Large sets of t-designs through partitionable sets: A survey[1]

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Abstract

The method of partitionable sets for constructing large sets of t-designs have now been used for nearly a decade. The method has resulted in some powerful recursive constructions and also existence results especially for large sets of prime sizes. Perhaps the main feature of the approach is its simplicity. In this paper, we describe the approach and show how it is employed to obtain some of the recursive theorems. We also review the existence results and recursive constructions which have been found by this method.

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1 Introduction

A large set of t-(v, k, λ) designs of size N is a partition of the set of all k-subsets of a v-set into block sets of t-(v, k, λ) designs, where $N = \binom{v-t}{k-t}/\lambda$. Large sets by themselves are not only interesting combinatorial arrangements, but also they provide a possible setting for the study of the existence problem of t-designs. The celebrated theorem of Teirlink on the existence of t-designs for all t involves constructing large sets of t-designs.

The known existence results on large sets have been obtained by various methods which are very different in nature. In 1975, Baranyai settled the existence of large sets of Steiner 1-designs [7]. Later, Hartman using this result established the existence of large sets of 1-designs in general [17]. During the seventies of the last century, many combinatorialists worked on the problem of large sets of Steiner triple systems. But it was Lu who finally solved the problem in 1984 [28] with

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a few exceptions which later on were completed by Teirlink [37]. Later, the existence problem for large sets of designs with $t = 2$ and $k = 3$ was solved [28, 29, 32, 33, 37, 41]. The next great achievement was obtained by Teirlink who showed that large sets of t-designs exist for all t [39]. In 1987, an important conjecture by Hartman (also known as halving conjecture) which asserts that large sets of size 2 exist for all parameter sets satisfying the trivial necessary conditions appeared [17]. This conjecture inspired the researchers in this field and initiated many new results on the existence problem of large sets. A new approach sprouted out from these efforts now known as the method of *partitionable sets*. The best result found by this method is due to Ajoodani-Namini who showed that the halving conjecture is true for 2-designs [1]. After that, the method was used for constructing large sets of prime sizes. At present most of the results obtained by the approach of partitionable sets is for large sets of prime sizes, although some important recursive constructions have also been found for large sets in the general case. One of the main features of this approach is its simplicity. For example, Teirlink's long and complicated proof of the existence of t-designs for all t can be established in less than a page by the use of partitionable sets. The approach has also provided some extension theorems which are unique in design theory in the sense that no further conditions are imposed on the parameters. In this paper, after definitions and review of the known results by other methods, we first describe the approach and review the results which have been found for large sets of any sizes. Then we pay our attention to large sets of prime sizes. There are nice results on large sets of prime sizes including the notion of root cases which is discussed in Sections 7 and 8. Throughout the paper, we provide proofs for some theorems for clarification and instructional purposes. Large sets of sizes 2 and 3 are of special interest and there are more comprehensive results for them. We devote a separate section to these cases. The existence results obtained by the approach are reviewed in Section 9. We finish the paper with some open problems.

2 Definitions and Preliminaries

Let t, k, v and λ be integers such that $0 \le t \le k \le v$ and $\lambda > 0$. Let X be a v-set and $P_k(X)$ denote the set of all k-subsets of X. A t-(v, k, λ) *design* (briefly a t-design) is a pair $\mathbf{D} = (X, \mathcal{D})$ in which $\mathcal{D}$ is a collection of elements of $P_k(X)$ (called *blocks*) such that every t-subset of X appears in exactly λ blocks. If $\mathcal{D}$ has no repeated blocks, then $\mathbf{D}$ is called *simple*. Here we are concerned only with simple designs. Note that $(X, P_k(X))$ is a t-$(v, k, \binom{v-t}{k-t})$ design which is called the *complete* design.

A simple counting argument shows that a t-(v, k, λ) design is also an i-(v, k, λ_i) design for $0 \le i \le t$, where $\lambda_i = \lambda\binom{v-i}{t-i}/\binom{k-i}{t-i}$. In particular, λ_0 is the number of blocks in the design.

Hence, a set of necessary conditions for the existence of a t-(v,k,λ) design is

$$\lambda\binom{v-i}{t-i} \equiv 0 \quad \left(\operatorname{mod}\binom{k-i}{t-i}\right), \qquad 0 \le i \le t. \tag{2.1}$$

Using $\binom{v-i}{t-i}\binom{v-t}{k-t} = \binom{v-i}{k-i}\binom{k-i}{t-i}$, one can easily see that the conditions (2.1) are equivalent to

$$\lambda\binom{v-i}{k-i} \equiv 0 \quad \left(\operatorname{mod}\binom{v-t}{k-t}\right), \qquad 0 \le i \le t. \tag{2.2}$$

The minimum value of λ satisfying (2.1) is denoted by $\lambda_{\min}$ and any other feasible λ is clearly an integral multiple of $\lambda_{\min}$. The λ of the complete design is denoted by $\lambda_{\max}$.

