On the existence of large sets of t-designs of prime sizes[1]

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Abstract

In this paper, we employ the known recursive construction methods to obtain some new existence results for large sets of t-designs of prime sizes. We also present a new recursive construction which leads to more comprehensive theorems on large sets of sizes two and three. As an application, we show that for infinitely many values of block size, the trivial necessary conditions for the existence of large sets of 2-designs of size three are sufficient.

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1 Introduction

Let t, k, v, and λ be integers such that $0 \le t \le k \le v$ and $\lambda \ge 1$. Let X be a v-set and $P_k(X)$ denote the set of all k-subsets of X. A t-(v, k, λ) *design* (briefly a t-design) on X is a collection $\mathcal{D}$ of the elements of $P_k(X)$ such that every t-subset of X is contained in exactly λ elements of $\mathcal{D}$. We always implicitly assume that $0 \le t < k < v$ to avoid trivial cases. If $\mathcal{D}$ has no repeated block, it is called a *simple* design. Here we are concerned only with simple designs. A *large set* of t-(v, k, λ) designs of size N on X, denoted by $\mathrm{LS}[N](t, k, v)$, is a partition of $P_k(X)$ into N disjoint t-(v, k, λ) designs, where $N = \binom{v-t}{k-t}/\lambda$. We always assume that $N > 1$. It is well known that a set of trivial necessary conditions for the existence of a $\mathrm{LS}[N](t, k, v)$ is that

$$\binom{v-i}{k-i} \equiv 0 \pmod{N}, \tag{1}$$

for $0 \le i \le t$. Every quadruple $(t, k, v; N)$ satisfying (1) is called a *feasible set of parameters.*

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In this note, we use the known recursive methods to obtain new existence results for large sets of t-designs of prime sizes. A new recursive construction is also given for these large sets. We then focus on large sets of sizes two and three. It is conjectured that in these cases a large set exists for every feasible set of parameters [4, 6]. We demonstrate the correctness of the conjectures for infinitely many values of k by assuming the existence of a few number of large sets. As a consequence, we show that if $2.3^{n-1} + 3^{n-4} \leq k < 3^n$ for any $n \geq 4$, then a LS[3]$(2, k, v)$ exists for any feasible value of v.

2 Recursive constructions

We first recall the following well known theorems for a later usage.

Theorem 1 *If there exists a LS*$[N](t, k, v)$*, then there exists a LS*$[N](t, v-k, v)$*.*

Theorem 2 **[3, 7]** *If there exists a LS*$[N](t, k, v)$*, then there exists a LS*$[N](t-i, k-j, v-l)$ *for* $0 \leq j \leq l \leq i \leq t$*.*

Theorem 3 **[5, 7]** *If there exists a LS*$[N](t, k+i, v)$ *for* $0 \leq i \leq l$*, then there exists a LS*$[N](t, k+i, v+j)$ *for* $0 \leq j \leq i \leq l$*.*

Most of the recursive methods for construction of large sets are obtained through the notion of (N, t)-partitionable sets which is in fact a generalization of the notion of large sets. Let X be a v-set. We say that $\mathcal{B} \subseteq P_k(X)$ is (N, t)-*partitionable* if there exists a partition $\{\mathcal{B}_i\}_{i=1}^N$ of $\mathcal{B}$ such that for any t-subset T of X, the occurrence of T in $\mathcal{B}_i$ is the same for all $1 \leq i \leq N$. Note that a LS$[N](t, k, v)$ exists if and only $P_k(X)$ is an (N, t)-partitionable set. Let X_1 and X_2 be two disjoint sets and let $\mathcal{B}_i \subseteq P_{k_i}(X_i)$ for $i = 1, 2$. Then we define

$$\mathcal{B}_1 * \mathcal{B}_2 = \{B_1 \cup B_2 | \ B_1 \in \mathcal{B}_1, \ B_2 \in \mathcal{B}_2\}.$$

Lemma 1 **[5]** *The union of disjoint* (N, t)*-partitionable sets is again an* (N, t)*-partitionable set.*

Lemma 2 **[5]** *Let* X_1 *and* X_2 *be two disjoint sets and let* $\mathcal{B}_i \subseteq P_{k_i}(X_i)$ *for* $i = 1, 2$*. Suppose that* $\mathcal{B}_1$ *is* (N, t_1)*-partitionable. Then*

