Some results on the existence of large sets of t-designs*

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Abstract

A set of trivial necessary conditions for the existence of a large set of t-designs, $\mathrm{LS}[N](t,k,v)$, is $N\big|\binom{v-i}{k-i}$ for $i=0,\ldots,t$. There are two conjectures due to Hartman and Khosrovshahi which state that the trivial necessary conditions are sufficient in the cases $N=2$ and $N=3$, respectively. Ajoodani-Namini has established the truth of Hartman's conjecture for $t=2$. Apart from this celebrated result, we know the correctness of the conjectures for a few small values of k when $N=2$ and $t\leq 6$ and also when $N=3$ and $t\leq 4$. In this paper, we show that similar results can be obtained for infinitely many values of k.

Keywords: t-designs, large sets of t-designs, (N,t)-partitionable sets.

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1. Introduction

A t-(v,k,λ) design is a collection of k-subsets of a given v-set such that every t-subset of the v-set is exactly contained in λ elements of the collection. A large set of t-(v,k,λ) designs of size N, denoted by $\mathrm{LS}[N](t,k,v)$, is a partition of all k-subsets of a given v-set into N disjoint t-(v,k,λ) designs, where $N = \binom{v-t}{k-t}/\lambda$. A set of trivial necessary conditions for the existence of a $\mathrm{LS}[N](t,k,v)$ is that $N \big| \binom{v-i}{k-i}$ for $i = 0,\ldots,t$.

We investigate the existence problem of large sets of sizes two and three. In 1987, Hartman [6] conjectured that the trivial necessary conditions are sufficient for the existence of large sets of size $N = 2$. Then, Khosrovshahi proposed similar conjecture for $N = 3$ [2]. Ajoodani-Namini established the truth of Hartman's conjecture for $t = 2$ [1]. We also know that Hatrman's conjecture is true if $t \leq 5$ and $k \leq 16$ or $t = 6$ and $k \leq 9$ [3,10]. For $N = 3$, the problem has been solved when $t = 2$ and $k \leq 80$ [7] or $t \leq 4$ and $k \leq 8$ [12].

There is a generalization of the notion of large sets of t-designs, called (N,t)-partitionable sets which provide a powerful approach for constructing large sets. Let j_1, j_2, and n be nonnegative integers such that $j_1 \leq t/2$ and $j_2 \leq t$. The approach of (N,t)-partitionable sets, which has been employed in [1,7], shows that the families of large sets $\mathrm{LS}[2](t, 2^n + j_1, 2^{n+1} + t)$ and $\mathrm{LS}[3](t, 3^n + j_2, 3^{n+1} + t)$ can produce all other possible large sets of sizes 2 and 3. We call these families of large sets as *root cases* . The existence of some of these root cases gives some partial results which were stated in the paragraph above. In this paper, we use the results of [7] to obtain new existence theorems on the large sets of sizes 2 and 3 for infinitely many values of block sizes. We also present a different family of root cases to produce large sets.

2. Definitions and Preliminaries

Let t, k, v and λ be integers such that $0 \le t \le k \le v$ and $\lambda \ge 1$ and let X be a v-set. We denote the set of all k-subsets of X by $P_k(X)$. A t-(v, k, λ) *design* (briefly a t-design) on X is a collection $\mathcal{D}$ of k-subsets of X (called *blocks*) such that every t-subset of X is contained in exactly λ blocks of $\mathcal{D}$. Hereafter we always assume that $0 \le t < k < v$ to avoid trivial cases. A t-design with no repeated block is called a *simple* t-design. Here we are only concerned with simple t-designs. $P_k(X)$ is trivially a t-$(v, k, \binom{v-t}{k-t})$ design which is called the *complete* design.

A *large set* of t-(v, k, λ) designs on X, denoted by $\mathrm{LS}_\lambda(t, k, v)$ or $\mathrm{LS}[N](t, k, v)$, is a partition of $P_k(X)$ into N disjoint t-(v, k, λ) designs, where $N = \binom{v-t}{k-t}/\lambda$. By convention, we always assume that $N > 1$. A $\mathrm{LS}[2](t, k, v)$ which provides a halving of the complete design is called a *halving*.