Some more notation. Let

$$\begin{aligned}
\mathcal{D}^d(x) &= \{B \setminus \{x\} |\ x \in B \in \mathcal{D}\},\\
\mathcal{D}^r(x) &= \{B |\ x \notin B \in \mathcal{D}\},\\
\mathcal{D}^c(x) &= \{X \setminus B |\ B \in \mathcal{D}\},\\
\mathcal{D}^s &= \{B |\ B \notin \mathcal{D}\}.
\end{aligned}$$

Then $\mathbf{D}^d(x) = (X \setminus \{x\}, \mathcal{D}^d(x))$ and $\mathbf{D}^r(x) = (X \setminus \{x\}, \mathcal{D}^r(x))$ are $(t-1)$-$(v-1, k-1, \lambda)$ and $(t-1)$-$(v-1, k, \lambda_{t-1} - \lambda)$ designs, respectively, and are called *derived* and *residual* designs of $\mathbf{D}$ with respect to x. By the inclusion-exclusion principle, it is also seen that for $t \le v-k$, $\mathbf{D}^c = (X, \mathcal{D}^c)$ is a t-$(v, v-k, \lambda^c)$ design, where $\lambda^c = \sum_{i=0}^{t}(-1)^t\binom{t}{i}\lambda_i$ and is called the *complement* of $\mathbf{D}$. The *supplement* of $\mathbf{D}$, $\mathbf{D}^s = (X, \mathcal{D}^s)$, is a t-$(v,k,\lambda_{\max} - \lambda)$ design.

Let $N \ge 1$. A *large set* of t-(v,k,λ) designs of size N, denoted by $\mathrm{LS}[N](t,k,v)$, is a set $\mathbf{L}$ of N disjoint t-(v,k,λ) designs $\mathbf{D}_i = (X, \mathcal{D}_i)$ such that $\{\mathcal{D}_i |\ 1 \le i \le N\}$ is a partition of $P_k(X)$. Note that we have $N = \binom{v-t}{k-t}/\lambda$. Sometimes $\mathrm{LS}[N](t,k,v)$ is denoted by $\mathrm{LS}_\lambda(t,k,v)$ to show λ. If λ is one, it can be omitted. By (2.2), we observe that a set of necessary conditions for the existence of an $\mathrm{LS}[N](t,k,v)$ is

$$N \Big| \binom{v-i}{k-i}, \qquad 0 \le i \le t. \tag{2.3}$$

The *derived, residual* and *complementary* large sets of $\mathbf{L} = \{\mathbf{D}_i\}$ with respect to x are defined as $\mathbf{L}^d(x) = \{\mathbf{D}_i^d(x)\}$, $\mathbf{L}^r(x) = \{\mathbf{D}_i^r(x)\}$ and $\mathbf{L}^c = \{\mathbf{D}_i^c\}$ (when $t \le v-k$) which are $\mathrm{LS}[N](t-1, k-1, v-1)$, $\mathrm{LS}[N](t-1, k, v-1)$ and $\mathrm{LS}[N](t, v-k, v)$, respectively. Note that we can obtain more large sets from a given large set as the following theorem suggests using derived and residual large sets.

Theorem 2.1 **[3, 20]** *If there exists an LS*$[N](t,k,v)$*, then there exist LS*$[N](t-i, k-j, v-l)$ *for all* $0 \le j \le l \le i \le t$*.*

Notation Let N, t, and k be given integers such that $N > 0$ and $0 \le t \le k$. The set of all v for which an LS$[N](t, k, v)$ exists is denoted by $A[N](t, k)$. The set of all v which satisfy the necessary conditions (2.3) is denoted by $B[N](t, k)$. Any quadruple $(N; t, k, v)$ satisfying (2.3) is called an *admissible set of parameters*. Throughout this paper, when we speak of quadruples such as $(N; t, k, v)$, we implicity suppose that $N > 0$ and $0 \le t \le k \le v$. Hereafter, we let p^α be a prime power where p is prime. Let m and n be positive integers. We denote the quotient and remainder of division m by n by $[m/n]$ and (m/n), respectively.

Example The block sets of two designs of the unique LS$[2](2, 3, 6)$ are as follows.

$$\mathcal{D}_1 = \{123, 124, 135, 146, 156, 236, 245, 256, 345, 346\},$$

$$\mathcal{D}_2 = \{125, 126, 134, 136, 145, 234, 235, 246, 356, 456\},$$

where 123 stands for $\{1, 2, 3\}$, etc.

Example The necessary conditions (2.3) are not always sufficient. A hundred and fifty years ago, Cayley showed that it is possible to have two disjoint 2-(7,3,1) designs and no more [10]. So there are no LS$(2, 3, 7)$ and LS$(3, 4, 8)$.

3 Review of the known large sets

In this section we give a brief account of the known results on the existence of large sets of t-designs found by various methods. The results obtained by the approach of partitionable sets which is the main subject of this paper will be presented in the final sections. Some parts of this section has been taken from [24].

