also [3, page 385]) *The list of all connected graphs with index in the interval* $(2, \sqrt{2+\sqrt{5}}]$ *includes precisely the following graphs:*
(i) $T(a,b,c)$ *for* $a=1$, $b=2$, $c>5$ *or* $a=1$, $b>2$, $c>3$ *or* $a=2$, $b=2$, $c>2$ *or* $a=2$,

$b = 3,\ c = 3,$
(ii) $Q(a, b, c)$ *for* $(a, b, c) \in \{(2, 1, 3), (3, 4, 3), (3, 5, 4), (4, 7, 4), (4, 8, 5)\}$ *or* $a > 1, c > 1$, $b \geq b^*(a, c)$, *where* $(a, c) \neq (2, 2)$ *and*

$$b^*(a,c) = \begin{cases} a+c & a > 3, \\ 2+c & a = 3, \\ -1+c & a = 2. \end{cases}$$

The following lemma is well known and a proof of it can be found in [4].

Lemma 1 *For $n \times n$ symmetric matrices A and B, the following are equivalent.*
(i) *A and B are cospectral.*
(ii) $tr(A^i) = tr(B^i)$ *for* $i = 1, \ldots, n$.

If A is the adjacency matrix of a graph, then $\mathrm{tr}(A^i)$ gives the total number of closed walks of length i. So by the above lemma, two cospectral graphs have the same number of closed walks of a given length i. In particular, they have the same number of edges and triangles. The following useful lemma and theorem are trivial.

Lemma 2 *In any graph, the number of closed walks of length 4 equals twice the number of edges plus four times the number of induced paths of length two plus eight times the number of 4-cycles.*

Theorem 5 [3, page 59] *Let x_1 be a vertex of degree 1 in the graph G and x_2 be the vertex adjacent to x_1. Let G_1 and G_2 be the induced subgraphs obtained from G by deleting x_1 and x_1, x_2, respectively. Then*

$$P(G, \lambda) = \lambda P(G_1, \lambda) - P(G_2, \lambda).$$

Theorem 6 [9] *Any graph of type $T(a, b, c)$ $(a \leq b \leq c)$ is DS if and only if $(a, b, c) \neq (l, l, 2l - 2)$ for any $l > 1$.*

This theorem along with Theorem 4 implies the following result.

Lemma 3 *Any graph of type $T(a,b,c)$ with the index in the interval $(2, \sqrt{2+\sqrt{5}}]$ is DS.*

Lemma 4 *We have*
(i) $P(P_a, 2) = a + 1$,
(ii) $P(T(a,b,c), 2) = a + b + c + 2 - abc$,
(iii) $P(Q(a,b,c), 2) = 4a + 4b + 4c - 4ac - 2bc - 2ab + abc$.

Proof. We use Theorem 5 to prove (i) and (ii). We omit the proof of (i) and proceed to prove (ii). The proof is by an induction on $a+b+c$. Let $a+b+c \geq 7$. Then without loss of generality, we may assume that $a \geq 3$. By the induction hypothesis, we have

$$\begin{aligned} P(T(a,b,c),2) &= 2P(T(a-1,b,c),2) - P(T(a-2,b,c),2) \\ &= 2(a+b+c+1-(a-1)bc) - (a+b+c-(a-2)bc) \\ &= a+b+c+2-abc. \end{aligned}$$

If $a+b+c < 7$, then it is an easy task to verify (ii) using the table of characteristic polynomials of trees of small orders (Table 2 of [3]).

In order to prove (iii), we again use Theorem 5. We have

$$P(Q(a,b,c),\lambda) = \lambda P(T(a+b-1,1,c-1),\lambda) - P(P_{a-1},\lambda)P(T(b-1,1,c-1),\lambda).$$

Putting $\lambda = 2$ and using (i) and (ii), the result follows. □

Lemma 5 *Let $\lambda > 2$ and let r_1 and r_2 be the roots of $r^2 - \lambda r + 1 = 0$. Then*

$$P(T(m,1,n),\lambda) = \frac{1}{(r_1-r_2)^2} \sum_{i=1}^{2} (r_i^{m+n+4} + r_i^{m-n} - r_i^{m+n} - r_i^{m+n+2}). \tag{1}$$