It is well known that a set of trivial necessary conditions for the existence of a $\mathrm{LS}[N](t, k, v)$ is that

$$\binom{v-i}{k-i} \equiv 0 \pmod{N}, \qquad 0 \le i \le t. \tag{1}$$

By Baranyai's theorem [5], these conditions are also sufficient for $t = 1$.

Notation. Let N, t, and k be given. The set of all v for which $\mathrm{LS}[N](t, k, v)$ exist is denoted by $\mathrm{A}[N](t, k)$. The set of all v which satisfy the necessary conditions (1) is denoted by $\mathrm{B}[N](t, k)$. Every quadruple $(N; t, k, v)$ satisfying (1) is called an *admissible set of parameters*.

Let m and n be positive integers. We denote the quotient and remainder of division m by n by $[m/n]$ and (m/n), respectively. Let p^α be a prime power. We recall the following lemma from [7] for characterizing the admissible sets of parameters.

Lemma 1 [7]. *The quadruple $(p^\alpha; t, k, v)$ is admissible if and only if there exist distinct positive integers l_i for $1 \leq i \leq \alpha$ such that $t \leq (v/p^{l_i}) < (k/p^{l_i})$ for all i.*

Using Lemma 1, we can easily determine all the admissible sets of parameters for prime number $N = p$:

$$(p; t, k, v) = (p; t, mp^z + r, np^z + s), \tag{2}$$

where $0 \leq t \leq s < r < p^z$, $0 \leq m < n$. We can also assume that z is the smallest or the greatest number with the properties above to be assured of the uniqueness of the representation (2). This assumption forces other conditions on r and s. Further results concerning the admissible parameters can be found in [7].

Note that we can obtain more large sets from a given large set as the following theorems show.

Theorem 1 [7]. *If there exists a $LS[N](t, k, v)$, then there exist $LS[N](t - i, k - j, v - l)$ for all $0 \leq j \leq l \leq i \leq t$.*

Theorem 2. *If there exists a $LS[N](t, k, v)$, then there exists a $LS[N](t, v - k, v)$.*

The following well known extension theorems are among the important tools in constructing large sets.

Theorem 3 [7]. *If there exist $LS[N](t, k + i, v)$ for all $0 \leq i \leq l$, then there exist $LS[N](t, k + i, v + j)$ for all $0 \leq j \leq i \leq l$.*

Theorem 4 [4]. *Let t be even and N be arbitrary or let t be odd and $N = 2$. If there exists a $LS[N](t, k, 2k+1)$, then there exists a $LS[N](t+1, k+1, 2k+2)$.*

3. Recursive Constructions through (N, t)-Partitionable Sets

The most important tool in the construction of large sets is the notion of (N, t)-partitionable sets which was introduced in [3]. This idea is indeed a generalization of the notion of large sets, where we consider t-balanced partition of a subset $\mathcal{B}$ of $P_k(X)$ instead of the whole set $P_k(X)$. This notion can also be considered as a generalization of trades. Let $\mathcal{B}_1, \mathcal{B}_2 \subseteq P_k(X)$. We say that $\mathcal{B}_1$ and $\mathcal{B}_2$ are *t-equivalent* if every t-subset of X appears in the same number of blocks of $\mathcal{B}_1$ and $\mathcal{B}_2$. If there exists a partition of $\mathcal{B} \subseteq P_k(X)$ into N mutually t-equivalent subsets, then $\mathcal{B}$ is called (N, t)-*partitionable* set. Let X_1 and X_2 be two disjoint sets and let $\mathcal{B}_i \subseteq P_{k_i}(X_i)$ for $i = 1, 2$. Then we define

$$\mathcal{B}_1 * \mathcal{B}_2 = \{B_1 \cup B_2 |\ B_1 \in \mathcal{B}_1,\ B_2 \in \mathcal{B}_2\}.$$

There are two important lemmas on (N, t)-*partitionable* sets.