In 1975, Baranyai showed that there exists an LS$(1, k, v)$ if and only if $k|v$. The proofs related to this result employ the integrality theorem on flows in transportation networks. Two proofs can be found in [8, 42]. Hartman has extended the result for all values of k and v as stated in the following theorem.

Theorem 3.1 **[7, 17]** *$A[N](1, k) = B[N](1, k)$ for all positive integers N and k.*

Another celebrated theorem was obtained by Lu and Teirlink who showed that LS$(2, 3, v)$ exists if and only if $v > 7$ and $v \equiv 1, 3 \pmod 6$. This result was obtained after a lot of works done by many researchers. The whole story about triple systems is given in the following theorem.

Theorem 3.2 **[28, 29, 32, 33, 37, 41]** *$A[N](2,3) = B[N](2,3) \setminus \{7\}$ for all positive integers N.*

In 1987, Teirlink proved the following theorem which was greatly acknowledged at the time since it did offer a proof of existence of t-designs for all values of t.

Theorem 3.3 **[39]** *For all positive integers N and t, there is an integer v such that an $LS[N](t, t+1, v)$ exists.*

Note As Theorems 3.1 and 3.2 show all admissible $\mathrm{LS}(1,k,v)$ and all admissible $\mathrm{LS}(2,3,v)$ except for $v = 7$ exist. Beyond these cases the only known $\mathrm{LS}(t,k,v)$ is an $\mathrm{LS}(2,4,13)$ constructed in [13]. Etzion and Hartman have constructed $v-5$ disjoint 3-$(v,4,1)$ designs for $v = 5.2^n$. This leaves only two more to go for an $\mathrm{LS}(3,4,v)$ [16].

Some other miscellaneous results on the existence of large sets are as follows.

(i) An $\mathrm{LS}_{\lambda_{\min}}(3,4,v)$ exists if $v \equiv 0 \pmod 3$ [40].

(ii) An $\mathrm{LS}_{\lambda_{\min}}(4,5,20v+4)$ exists if $\gcd(v,30) = 1$ [38].

(iii) An $\mathrm{LS}_{60}(4,5,60v+4)$ exists if $\gcd(v,60) = 1,2$ [38].

Alltop [6] has proved a theorem on extending t-designs. We state a similar result for large sets. The proof is essentially the same.

Theorem 3.4 *Let t be even and N be a positive integer or, let t be odd and $N = 2$. If there exists an $LS[N](t,k,2k+1)$, then there exists an $LS[N](t+1,k+1,2k+2)$.*

The theorem has a useful consequence.

Corollary 3.1 *An $LS[2](2,k,2k)$ exists if and only if k is not a power of 2.*

Proof $\binom{2k-1}{k-1}$ and $\binom{2k-2}{k-2}$ are even if and only if k is not a power of 2 (see for example Theorem 4.1). Therefore, by Theorem 3.1, an $\mathrm{LS}[2](1,k-1,2k-1)$ exists if and only if k is not a power of 2. Now the assertion follows from Theorem 3.4. $\square$

Small cases of large sets play an important role in the constructions of large sets in general. They are initial points in recursive methods to produce infinite families of large sets. In [22], all parameter sets on less than or equal 12 points have been settled. In [12], a table on the existence of large sets with at most 18 points is presented, but it has to be updated. Most of small designs have been found by prescribing some groups as automorphism groups of designs. This approach was formulated for the first time by Kramer and Mesner [23]. The idea is simply that if there exist t-(v, k, λ) designs, then probably some of them have nontrivial automorphism groups. Therefore, we can reverse the procedure and try some suitable groups as automorphism groups of desired designs. This approach can be used both computationally and theoretically. Using computer and sometimes hand checking, many small designs and large sets have been constructed by the method. The results can be found in the literature. A reference list includes [11, 12, 14, 21, 22, 23, 26, 27]. The only remarkable theoretic works done so far are related to the groups $\mathrm{PSL}(2, q)$ and $\mathrm{PGL}(2, q)$. Here, we do not have the intention to present those results. The reader can consult [9, 15, 18, 19, 27, 31]

4 The necessary conditions

In this section, the necessary conditions for the existence of $\mathrm{LS}[N](t, k, v)$ as given in (2.3) are dealt with. It is possible to give an alternative description of (2.3) when N is a prime power. If N is not a prime power, then we can factorize it into prime powers and apply our results to its prime power factors. The main theorem is as follows.

Theorem 4.1 **[20]** *The quadruple $(p^\alpha; t, k, v)$ is admissible if and only if there exist distinct positive integers ℓ_i $(1 \leq i \leq \alpha)$ such that $t \leq (v/p^{\ell_i}) < (k/p^{\ell_i})$.*

Example By Theorem 4.1, $\mathrm{LS}[55](2, 4, 13)$ is admissible. Since we have $2 \leq (13/5) < (4/5)$ and $2 \leq (13/11) < (4/11)$.