Proof. By Theorem 5, we have

$$P(P_n,\lambda) = \lambda P(P_{n-1},\lambda) - P(P_{n-2},\lambda).$$

Solving this recurrence equation, we find

$$P(P_n,\lambda) = c_1 r_1^n + c_2 r_2^n, \tag{2}$$

where $c_1 = r_1/(r_1 - r_2)$ and $c_2 = r_2/(r_2 - r_1)$. Again by Theorem 5, we have

$$P(T(m,1,n),\lambda) = \lambda P(P_{m+n+1},\lambda) - P(P_m,\lambda)P(P_n,\lambda).$$

Using (2), we obtain the desired result. $\square$

Lemma 6 *If two non-isomorphic trees of type $Q(a,b,c)$ $(a \le c)$ are cospectral, then $(a,b,c) \in \{(2,1,10),(4,3,6),(2,1,6),(3,2,4),(3,2,12),(6,3,8)\}$.*

Proof. First we find $P(Q(a,b,c),\lambda)$ for $\lambda > 2$. Let $n = a+b+c$. By Theorem 5, we have

$$P(Q(a,b,c),\lambda) = \lambda P(T(a+b-1,1,c-1),\lambda) - P(P_{a-1},\lambda)P(T(b-1,1,c-1),\lambda).$$

(Note that (1) also holds for $m = 0$.) Therefore, using Lemma 5, we obtain

$$\begin{aligned}(r_1 - r_2)^2 P(Q(a,b,c),\lambda) =& \lambda\sum_{i=1}^{2}(r_i^{n+2} + r_i^{n-2c} - r_i^{n-2} - r_i^{n}) \\ & - (c_1 r_1^{a-1} + c_2 r_2^{a-1})\sum_{i=1}^{2}(r_i^{n-a+2} + r_i^{b-c} - r_i^{n-a-2} - r_i^{n-a}),\end{aligned}$$

which in turn implies

$$\begin{aligned}(r_1 - r_2)^3 P(Q(a,b,c),\lambda) =& \varphi_n(\lambda) - \sum_{i=1}^{2}(-1)^i(r_i^{n-2c+2} - r_i^{n-2c} - r_i^{n-2c-2} \\ & + r_i^{n-2a+2} - r_i^{n-2a} - r_i^{n-2a-2} + r_i^{n-2a-2c}),\end{aligned} \tag{3}$$

where r_1 and r_2 are the roots of $r^2 - \lambda r + 1 = 0$ and $\varphi_n(\lambda)$ is a polynomial of λ.

Let $G = Q(a,b,c)$ and $G' = Q(a',b',c')$ be cospectral. We have $a+b+c = a'+b'+c' = n$ and

$$P(Q(a,b,c),\lambda) = P(Q(a',b',c'),\lambda).$$

Using (3), we obtain

$$\begin{aligned}0 =& \sum_{i=1}^{2}(-1)^i(r_i^{n-2c+2} - r_i^{n-2c} - r_i^{n-2c-2} + r_i^{n-2a+2} - r_i^{n-2a} - r_i^{n-2a-2} + r_i^{n-2a-2c}) - \\ & \sum_{i=1}^{2}(-1)^i(r_i^{n-2c'+2} - r_i^{n-2c'} - r_i^{n-2c'-2} + r_i^{n-2a'+2} - r_i^{n-2a'} - r_i^{n-2a'-2} + r_i^{n-2a'-2c'}).\end{aligned}$$

Substituting $r_2 = r_1^{-1}$ and multiplying both sides by r_1^n and using the fact that the set of powers of r_1 with positive coefficients is equal to the set of powers of r_1 with negative coefficients, we find that the following multisets A and B must be identical:

$$A = \{b+c+1, a+1, a, a+b+1, c+1, c, b, a'-1, b'+c'-1, b'+c', c'-1, a'+b'-1, a'+b', a'+c'\},$$
$$B = \{b'+c'+1, a'+1, a', a'+b'+1, c'+1, c', b', a-1, b+c-1, b+c, c-1, a+b-1, a+b, a+c\}.$$

First let $a = a'$. Then by omitting common elements from A and B, we are left with two multisets $A' = \{a+b+1, c+1, c, b, c'-1, a'+b'-1, a'+b', a'+c'\}$ and $B' = \{a'+b'+1, c'+1, c', b', c-1, a+b-1, a+b, a+c\}$ which should be identical. If $b = b'$, then $c = c'$ and we are done. Otherwise, without loss of generality we may assume that $b < b'$. Since $\max A' = \max B'$, we have $\max\{c+1, a+b+1, a'+b', a'+c'\} = \max\{a'+b'+1, a+c\}$. But this is a contradiction since the left max is less than the right max.