Lemma 2 [3]. (i) *t-equivalence implies i-equivalence for all $0 \leq i \leq t$.*
(ii) *The union of disjoint (N, t)-partitionable sets is again an (N, t)-partitionable set.*

Lemma 3 [3]. *Let X_1 and X_2 be two disjoint sets and let $\mathcal{B}_i \subseteq P_{k_i}(X_i)$ for $i = 1, 2$. Suppose that $\mathcal{B}_1$ is (N, t_1)-partitionable. Then*

(i) *$\mathcal{B}_1 * \mathcal{B}_2$ is (N, t_1)-partitionable.*

(ii) *If* $\mathcal{B}_2$ *is* (N, t_2)*-partitionable, then* $\mathcal{B}_1 * \mathcal{B}_2$ *is* (N, t_1+t_2+1)*-partitionable.*

By Lemma 2(ii), if we are able to partition $P_k(X)$ into disjoint (N, t)-partitionable sets, then we obtain a large set. This technique in combination with Lemma 3 provides a general approach for the recursive and direct constructions of large sets. We first outline the approach by a simple example.

Example. Construction of a LS[3](2, 12, 29) from LS[3](2, 3, 11), LS[3](2, 7, 15), LS[3](2, 8, 16) and LS[3](2, 12, 20). Let X be a 29-set and consider the following partitioning of $P_{12}(X)$:

$$
\begin{aligned}
\mathcal{B}_0 &= P_{12}(\{1, \ldots, 20\}), \\
\mathcal{B}_1 &= P_{11}(\{1, \ldots, 19\}) * P_1(\{21, \ldots, 29\}), \\
\mathcal{B}_2 &= P_{10}(\{1, \ldots, 18\}) * P_2(\{20, \ldots, 29\}), \\
\mathcal{B}_3 &= P_9(\{1, \ldots, 17\}) * P_3(\{19, \ldots, 29\}), \\
\mathcal{B}_4 &= P_8(\{1, \ldots, 16\}) * P_4(\{18, \ldots, 29\}), \\
\mathcal{B}_5 &= P_7(\{1, \ldots, 15\}) * P_5(\{17, \ldots, 29\}), \\
\mathcal{B}_6 &= P_6(\{1, \ldots, 14\}) * P_6(\{16, \ldots, 29\}), \\
\mathcal{B}_7 &= P_5(\{1, \ldots, 13\}) * P_7(\{15, \ldots, 29\}), \\
\mathcal{B}_8 &= P_4(\{1, \ldots, 12\}) * P_8(\{14, \ldots, 29\}), \\
\mathcal{B}_9 &= P_3(\{1, \ldots, 11\}) * P_9(\{13, \ldots, 29\}), \\
\mathcal{B}_{10} &= P_2(\{1, \ldots, 10\}) * P_{10}(\{12, \ldots, 29\}), \\
\mathcal{B}_{11} &= P_1(\{1, \ldots, 9\}) * P_{11}(\{11, \ldots, 29\}), \\
\mathcal{B}_{12} &= P_{12}(\{10, \ldots, 29\}).
\end{aligned}
$$

$\mathcal{B}_0$ and $\mathcal{B}_{12}$ are (3,2)-partitionable sets because there is a LS[3](2, 12, 20). By Theorem 1, there exist LS[3](1, 2, 10), LS[3](0, 1, 9), LS[3](1, 6, 14), and LS[3](1, 11, 19). Using Lemma 3, it is an easy task to see that all the remaining $\mathcal{B}_i$ are also (3,2)-partitionable sets. For example, consider $\mathcal{B}_2$ and $\mathcal{B}_3$. $P_{10}(\{1,\ldots,18\})$ is (3,0)-partitionable because $\binom{18}{10}$ is divisible by 3. $P_2(\{20,\ldots,29\})$ is (3,1)-partitionable since LS[3]$(1,2,10)$ exists. Now Lemma 3(ii) shows that $\mathcal{B}_2$ is (3,2)-partitionable . Because of the existence of a LS[3]$(2,3,11)$, $P_3(\{19,\ldots,29\})$ is a (3,2)-partitionable set and so is $\mathcal{B}_3$ by Lemma 3(i). The remaining $\mathcal{B}_i$ are dealt with in similar ways. Hence, by Lemma 2, we obtain that $P_{12}(X)$ is a (3,2)-partitionable set and a LS[3]$(2,12,29)$ is constructed.