Example What is the largest value of t for which $\mathrm{LS}[13](t, 9, 18)$ is admissible? By Theorem 4.1, we must have $t \leq (18/13^\alpha) < (9/13^\alpha)$ and hence $\alpha = 1$ and $t_{\max} = 5$.

Using this theorem, we can easily determine all the admissible sets of parameters for $N = p$:

$$(p; t, k, v) = (p; t, mp^z + r, np^z + s), \tag{4.1}$$

where $0 \leq t \leq s < r < p^z$ and $0 \leq m < n$. We can also assume that z is the smallest or the greatest number with the properties above to be assured of the uniqueness of the representation (4.1). By Theorem 4.1, we are also able to identify $B[N](t, t+1)$ completely.

Theorem 4.2 **[20]** *Let $\prod_{i=1}^{s} p_i^{\alpha_i}$ be the prime power factorization of N. For $1 \leq i \leq s$, suppose* *that* $p_i^{s_i-1} \leq t+1 < p_i^{s_i}$. *Then*

$$B[N](t,t+1) = \left\{ v \middle| \ v \equiv t \mod \prod_{i=1}^{s} p_i^{\alpha_i+s_i-1} \right\}.$$

The following result is due to Teirlink and it can be obtained from Theorem 4.2.

Theorem 4.3 **[36]** *For* $k = t+1$, *we have*

$$\lambda_{min} = gcd(v-t, lcm(1,\ldots,t+1)).$$

Proof Let $\prod_{i=1}^{s} p_i^{\alpha_i}$ be the prime power factorization of $v-t$ and let $p_i^{s_i-1} \leq t+1 < p_i^{s_i}$ for $1 \leq i \leq s$. If $v \in B[N](t,t+1)$, then by Theorem 4.2, N is at most equal to $\prod_{i=1}^{s} p_i^{\alpha_i - s_i + 1}$. Therefore, $\lambda_{\min} = \lambda_{\max}/N = \prod_{i=1}^{s} p_i^{s_i-1}$. This proves the assertion. □

We bring this section to an end by presenting another useful application of Theorem 4.1.

Theorem 4.4 **[20]** *Let* $0 \leq t < k$. *Then the minimal element of* $B[p^\alpha](t,k)$ *is equal to*

$$v_{min} = ([k/p^{\ell+\alpha-1}]+1)p^{\ell+\alpha-1} + t$$

in which ℓ *is the smallest positive integer such that* $(k/p^\ell) > t$.

5 The approach of partitionable sets

A powerful approach for the construction of large sets is obtained from the notion of (N,t)-partitionable sets which was first introduced in [5]. This idea is indeed a generalization of the notion of large sets, where we consider t-balanced partition of a subset $\mathcal{B}$ of $P_k(X)$ instead of the whole set $P_k(X)$. Let $\mathcal{B}_1, \mathcal{B}_2 \subseteq P_k(X)$. We say that $\mathcal{B}_1$ and $\mathcal{B}_2$ are *t-equivalent* if every t-subset of X appears in the same number of blocks of $\mathcal{B}_1$ and $\mathcal{B}_2$. If there exists a partition of $\mathcal{B} \subseteq P_k(X)$ into N mutually t-equivalent subsets, then $\mathcal{B}$ is called an *(N,t)-partitionable* set. In the literature of design theory, *$(2,t)$-partitionable sets* are very well known objects called *trades*. So one can also consider (N,t)-partitionable sets as a generalization of trades. Let X_1 and X_2 be two disjoint sets and let $\mathcal{B}_i \subseteq P_{k_i}(X_i)$ for $i = 1,2$. Then we define

$$\mathcal{B}_1 * \mathcal{B}_2 = \{B_1 \cup B_2 | \ B_1 \in \mathcal{B}_1, \ B_2 \in \mathcal{B}_2\}.$$

There are two important lemmas concerning (N,t)-partitionable sets. The first one is trivial while the other one is a very unexpected.

Lemma 5.1 [5] (i) *t-equivalence implies i-equivalence for all $0 \leq i \leq t$.*
(ii) *The union of disjoint (N, t)-partitionable sets is again an (N, t)-partitionable set.*

Lemma 5.2 [5] *Let X_1 and X_2 be two disjoint sets and let $\mathcal{B}_i \subseteq P_{k_i}(X_i)$ for $i = 1, 2$. Suppose that $\mathcal{B}_1$ is (N, t_1)-partitionable. Then*

(i) *$\mathcal{B}_1 * \mathcal{B}_2$ is (N, t_1)-partitionable.*
(ii) *If $\mathcal{B}_2$ is (N, t_2)-partitionable, then $\mathcal{B}_1 * \mathcal{B}_2$ is $(N, t_1 + t_2 + 1)$-partitionable.*