Now let $a \neq a'$. Without loss of generality, we may assume that $a < a' \leq c'$ and $a \leq c$. It easy to see that the only element of A that is equal to $a-1 \in B$ is b and so $a = b+1$. Also the elements of B that can be equal to $a \in A$ are $b+c-1, a+b-1, c-1, b'$. Hence we have the following four cases:

(i) $a = b+c-1$. Then $a = c = 2$ and $b = 1$. This yields $n = 5$ which is contrary to $c', a' \geq 3$.

(ii) $a = a+b-1$. Then $a = 2$ and $b = 1$. By omitting common elements from A and B it turns out that $b'+c'+1 \in B$ should be equal to $a'+c' \in A$. So $a' = b'+1 \geq 3$. The element of A that can be equal to $a' \in B$ is $c'-1$ or 4. If $a' = 4$, then $b' = 3, c' = 6$ and so $c = 10$ which implies that $Q(2,1,10)$ and $Q(4,3,6)$ are cospectral. If $a' = c'-1$, then $a' = 3$ and so $b' = 2, c' = 4, c = 6$ and two graphs $Q(2,1,6)$ and $Q(3,2,4)$ are cospectral.

(iii) $a = c-1$. Then $a = n/3$ and so $b' < n/3-1$. But there is no element corresponding to $b' \in B$ in A, a contradiction.

(iv) $a = b'$. Then the element $b'+c'+1 \in B$ can be equal to c, $c+1$ or $a'+c'$ of A. If $b'+c'+1 = c$, then $(a,b,c) = (3,2,12)$, $(a',b',c') = (6,3,8)$ and therefore $Q(3,2,12)$ and $Q(6,3,8)$ are cospectral. If $b'+c'+1 = c+1$, then $b'+c' = c = a'+b'+1 = b+c-1$ and so $(a,b,c) = (2,1,6)$ and $(a',b',c') = (3,2,4)$ which has been also dealt with in (ii). If $b'+c'+1 = a'+c'$, then $c' = c-2$ and so $(a,b,c) = (2,1,6)$ and $(a',b',c') = (3,2,4)$ which is already dealt with. $\square$

Lemma 7 *Let $c \geq a > 0$, $b > 0$ and*

$$4a + 4b + 4c - 4ac - 2bc - 2ab + abc = 0.$$

Then $(a, b, c) \in \{(1, 4, 3), (1, 2, 4), (1, 1, 6), (3, 12, 3), (3, 10, 4), (3, 9, 6), (4, 8, 4), (4, 7, 6), (5, 6, 8), (6, 6, 6), (7, 5, 22), (8, 5, 14), (10, 5, 10)\}$ *or* $(a, b, c) = (2, b, 2)$.

Proof. We have

$$b(4 + ac - 2a - 2c) = 4ac - 4a - 4c.$$

If $4 + ac - 2a - 2c = 0$, then $4ac - 4a - 4c = 0$ and hence $a = c = 2$ and $(2, b, 2)$ is a solution.

Now suppose that $4 + ac - 2a - 2c \neq 0$. Then $a \neq 2$ and

$$b = 4 + \frac{4a + 4c - 16}{ac + 4 - 2a - 2c}.$$

If $a \geq 12$, then

$$0 < b - 4 = \frac{4a + 4c - 16}{ac + 4 - 2a - 2c} \leq \frac{4c + 4c - 16}{12c + 4 - 2c - 2c} < 1,$$

which is a contradiction. Therefore, $a < 12$ and we have the following cases:
(i) $a = 1$. Then $b = 4/(c - 2)$ which implies $(a, b, c) = (1, 4, 3), (1, 2, 4), (1, 1, 6)$.
(ii) $a = 3$. Then $b = 8 + 4/(c - 2)$ and so $(a, b, c) = (3, 12, 3), (3, 10, 4), (3, 9, 6)$.
(iii) $a = 4$. Then $b = 6 + 4/(c - 2)$ and so $(a, b, c) = (4, 8, 4), (4, 7, 6)$.
(iv) $a = 5$. Then $b = 5 + (c + 10)/(3c - 6)$ which implies $(a, b, c) = (5, 6, 8)$.
(v) $a = 6$. Then $b = 5 + 4/(c - 2)$ and so $(a, b, c) = (6, 6, 6)$.
(vi) $a = 7$. Then $b = 4 + (4c + 12)/(5c - 10)$ and so $(a, b, c) = (7, 5, 22)$.
(vii) $a = 8$. Then $b = 4 + (4c + 16)/(6c - 12)$ and so $(a, b, c) = (8, 5, 14)$.
(iix) $a = 9$. Then $b = 4 + (4c + 20)/(7c - 14)$ which has no solution.
(ix) $a = 10$. Then $b = 4 + (c + 6)/(2c - 4)$ and so $(a, b, c) = (10, 5, 10)$.
(x) $a = 11$. Then $b = 4 + (4c + 28)/(9c - 18)$ which has no solution. $\square$