The general form of the specific partitioning of $P_k(X)$ which appeared in the example above is as follows.

Lemma 4. *Let $X=\{1,\ldots,u+v+1\}$ and let $X_j=\{1,\ldots,j\}$ and $Y_j=X\setminus X_j$ for $j=1,\ldots,u+v+1$. Assume that*

$$\mathcal{B}_i=P_{k-i}(X_{u-i})*P_i(Y_{u-i+1}),\qquad 0\le i\le k.$$

Then the sets $\mathcal{B}_i$ partition $P_k(X)$.

Using the same approach of the example and induction, a powerful recursive construction method is obtained. It is presented in the following theorem.

Theorem 6 [3]. *If LS$[N](t,i,v+i)$ exist for all $t+1\le i\le k$ and a LS$[N](t,k,u)$ also exists, then LS$[N](t,k,u+l(v+1))$ exist for all $l\ge 1$.*

For prime number $N=p$, we have a stronger version of Theorem 6.

Theorem 7 [7]. *Let l be a positive number such that $(k/p^l)>t$. Suppose that $([k'/p^l]+1)p^l+t\in A[p](t,k')$ for all k' provided that $t+1\le k'\le k$ and $(k'/p^l)>t$. Then $np^l+t\in A[p](t,k)$ for all $n>[k/p^l]$.*

Theorem 7 leads to the following comprehensive result.

Theorem 8 [7]. *Let l be a positive integer such that $(k/p^l) > t$. Suppose that $p^l + t \in A[p](t, k')$ for all k' provided that $t+1 \leq k' \leq min(k, (p^l+t)/2)$. If $t \leq (v/p^l) < (k/p^l)$, then $v \in A[p](t, k)$.*

The advantage of Theorem 8 is best understood by considering (2). By (2), every admissible set of parameters is in the form $(p; t, mp^z + r, np^z + s)$, where $t \leq s < r < p^z$ and $0 \leq m < n$. The question is: for given z, which of the large sets in the family of $\mathrm{LS}[p](t, mp^z + r, np^z + s)$ are sufficient to produce the whole family? By Theorem 8, the existence of $\mathrm{LS}[p](t, r, p^z + t)$ for all $t < r \leq (p^z + t)/2$ will be enough for this purpose.

4. Main Results

Using Theorem 8, we can determine the special large sets which are used in the recursive constructions of the remaining large sets. The proofs of the next two theorems are similar and hence we only present the proof of Theorem 10.

Theorem 9. *Let s, t, and l be positive integers such that $2^s - 1 \leq t < 2^{s+1} - 1$ and $l > s$. Suppose that for every j and n such that $0 \leq j \leq [t/2], s \leq n < l$, and $2^n + j > t$, there exists a LS$[2](t, 2^n + j, 2^{n+1} + t)$. Then $A[2](t, k) = B[2](t, k)$ for all $k < 2^l$. Moreover, if $t \leq (v/2^f) < (k/2^f)$ for any v, k, and f such that $f \leq l$, then $v \in A[2](t, k)$.*

Theorem 10. *Let s, t, and l be positive integers such that $3^s - 1 \leq t < 3^{s+1} - 1$ and $l > s$. Suppose that the following conditions hold:*

(i) *There exists a* $LS[3](t, 3^s+j, 3^{s+1}+t)$ *for every* $t-3^s+1 \leq j \leq (3^s+t)/2$.

(ii) *There exists a* $LS[3](t, 3^n + j, 3^{n+1} + t)$ *for every* j *and* n *such that* $0 \leq j \leq t$ *and* $s < n < l$.

Then $A[3](t,k) = B[3](t,k)$ *for all* $k < 3^l$. *Moreover, if* $t \leq (v/3^f) < (k/3^f)$ *for any* v, k, *and* f *such that* $f \leq l$, *then* $v \in A[3](t,k)$.