(i) $\mathcal{B}_1 * \mathcal{B}_2$ *is* (N, t_1)*-partitionable.*

(ii) If $\mathcal{B}_2$ *is also* (N, t_2)*-partitionable, then* $\mathcal{B}_1 * \mathcal{B}_2$ *is* $(N, t_1 + t_2 + 1)$*-partitionable.*

A few powerful recursive constructions are obtained by the use of Lemmas 1 and 2. The key idea is that we try to partition $P_k(X)$ in such a suitable way that enables us to show that each part of the partition is (N,t)-partitionable by Lemma 2. Then obviously, we find a large set by Lemma 1. An example of this approach is given in the proof of Theorem 8. We review the recursive constructions obtained by this approach for large sets of arbitrary sizes in the following theorems. Then we will focus on large sets of prime sizes for which more powerful and comprehensive results are obtained.

Notation. Let N, t, and k be given. The set of all v for which $\mathrm{LS}[N](t,k,v)$ exist is denoted by $\mathrm{A}[N](t,k)$. The set of all v which satisfy the necessary conditions (1) is denoted by $\mathrm{B}[N](t,k)$. Let m and n be positive integers. We denote the quotient and remainder of division m by n by $[m/n]$ and (m/n), respectively.

Theorem 4 [2] *Let a, b, c, d, t, s, k, v_1, and v_2 be nonnegative integers such that $t \leq s < k \leq min\{v_1, v_2\}$ and $s = k-1-a-b = t+c+d$. Let $v_1 \in \cap_{i=k-a}^{k} A[N](t,i)$, $v_2 \in \cap_{i=k-b}^{k} A[N](t,i)$, $v_1 - l \in A[N](t, k-a-l)$ for $1 \leq l \leq c$, and $v_2 - l \in A[N](t, k-b-l)$ for $1 \leq l \leq d$. Then $v_1 + v_2 - s \in A[N](t,k)$.*

Corollary 1 [5] *If a $LS[N](t,i,v)$ exists for $t+1 \leq i \leq k$ and a $LS[N](t,k,u)$ also exists, then a $LS[N](t,k,u+l(v-t))$ exists for all $l \geq 1$.*

Corollary 2 [2] *If a $LS[N](t,i,v+i)$ exists for $t+1 \leq i \leq k$ and a $LS[N](t,k,u)$ also exists, then a $LS[N](t,k,u+l(v+1))$ exists for all $l \geq 1$.*

Corollary 3 [5] *If a $LS[N](t,t+1,v+t)$ exists, then a $LS[N](t,t+1,lv+t)$ exists for all $l \geq 1$.*

For large sets of prime sizes there are stronger results which we present in the following theorems. Note that for this special category of large sets we have a nice interpretation of feasible parameter sets by Theorem 5. In what follows, we assume that t, k, v and p are given and p is prime.

Theorem 5 [7] *$v \in B[p^{\alpha}](t,k)$ if and only if there exist distinct positive integers ℓ_i for $1 \leq i \leq \alpha$ such that $t \leq (v/p^{\ell_i}) < (k/p^{\ell_i})$.*

Theorem 6 [3, 8] *If a $LS[p](t,k,v-1)$ exists, then a $LS[p](t,pk+i,pv+j)$ exists for $-p \leq j < i \leq p-1$.*

Theorem 7 **[3]** *If a* $LS[p](t,k,v-1)$ *exists, then a* $LS[p](t+1,pk+i,pv+j)$ *exists for* $0 \le j < i \le p-1$.

Theorem 8 *Let* f *be a positive integer such that* $k > p^f$ *and* $t \le (v/p^f) < (k/p^f)$. *For every* $u < v$, *suppose that the following holds:*

(i) If $t \le (u/p^f) < p^f - 1$, *then* $u \in A[p](t, p^f - 1)$,

(ii) If $t \le (u/p^f) < (k/p^f)$, *then* $u \in A[p](t, k - p^f)$.

Then $v \in A[p](t,k)$.

Proof Let $X = \{1,\dots,v\}$ and let $X_j = \{1,\dots,j\}$ and $Y_j = X \setminus X_j$ for $j = 1,\dots,v$. Assume that

$$\mathcal{B}_h = P_{p^f-1}(X_h) * \{\{h+1\}\} * P_{k-p^f}(Y_{h+1}), \qquad p^f - 1 \le h < v - k + p^f.$$

It is not hard to see that the sets $\mathcal{B}_h$ partition $P_k(X)$. By Lemma 1, it suffices to show that each $\mathcal{B}_h$ is (N,t)-partitionable.