The importance of Lemma 5.2 is seen at the first glance. In the theory of t-designs, extension theorems which increase the value of t are very rare (one example is Theorem 3.4). If Lemma 5.2 is employed in a clever way, then very useful extension theorems can be found. We can now state our method for constructing large sets based on Lemmas 5.1 and 5.2. Suppose that we are looking for an LS$[N](t, k, v)$ on a v-set X. We try to partition $P_k(X)$ in a such a way that each part of the partition is an (N, t)-partitionable set. If this done, then by Lemma 5.1, $P_k(X)$ will be an (N, t)-partitionable set which means that we have obtained an LS$[N](t, k, v)$. Each part $\mathcal{B}$ in the partition is usually of the form $P_{k_1}(X_1) * P_{k_2}(X_2)$ where X_1 and X_2 are disjoint subsets of X and $k = k_1 + k_2$. If there exist LS$[N](t_1, k_1, v_1)$ and LS$[N](t_2, k_2, v_2)$ and $t = t_1 + t_2 + 1$, then by Lemma 5.2, $\mathcal{B}$ is (N, t)-partitionable. The approach is understood better with the following simple example.

Example. Construction of an LS$[2](2, 3, 10)$ from an LS$[2](2, 3, 6)$. Let $X = \{1, 2, \ldots, 10\}$ and consider the following partitioning of $P_3(X)$:

$$\begin{aligned}
\mathcal{B}_1 &= P_3(\{1, \ldots, 6\}), \\
\mathcal{B}_2 &= P_2(\{1, \ldots, 5\}) * P_1(\{7, \ldots, 10\}), \\
\mathcal{B}_3 &= P_1(\{1, \ldots, 4\}) * P_2(\{6, \ldots, 10\}), \\
\mathcal{B}_4 &= P_3(\{5, \ldots, 10\}).
\end{aligned}$$

$\mathcal{B}_1$ and $\mathcal{B}_4$ are (2,2)-partitionable by the assumption. By Theorem 2.1, there exist LS$[2](1, 2, 5)$ and LS$[2](0, 1, 4)$. Therefore, $\mathcal{B}_2$ and $\mathcal{B}_3$ are (2,2)-partitionable sets by Lemma 5.2. Now Lemma 5.1 shows that $P_3(X)$ is (2,2)-partitionable set, i. e. an LS$[2](2, 3, 10)$ is constructed.

The general form of the partitioning given in the examples above is as follows.

Lemma 5.3 [5] *Let $X = \{1, 2, \ldots, u + v\}$ and also for $1 \leq j \leq u + v$, let $X_j = \{1, 2, \ldots, j\}$ and $Y_j = X \setminus X_j$. For $0 \leq i \leq k$, define*

$$\mathcal{B}_i = P_{k-i}(X_{u-i}) * P_i(Y_{u-i+1}).$$

Then $\mathcal{B}_i$ provide a partitioning of $P_k(X)$.

A more complicated generalization of Lemma 5.3 is given in the following lemma.

Lemma 5.4 **[2]** *Let a, b, s, k, v_1 and v_2 be nonnegative integers such that $s < k \leq min\{v_1, v_2\}$ and $s = k - 1 - a - b$. Let $X = \{1, 2, \ldots, v_1 + v_2 - s\}$ and also for $1 \leq j \leq v_1 + v_2 - s$, let $X_j = \{1, 2, \ldots, j\}$ and $Y_j = X \setminus X_j$. Consider the following subsets of $P_k(X)$:*

$$\mathcal{A}_i = P_{k-i}(X_{v_1}) * P_i(Y_{v_1}), \quad 0 \leq i \leq a,$$
$$\mathcal{B}_j = P_{k-a-j}(X_{v_1-j}) * P_{a+j}(Y_{v_1-j+1}), \quad 1 \leq j \leq s,$$
$$\mathcal{C}_l = P_l(X_{v_1-s}) * P_{k-l}(Y_{v_1-s}), \quad 0 \leq l \leq b.$$

Then $\mathcal{A}_i, \mathcal{B}_j$ and $\mathcal{C}_l$ partition $P_k(X)$.

Another useful partitioning is given in the next lemma. Before stating the lemma, we give an example of this partitioning.

Example. An LS[2](2, 7, 10) (and therefore an LS[2](2, 3, 10)) may be constructed from an LS[2](2, 3, 6). Let $X = \{1, 2, \ldots, 10\}$ and consider the following partitioning of $P_7(X)$:

$$\mathcal{B}_3 = P_3(\{1, 2, 3\}) * \{\{4\}\} * P_3(\{5, \ldots, 10\}),$$
$$\mathcal{B}_4 = P_3(\{1, \ldots, 4\}) * \{\{5\}\} * P_3(\{6, \ldots, 10\}),$$
$$\mathcal{B}_5 = P_3(\{1, \ldots, 5\}) * \{\{6\}\} * P_3(\{7, \ldots, 10\}),$$
$$\mathcal{B}_6 = P_3(\{1, \ldots, 6\}) * \{\{7\}\} * P_3(\{8, 9, 10\}).$$

$\mathcal{B}_3$ and $\mathcal{B}_6$ are (2,2)-partitionable by the assumption and Lemma 5.2. By Theorem 2.1, there exist LS[2](0, 3, 4) and LS[2](1, 3, 5). Therefore, $\mathcal{B}_4$ and $\mathcal{B}_5$ are (2,2)-partitionable sets by Lemma 5.2. Now Lemma 5.1 shows that $P_7(X)$ is (2,2)-partitionable set, i. e. an LS[2](2, 7, 10) is constructed.