Proof First we show that $\mathrm{A}[3](t,k) = \mathrm{B}[3](t,k)$ for all $k < 3^l$. We use an induction on k. First let $k = t+1$. We have $\min(k, (3^{s+1}+t)/2) = k$. Hence, by Lemma 1 and Theorem 8, the only necessary large set to establish the assertion is $\mathrm{LS}[3](t, k, 3^{s+1} + t)$ which exists by the assumption (i). Now let $t + 1 < k < 3^l$. Let h be the smallest positive integer such that $(k/3^h) > t$. Assume that we have shown that

$$3^h + t \in \mathrm{A}[3](t, k'), \text{ for all } t + 1 \leq k' \leq \min(k, (3^h + t)/2). \tag{3}$$

Let $v \in \mathrm{B}[3](t,k)$. By Lemma 1, there exists $r \geq h$ such that $t \leq (v/3^r) < (k/3^r)$. We have $[v/3^r]3^r+t \in \mathrm{A}[3](t, [k/p^r]\, 3^r+j)$ for all $(k/3^r)-(v/3^r)+t \leq j \leq (k/3^r)$. Since if $j < (k/3^r)$, we are done by the induction hypothesis. If $j = (k/3^r)$, then we are done by (3) and Theorem 8. Hence, by Theorem 3, $v = [v/3^r]3^r + (v/3^r) \in \mathrm{A}[3](t,k)$. To prove (3), note that if $k > (3^h + t)/2$, then we are done by the induction. So let

$$k \leq (3^h + t)/2. \tag{4}$$

By the induction hypothesis, it is sufficient to establish the existence of an $\mathrm{LS}[3](t, k, 3^h + t)$. From (4), we have $[k/3^h] = 0$. Therefore, $h \geq s + 1$. If $h = s + 1$, then by (i), we have an $\mathrm{LS}[3](t, k, 3^{s+1} + t)$. So suppose that $h > s+1$. By (4), we have $k = 3^{h-1} + j$, where $j \leq t$. Now $\mathrm{LS}[3](t, k, 3^h + t)$ exists by (ii).

The next part of the assertion follows from Theorem 8. □

The families of large sets $\mathrm{LS}[2](t, 2^n + j_1, 2^{n+1} + t)$ and $\mathrm{LS}[3](t, 3^n + j_2, 3^{n+1} + t)$ in which j_1, j_2, and n are nonnegative integers such that $j_1 \leq t/2$ and $j_2 \leq t$ are called *root cases*. Their constructions will be sufficient to establish the conjectures. Ajoodani-Namini has constructed the root cases $\mathrm{LS}[2](2, 2^n, 2^{n+1} + 2)$ for every n by the use of (N, t)-partitionable sets [1]. The family of large sets $\mathrm{LS}[2](2, 2^n + 1, 2^{n+1} + 2)$ can be constructed by means of Theorem 4 and Baranyai's theorem [5]. Therefore, Hartman's conjecture is true for $t = 2$. We now improve the partial results that exist beyond this case.

Theorem 11. *If $k < 81$, then $A[3](2, k) = B[3](2, k)$. Moreover, if $2 \leq (v/3^f) < (k/3^f)$ for any v, k, and f such that $f \leq 4$, then $v \in A[3](2, k)$.*

Proof. By Theorem 10, we have to establish the existence of the following large sets:

(i) $\mathrm{LS}[3](2, 9, 29)$, (ii) $\mathrm{LS}[3](2, 10, 29)$, (iii) $\mathrm{LS}[3](2, 11, 29)$,

(iv) $\mathrm{LS}[3](2, 27, 83)$, (v) $\mathrm{LS}[3](2, 28, 83)$, (vi) $\mathrm{LS}[3](2, 29, 83)$.