First suppose that $(h/p^f) = p^f - 1$. Then $((v-1-h)/p^f) = (v/p^f)$ and hence $P_{k-p^f}(Y_{h+1})$ is (p,t)-partitionable by the assumption which in turn we conclude that $\mathcal{B}_h$ is (p,t)-partitionable by Lemma 2. If $t \le (h/p^f) < p^f - 1$, then $P_{p^f-1}(X_h)$ is (p,t)-partitionable by the assumption and so is $\mathcal{B}_h$ by Lemma 2. Now let $(h/p^f) = r < t$. Then $P_{p^f-1}(X_{h+t-r})$ is (p,t)-partitionable by the assumption. It yields that $P_{p^f-1}(X_h)$ is (p,r)-partitionable by Theorem 2. We also have $((v-h+r)/p^f) = (v/p^f)$. Therefore, $P_{k-p^f}(Y_{h-r})$ is (p,t)-partitionable by the assumption. By Theorem 2, we obtain that $P_{k-p^f}(Y_{h+1})$ is $(p, t-r-1)$-partitionable. Therefore, by Lemma 2, $\mathcal{B}_h$ is a (p,t)-partitionable set. $\square$

The following theorem was proved in [7]. Here we present a simpler proof using Corollary 1 and Theorem 8.

Theorem 9 **[7]** *Let* f *be a positive integer such that* $(k/p^f) > t$. *Suppose that* $p^f + t \in A[p](t,i)$ *for* $t+1 \le i \le min(k, (p^f+t)/2)$. *If* $t \le (v/p^f) < (k/p^f)$, *then* $v \in A[p](t,k)$.

Proof By an induction argument on i we show that if $t+1 \le i \le k$ and $(i/p^f) > t$, then $v \in \mathrm{A}[p](t,i)$ for all v such that $t \le (v/p^f) < (i/p^f)$. If $(p^f+t)/2 < i < p^f$, then $p^f + t - i < (p^f+t)/2$ and so $p^f + t \in \mathrm{A}[p](t, p^f + t - i)$ by the assumption. Hence, by Theorem 1, we have $p^f + t \in \mathrm{A}[p](t,i)$. By Corollary 1 we obtain that $lp^f + t \in \mathrm{A}[p](t,i)$ for $t < i < p^f$ and $l \ge 1$. Now Theorems 3 and 5 show that the assertion is true for $t < i < p^f$.

Now let $p^f \leq i \leq k$, $(i/p^f) > t$, and $t \leq (v/p^f) < (i/p^f)$. Then $((i-p^f)/p^f) > t$ and by the induction hypothesis for every u such that $t \leq (u/p^f) < (i/p^f)$, we have $u \in \mathrm{A}[p](t, i-p^f)$. Therefore, Theorem 8 shows that $v \in \mathrm{A}[p](t,k)$. $\square$

The theorems above have been utilized to show that some special classes of large sets (which we call *root cases* after [7]) can be used to produce all possible large sets. The following theorems identify these root cases.

Theorem 10 **[1, 7]** *Let $2^s - 1 \leq t < 2^{s+1} - 1$. Suppose that for every j and n such that $0 \leq j \leq [t/2]$ and $t+1 \leq 2^n + j \leq k$, there exists a $LS[2](t, 2^n+j, 2^{n+1}+t)$. Then $A[2](t_1,k_1) = B[2](t_1,k_1)$ for all $2^s - 1 \leq t_1 \leq t$ and $k_1 \leq k$.*

Theorem 11 [7] *Let p be an odd prime and let $p^s - 1 \leq t < p^{s+1} - 1$. Suppose that the following conditions hold:*

(i) There exists a $LS[p](t, k', p^{s+1}+t)$ for every $t+1 \leq k' \leq min(k, (p^{s+1}+t)/2)$,

(ii) There exists a $LS[p](t, ip^n+j, p^{n+1}+t)$ for every i, j, and n such that $0 \leq j \leq t, 1 \leq i \leq (p-1)/2, ip^n + j \leq k$, and $n > s$.

Then $A[p](t_1,k_1) = B[p](t_1,k_1)$ for all $p^s - 1 \leq t_1 \leq t$ and $k_1 \leq k$.

3 Large sets of prime sizes

In this section we use the previously known results to obtain new theorems on large sets of prime sizes. We note that the following results are in fact generalizations of similar theorems for large sets of size two in [2].