Lemma 5.5 **[35]** *Let $X = \{1, 2, \ldots, v\}$ and also for $1 \leq j \leq v$, let $X_j = \{1, 2, \ldots, j\}$ and $Y_j = X \setminus X_j$. For $a \leq i \leq v - b - 1$, define*

$$\mathcal{B}_i = P_a(X_i) * \{\{i + 1\}\} * P_b(Y_{i+1}).$$

Then $\mathcal{B}_i$ provide a partitioning of $P_{a+b+1}(X)$.

We now use the approach to prove a simple recursive method which has been known for a long time at least for t-designs.

Lemma 5.6 *If there exist an LS[N](t, k, v) and an LS[N](t, k + 1, v), then there exists an LS[N](t, k + 1, v + 1).*

Proof Let X be a v-set and $x \notin X$. Consider the following partitioning of $P_{k+1}(X \cup \{x\})$:

$$\mathcal{B}_0 = P_{k+1}(X),$$
$$\mathcal{B}_1 = \{\{x\}\} * P_k(X).$$

By the assumption $\mathcal{B}_0$ is (N,t)-partitionable. Also $P_k(X)$ is an (N,t)-partitionable set by the assumption and therefore by Lemma 5.2, $\mathcal{B}_1$ is (N,t)-partitionable. Now the assertion follows from Lemma 5.1. □

6 General recursive constructions

In this section we present some recursive constructions for large sets of any size which are obtained by the approach of (N,t)-partitionable sets. Large sets of prime sizes will be tackled in the next section. It is worth to note that except for Theorem 3.4, all known recursive constructions for large sets were found through this approach. The first theorem is a result of Lemma 5.6 and an induction argument.

Theorem 6.1 **[20]** *If there exist $LS[N](t,k+i,v)$ for all $0 \leq i \leq l$, then there exist $LS[N](t,k+i,v+j)$ for all $0 \leq j \leq i \leq l$.*

Theorem 6.2 **[2]** *Let a, b, c, d, t, s, k, v_1 and v_2 be nonnegative integers such that $t \leq s < k \leq min\{v_1, v_2\}$ and $s = k-1-a-b = t+c+d$. Let $v_1 \in \cap_{i=k-a}^{k} A[N](t,i)$, $v_2 \in \cap_{i=k-b}^{k} A[N](t,i)$, $v_1 - l \in A[N](t,k-a-l)$ for $1 \leq l \leq c$ and $v_2 - l \in A[N](t,k-b-l)$ for $1 \leq l \leq d$. Then $v_1 + v_2 - s \in A[N](t,k)$.*

Proof Let X, X_j, Y_j, $\mathcal{A}_i, \mathcal{B}_j$ and $\mathcal{C}_l$ be as defined in Lemma 5.4. We show that $\mathcal{A}_i, \mathcal{B}_j$ and $\mathcal{C}_l$ are (N,t)-partitionable sets. Let $0 \leq i \leq a$ and $0 \leq l \leq b$. By the assumption, $P_{k-i}(X_{v_1})$ and $P_{k-l}(Y_{v_1-s})$ are (N,t)-partitionable sets and so are $\mathcal{A}_i$ and $\mathcal{C}_l$ by Lemma 5.2. Let $1 \leq j \leq s$. If $1 \leq j \leq c$, then by the assumption, $P_{k-a-j}(X_{v_1-j})$ is (N,t)-partitionable and so is $\mathcal{B}_j$ by Lemma 5.2. If $s-d < j \leq s$, then by the assumption, $P_{a+j}(Y_{v_1-j+1})$ is (N,t)-partitionable and so is $\mathcal{B}_j$ by Lemma 5.2. Now let $c < j \leq s-d$. Then, by Theorem 2.1, $P_{k-a-j}(X_{v_1-j})$ and $P_{a+j}(Y_{v_1-j+1})$ are $(N,t-j+c)$-partitionable and $(N,j-c-1)$-partitionable, respectively. Therefore, by Lemma 5.2, $\mathcal{B}_j$ is (N,t)-partitionable. □

Corollary 6.1 *If $LS[N](t,i,v)$ exist for $t+1 \leq i \leq k$ and an $LS[N](t,k,u)$ also exists, then $LS[N](t,k,u+l(v-t))$ exist for all $l \geq 1$.*