There exist $\mathrm{LS}[3](2, 9, 29)$ and $\mathrm{LS}[3](3, 11, 30)$ by [11]. By the last large set we obtain a $\mathrm{LS}[3](2, 10, 29)$ and a $\mathrm{LS}[3](2, 11, 29)$ using Theorem 1. Large sets (iv)-(vi) are constructed by Theorem 6 in [11]. □

Theorem 12. *Let $3 \leq t \leq 5$. If $k < 16$, then $A[2](t, k) = B[2](t, k)$. Moreover, if $t \leq (v/2^f) < (k/2^f)$ for any v, k, and f such that $f \leq 4$, then $v \in A[2](t, k)$.*

Proof. There exist large sets $\mathrm{LS}[2](6,7,14), \mathrm{LS}[2](6,8,22), \mathrm{LS}[2](6,9,22)$, and $\mathrm{LS}[2](5,10,21)$ by [9], [10], [10], and [10], respectively. Now the assertion follows from Theorems 1, 8, and 9. $\square$

Theorem 13. *Let $t \leq 4$. If $k < 9$, then $A[3](t,k) = B[3](t,k)$. Moreover, if $t \leq (v/9) < (k/9)$ for any v and k, then $v \in A[3](t,k)$.*

Proof. Large sets $\mathrm{LS}[3](4,5,13)$ and $\mathrm{LS}[3](4,6,13)$ exist by [8] and [11], respectively. Therefore, by Theorems 1 and 10, we are done. $\square$

5. Another Class of Root Cases

In the previous section, we showed that one can construct all possible large sets of size two and large sets of size three from the root cases $\mathrm{LS}[2](t, 2^n + j_1, 2^{n+1} + t)$ and $\mathrm{LS}[3](t, 3^n + j_2, 3^{n+1} + t)$, respectively, where j_1, j_2, and n are nonnegative integers such that $j_1 \leq t/2$ and $j_2 \leq t$. It is possible to introduce a different class of large sets as the root cases. These classes are $\mathrm{LS}[2](2^n - 2, 2^n - 1, 2^{n+1} - 2)$ and $\mathrm{LS}[3](3^n - 2, 3^n - 1, 2.3^n - 2)$, where n is a positive integer. The following theorems have similar proofs and we only present the proof of Theorem 15.

Theorem 14. *If there exists $LS[2](2^n - 2, 2^n - 1, 2^{n+1} - 2)$ for every positive integer n, then $A[2](t,k) = B[2](t,k)$ for every t and k.*

Theorem 15. *If there exists $LS[3](3^n - 2, 3^n - 1, 2.3^n - 2)$ for every positive integer n, then $A[3](t,k) = B[3](t,k)$ for every t and k.*

Proof. For a given t, assume that $3^s - 1 \leq t < 3^{s+1} - 1$. By Theorem 10, it is sufficient to show the existence of the following families of large sets:

(i) $\mathrm{LS}[3](t, 3^s + j_0, 3^{s+1} + t)$, where $t - 3^s + 1 \leq j_0 \leq (3^s + t)/2$,

(ii) $\mathrm{LS}[3](t, 3^n + j_0, 3^{n+1} + t)$, where $n > s$ and $0 \leq j_0 \leq t$.

The large sets in the class (i) can be obtained from $\mathrm{LS}[3](3^{s+1} - 2, 3^{s+1} - 1, 2.3^{s+1} - 2)$ using Theorem 1. Simply, suppose that $i = 3^{s+1} - 2 - t, j = 2.3^s - 1 - j_0$, and $l = 3^{s+1} - 2 - t$. Similarly, the class (ii) can be constructed from $\mathrm{LS}[3](3^{n+1} - 2, 3^{n+1} - 1, 2.3^{n+1} - 2)$ by taking $i = 3^{n+1} - 2 - t, j = 2.3^n - 1 - j_0$, and $l = 3^{n+1} - 2 - t$ in Theorem 1. $\square$

Finally, we note that by Theorem 4, the large sets $\mathrm{LS}[2](2^n - 2, 2^n - 1, 2^{n+1} - 2)$ and $\mathrm{LS}[3](3^n - 2, 3^n - 1, 2.3^n - 2)$ can be considered as the unique extensions of $\mathrm{LS}[2](2^n - 3, 2^n - 2, 2^{n+1} - 3)$ and $\mathrm{LS}[3](3^n - 3, 3^n - 2, 2.3^n - 3)$, respectively. Therefore, it is possible to consider these latter classes as the root cases which have to be constructed.

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