Theorem 12 *Let $t < p^f - 1$ and suppose that $p^f + t \in A[p](t,i)$ for $t+1 \leq i \leq min(k, (p^f+t)/2)$. If $tp^{h-f} \leq (v/p^h) < (k/p^h)$ for any $h \geq f$, then $v \in A[p](t,k)$.*

Proof The proof is by induction on h. If $h = f$, then we are done by Theorem 9. Therefore, assume that $h > f$ and $tp^{h-f} \leq (v/p^h) < (k/p^h)$. Let $k = r_0p^h + r_1p + r_2$ and $v = s_0p^h + s_1p + s_2$, where $tp^{h-f} \leq s_1p + s_2 < r_1p + r_2 < p^h$ and $0 \leq s_2, r_2 < p$. Clearly, $tp^{h-1-f} \leq s_1 \leq r_1 < p^{h-1}$. Now we define

$$(v',k') = \begin{cases} (s_0p^{h-1} + s_1, r_0p^{h-1} + r_1) & \text{if} \quad s_1 < r_1, \\ (s_0p^{h-1} + s_1, r_0p^{h-1} + r_1 + 1) & \text{if} \quad s_1 = r_1 < p^{h-1} - 1, \\ (s_0p^{h-1} + s_1 - 1, r_0p^{h-1} + r_1) & \text{if} \quad s_1 = r_1 = p^{h-1} - 1. \end{cases}$$

Then $tp^{h-1-f} \leq (v'/p^{h-1}) < (k'/p^{h-1})$ and by the induction hypothesis we have $v' \in \mathrm{A}[p](t,k')$. Now Theorem 6 shows that $v \in \mathrm{A}[p](t,k)$. $\square$

Theorem 13 *Let $t \leq p^{f-1}/2$ and suppose that $p^f + t \in A[p](t,i)$ for $t+1 \leq i \leq min(k,(p^f + t)/2)$. Let $p^{n-1} \leq k < p^n$ $(n \geq f)$. Then the following holds:*

(i) If $v \in A[p](t,k)$, then $v + p^n \in A[p](t,k)$,

(ii) If $t \leq (v/p^n) < k$ and $v > 2p^n$, then $v \in A[p](t,k)$.

Proof By Theorem 12, $p^n + i - 1 \in \mathrm{A}[p](t,i)$ for $t+1 \leq i \leq k$. Therefore, by Corollary 2, the assertion (i) holds. We use an induction argument on n to prove (ii). By (i) and Theorem 12, it suffices to show that $2p^n + j \in \mathrm{A}[p](t,k)$ for $t \leq j < tp^{n-f}$. We make use of Theorem 4 to prove it. Let $a = k - p^{n-1}$, $b = p^{n-1} - 1 + t - 2tp^{n-f}$, $c = d = tp^{n-f} - t$, and $s = 2tp^{n-f} - t$. Then $s = k - 1 - a - b = c + d + t$. Also let $v_1 = p^n + tp^{n-f}$ and $v_2 = p^n + tp^{n-f} + j - t$, where $t \leq j < tp^{n-f}$. Since $t \leq p^{f-1}/2$, we have $b \geq 0$. By (i) and the induction hypothesis, we have $v_1 \in \cap_{i=p^{n-1}}^{k} \mathrm{A}[p](t,i)$, $v_1 - l \in \mathrm{A}[p](t, p^{n-1} - l)$ for $1 \leq l \leq c$, $v_2 \in \cap_{i=k-b}^{k} \mathrm{A}[p](t,i)$, and $v_2 - l \in \mathrm{A}[p](t, k-b-l)$ for $1 \leq l \leq d$. Therefore, by Theorem 4, $v_1 + v_2 - s = 2p^n + j \in \mathrm{A}[p](t,k)$ for $t \leq j < tp^{n-f}$. $\square$

4 Large sets of sizes two and three

In the previous section we presented new results on the existence of large sets of prime sizes. We now show that more comprehensive results can be obtained for large sets of sizes two and three. It is conjectured that the necessary conditions (1) are sufficient for the existence of large sets of sizes two and three. The following theorems indicate that if those conjectures are true for some small values of k, then they will be true for infinitely many values of k. We also provide two applications of our results.

Theorem 14 *Let $t \leq 2^{f-2}$ and suppose that $A[2](t,i) = B[2](t,i)$ for $t < i < 2^f$. Let $2^{n-1} \leq k < 2^n$ $(n > f)$. Then*

(i) $B[2](t,k) \setminus A[2](t,k) \subset \{2^n + j \mid t \leq j < t2^{n-f}\}$,

(ii) If $2^{n-1} + t2^{n-f} \leq k < 2^n$, then $A[2](t,k) = B[2](t,k)$.