Proof It suffices to prove the assertion for $l=1$. The statement then will follow by induction. In Theorem 6.2, put $a = k-t-1, b = c = d = 0, v_1 = v$ and $v_2 = u$. □

Corollary 6.2 *If $LS[N](t,i,v+i)$ exist for $t+1 \le i \le k$ and an $LS[N](t,k,u)$ also exists, then $LS[N](t,k,u+l(v+1))$ exist for all $l \ge 1$.*

Proof In Theorem 6.2, put $a=b=d=0, c=k-t-1, v_1=v+k$ and $v_2=u$. This proves the assertion for $l=1$. Now use induction. □

Corollary 6.3 *If an $LS[N](t,t+1,v+t)$ exists, then $LS[N](t,t+1,lv+t)$ exist for all $l \ge 1$.*

Proof This is an immediate result of Corollary 6.1 for $k=t+1$. □

7 Large sets of prime sizes

The approach of (N,t)-partitionable sets has been mainly used to obtain recursive constructions for large sets of prime sizes. Theorems 7.1 and 7.2 are due to Ajoodani-Namini and provide an alternative proof of Teirlink's result on the existence of t-designs for all t. Ajoodani-Namini's method has two merits: first it is simpler than Teirlink's, and secondly it provides designs with parameters which are much smaller than the parameters of those of Teirlink.

Theorem 7.1 **[3]** *If there exists an $LS[p](t,k,v-1)$, then there exist $LS[p](t+1,pk+i,pv+j)$ for all $0 \le j < i \le p-1$.*

Theorem 7.2 **[3, 34]** *If there exists an $LS[p](t,k,v-1)$, then there exist $LS[p](t,pk+i,pv+j)$ for all $-p \le j < i \le p-1$.*

Theorems 7.1 and 7.2 could be utilized to produce a large number of infinite families of large sets. Note that these theorems are unique in design theory in the sense that they impose no further conditions on the parameters. By this, we mean that if a large set with whatever parameters is given, then using it one can construct infinite families of large sets. This is true since any large set of size N leads to a large set of size p for any prime divisor p of N.

We include some applications of Theorems 7.1 and 7.2.

Theorem 7.3 *Let $t \ge 6$ and $m \ge 2$. Then there exists an $LS[2](t,2^{t-3}-1,m2^{t-3}-2)$. Especially, there exists a t-design for any t.*

Proof Using Theorem 7.1 and noting that there exists an LS$[2](6,7,14)$ [25], we obtain large sets LS$[2](6,7,8m-2)$ for all values of $m \ge 2$. □

Theorem 7.4 *Let $t \geq 0$ and let a_i and b_i $(0 \leq i \leq t)$ be integers such that $1 \leq b_i \leq a_i \leq p-1$ for $0 \leq i < t$ and $p|\binom{b_t-1}{a_t}$. Then there exists an LS[p]$(t, \sum_{i=0}^{t} a_i p^i, \sum_{i=0}^{t} b_i p^i - 1)$.*

Proof We use an induction on t. If $t = 0$, then there is an LS[p]$(0, a_0, b_0 - 1)$ since $p|\binom{b_0-1}{a_0}$. Now let $t > 0$. By the induction hypothesis, there is an LS[p]$(t-1, \sum_{i=0}^{t-1} a_{i+1} p^i, \sum_{i=0}^{t-1} b_{i+1} p^i - 1)$. Hence, by Theorem 7.1, an LS[p]$(t, \sum_{i=0}^{t} a_i p^i, \sum_{i=0}^{t} b_i p^i - 1)$ exists. $\square$

Theorem 7.2 is generalized in the following way.

Theorem 7.5 *Let a_i and b_i $(0 \leq i \leq n)$ be integers such that $-p < b_i \leq a_i < p$ for $0 \leq i < t$. If there exists an LS[p]$(t, a_n, b_n - 1)$, then there exists an LS[p]$(t, \sum_{i=0}^{n} a_i p^i, \sum_{i=0}^{n} b_i p^i - 1)$.*

Proof We use an induction on n. If $n = 0$, then there is nothing to be proved. So let $n > 0$. By the induction hypothesis, there is an LS[p]$(t, \sum_{i=0}^{n-1} a_{i+1} p^i, \sum_{i=0}^{n-1} b_{i+1} p^i - 1)$. Hence, by Theorem 7.2, an LS[p]$(t, \sum_{i=0}^{n} a_i p^i, \sum_{i=0}^{n} b_i p^i - 1)$ exists. $\square$

We now switch to the recursive theorems which are more specific and need more assumptions.

Theorem 7.6 **[35]** *Let t, k, v and f be positive integers such that $v > k > p^f$ and $t \leq (v/p^f) < (k/p^f)$. Suppose that for every $u < v$ the following holds:*
(i) *If $u \geq p^f - 1$ and $t \leq (u/p^f) < p^f - 1$, then $u \in A[p](t, p^f - 1)$,*
(ii) *If $u \geq k - p^f$ and $(u/p^f) = (v/p^f)$, then $u \in A[p](t, k - p^f)$.*

Then $v \in A[p](t, k)$.