Lemma 8 *Any graph $G = Q(a, b, c)$ with the index in the interval $(2, \sqrt{2 + \sqrt{5}}]$ is DS.*

Proof. Let $G = Q(a, b, c)$ be cospectral with G' and G' is not isomorphic to G. By Theorems 2 and 4, the components of G' are trees or cycles. By Theorem 4 and Lemmas

3 and 6, G' is not a tree. Therefore, G' has a component which is cycle. This yields $P(G',2) = 0$ and so by Lemma 4, $P(G,2) = 4a + 4b + 4c - 4ac - 2bc - 2ab + abc = 0$. By Lemma 7 and Theorem 4, $(a,b,c) \in \{(3,12,3),(3,10,4),(3,9,6),(4,8,4)\}$. Using the computer package **newGRAPH** [5], we find that all of the nonzero eigenvalues of G are simple. Since any cycle $C_m (m > 4)$ has at least one nonsimple nonzero eigenvalue, we conclude that $G' = C_4 + G''$, where $G'' = T(a',b',c')$ or $Q(a',b',c')$ and $a' + b' + c' = a + b + c - 4$. Using Lemma 2, we count the number of closed walks of length 4 in G and G'. This number is $6(a+b+c)+4$ for G and $6(a+b+c)+8$ or $6(a+b+c)+12$ for G' which is a contradiction to Lemma 1. $\square$

We summarize the results of this section in the following Theorem.

Theorem 7 *All connected graphs with index in the interval* $(2, \sqrt{2+\sqrt{5}}]$ *are DS.*

4 Concluding remarks

An important result of Schwenk [6] asserts that almost all trees have cospectral mates. A natural question which arises (and proposed in [4]) is that which tress are DS? Families of DS trees are give in [4, 7, 9]. In this paper, we have presented some other families of DS trees.

In [10], the structures of graphs with index at most $3\sqrt{2}/2$ is given. A good problem to tackle next is to determine all DS ones among the graphs characterized in [10].

References

[1] A. E. Brouwer and A. Neumaier, The graphs with spectral radius between 2 and $\sqrt{2+\sqrt{5}}$, *Linear Algebra Appl.* **114/115** (1989), 273–276.

[2] D. M. Cvetković, M. Doob and I. Gutman, On graphs whose spectral radius does not exceed $(2+\sqrt{5})^{1/2}$, *Ars Combin.* **14** (1982), 225–239.

[3] D. M. Cvetković, M. Doob and H. Sachs, *Spectra of graphs, Theory and applications*, Third edition, Johann Ambrosius Barth, Heidelberg, 1995.

[4] E. R. van Dam and W. H. Haemers, Which graphs are determined by their spectrum?, *Linear Algebra Appl.* **373** (2003), 241–272.

[5] newGRAP, http://www.mi.sanu.ac.yu/newgraph.

[6] A. J. Schwenk, Almost all trees are cospectral, in: F. Harary (Ed.), *New directions in the theory of graphs*, pp. 275–307, Academic Press, New York, 1973.

[7] X. Shen, Y. Hou and Y. Zhang, Graph Z_n and some graphs related to Z_n are determined by their spectrum, *Linear Algebra Appl.* **404** (2005), 58–68.

[8] J. H. Smith, Some properties of the spectrum of a graph, 1970 *Combinatorial Structures and their Applications*, pp. 403–406, Gordon and Breach, New York.

[9] W. Wang and C. -X. Xu, On the spectral characterization of T-shape trees, *Linear Algebra Appl.* **414** (2006), 492–501.

[10] R. Woo and A. Neumaier, On graphs whose spectral radius is bounded by $3\sqrt{2}/2$, preprint.