Proof The proof in by induction on n. Assume that $v \in \mathrm{B}[2](t,k)$ and $v \neq 2^n + j,\ t \leq j < t2^{n-f}$. Let $w = (v/2^n)$. If $w < k$, then by Theorems 12 and 13, $v \in \mathrm{A}[2](t,k)$. So let $k \leq w < 2^n$.

By Theorem 13, it suffices to show that $w \in \mathrm{A}[2](t,k)$, or by Theorem 1, $w \in \mathrm{A}[2](t, w-k)$. If $w-k < 2^{n-2}$, then we are done by the induction hypothesis. If $2^{n-2} \leq w-k < 2^{n-1}$, then $w \geq 2^{n-1} + 2^{n-2} \geq 2^{n-1} + t2^{n-1-f}$. Therefore, $w \in \mathrm{A}[2](t, w-k)$ by induction.

Now let $2^{n-1} + t2^{n-f} \leq k < 2^n$ and $v = 2^n + j$, $t \leq j < t2^{n-f}$. Then $v - k < 2^n + t2^{n-f} - 2^{n-1} - t2^{n-f} = 2^{n-1}$. Hence, by induction $v \in \mathrm{A}[2](t, v-k)$ which in turn yields that $v \in \mathrm{A}[2](t,k)$. □

Theorem 15 *Let $t \leq 3^{f-2}$ and suppose that $A[3](t,i) = B[3](t,i)$ for $t < i < 3^f$. Let $3^{n-1} \leq k < 3^n$ $(n > f)$. Then*

(i) $B[3](t,k) \setminus A[3](t,k) \subset \{3^n + j \mid t \leq j < t3^{n-f}\}$,

(ii) If $2.3^{n-1} + t3^{n-f} \leq k < 3^n$, *then* $A[3](t,k) = B[3](t,k)$.

Proof The proof is mostly similar to that of Theorem 14. We use an induction argument on k. Assume that $v \in \mathrm{B}[3](t,k)$ and $v \neq 3^n + j$, $t \leq j < t3^{n-f}$. Let $w = (v/3^n)$. If $w < k$, then by Theorems 12 and 13, $v \in \mathrm{A}[3](t,k)$. So let $k \leq w < 3^n$. By Theorem 13, it suffices to show that $w \in \mathrm{A}[3](t,k)$, or by Theorem 1, $w \in \mathrm{A}[3](t, w-k)$. If $w - k < 3^{n-2}$ or $3^{n-1} \leq w-k < k$, then we are done by the induction hypothesis. If $3^{n-2} \leq w-k < 3^{n-1}$, then $w \geq 3^{n-1} + 3^{n-2} \geq 3^{n-1} + t3^{n-1-f}$. Therefore, $w \in \mathrm{A}[3](t, w-k)$ by induction. Now suppose that $w - k > k$. Then we have $w = 2.3^{n-1} + i$ and $k = 3^{n-1} + j$, where $0 \leq j < i < 3^{n-1}$. By Theorem 5, there is $h < n-1$ such that $t \leq (w/3^h) < (k/3^h)$. Such an h is not necessarily unique and we can assume that $(k/3^h) \geq 3^{h-1}$. We now have $2.3^{n-2} + t3^{n-1-f} \leq k - 3^h < k$. Therefore, by the induction hypothesis and Theorem 8, it yields that $w \in \mathrm{A}[3](t,k)$.

Now let $2.3^{n-1} + t3^{n-f} \leq k < 3^n$ and $v = 3^n + j$, $t \leq j < t3^{n-f}$. Then $v - k < 3^n + t3^{n-f} - 2.3^{n-1} - t3^{n-f} = 3^{n-1}$. Hence, by induction $v \in \mathrm{A}[3](t, v-k)$ which concludes that $v \in \mathrm{A}[3](t,k)$. □

Theorem 16 **[2]** *If $2^{n-1} + 3.2^{n-4} \leq k < 2^n$ for any n, then $A[2](3,k) = B[2](3,k)$.*

Proof We know that $\mathrm{A}[2](3,i) = \mathrm{B}[2](3,i)$ for $i < 16$ [2] and so the assertion holds by Theorem 14. □

Theorem 17 *If $2.3^{n-1} + 2.3^{n-4} \leq k < 3^n$ for any n, then $A[3](2,k) = B[3](2,k)$.*

Proof In [7], it was shown that $\mathrm{A}[3](2,i) = \mathrm{B}[3](2,i)$ for $i < 81$. Therefore, the assertion follows from Theorem 15. □

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