Proof Let $X = \{1, \ldots, v\}$ and let $X_j = \{1, \ldots, j\}$ and $Y_j = X \setminus X_j$ for $j = 1, \ldots, v$. Assume that

$$\mathcal{B}_h = P_{p^f-1}(X_h) * \{\{h+1\}\} P_{k-p^f}(Y_{h+1}), \qquad p^f - 1 \leq h \leq v - k + p^f - 1.$$

By Lemma 5.5, the sets $\mathcal{B}_h$ partition $P_k(X)$. By Lemma 5.1, it suffices to show that each $\mathcal{B}_h$ is (N, t)-partitionable.

First suppose that $(h/p^f) = p^f - 1$. Then $((v-1-h)/p^f) = (v/p^f)$ and hence $P_{k-p^f}(Y_{h+1})$ is (p, t)-partitionable by the assumption which in turn concludes that $\mathcal{B}_h$ is (p, t)-partitionable by Lemma 5.2. If $t \leq (h/p^f) < p^f - 1$, then $P_{p^f-1}(X_h)$ is (p, t)-partitionable by the assumption and so is $\mathcal{B}_h$ by Lemma 5.2. Now let $(h/p^f) = r < t$. Then $P_{p^f-1}(X_{h+t-r})$ is (p, t)-partitionable by the assumption. It yields that $P_{p^f-1}(X_h)$ is (p, r)-partitionable by Theorem 2.1. We also have $((v - h + r)/p^f) = (v/p^f)$. Therefore, $P_{k-p^f}(Y_{h-r})$ is (p, t)-partitionable by the assumption. By Theorem 2.1, we obtain that $P_{k-p^f}(Y_{h+1})$ is $(p, t - r - 1)$-partitionable. Therefore, by Lemma 5.2, $\mathcal{B}_h$ is a (p, t)-partitionable set. $\square$

Theorem 7.6 is used to obtain the following results.

Theorem 7.7 **[35]** *Let t, k, v, f and h be positive integers such that $f \leq h$ and $tp^{h-f} \leq (v/p^h) < (k/p^h)$. Suppose that $p^f + t \in A[p](t, i)$ for $t + 1 \leq i \leq min(k, (p^f + t)/2)$. Then $v \in A[p](t, k)$.*

Theorem 7.8 **[35]** *Let t, k, f and n be positive integers such that $f \leq n$, $t \leq p^{f-1}/2$ and $p^{n-1} \leq k < p^n$. Suppose that $p^f + t \in A[p](t, i)$ for $t + 1 \leq i \leq min(k, (p^f + t)/2)$. Then the following holds:*

(i) *If $v \in A[p](t, k)$, then $v + p^n \in A[p](t, k)$,*

(ii) *If $t \leq (v/p^n) < k$ and $v > 2p^n$, then $v \in A[p](t, k)$.*

8 Root cases of large sets of prime sizes

Theorem 7.6 shows that many large sets of prime sizes can be constructed from smaller large sets. Theorem 7.7 demonstrates that for given t and k there are a finite number of certain large sets which suffice to produce large sets for every possible value of v. We call these large sets *root cases*. The root cases of large sets of size 2 have already been determined by Ajoodani-Namini [1]. He has also constructed them for $t = 2$ and arbitrary k. There are similar results for large sets of any prime size. The proofs of Theorems 8.1 and 8.2 below are similar and hence we only present the proof of the latter case.

Theorem 8.1 **[1]** *Let t, k and s be positive integers such that $2^s - 1 \leq t < 2^{s+1} - 1$ and $t < k$. Suppose that for every j and n such that $0 \leq j \leq [t/2]$ and $t + 1 \leq 2^n + j \leq k$, there exists an $LS[2](t, 2^n + j, 2^{n+1} + t)$. Then $A[2](t_1, k_1) = B[2](t_1, k_1)$ for all $2^s - 1 \leq t_1 \leq t$ and $t_1 < k_1 \leq k$.*

Theorem 8.2 **[20]** *Let p be an odd prime and let t, k and s be nonnegative integers such that $p^s - 1 \leq t < p^{s+1} - 1$ and $t < k$. Suppose that the following conditions hold:*

(i) *There exists an $LS[p](t, k', p^{s+1} + t)$ for every $t + 1 \leq k' \leq min(k, (p^{s+1} + t)/2)$,*

(ii) *There exists an $LS[p](t, ip^n + j, p^{n+1} + t)$ for every i, j and n such that $0 \leq j \leq t, 1 \leq i \leq (p - 1)/2, ip^n + j \leq k$ and $n > s$.*

Then $A[p](t_1, k_1) = B[p](t_1, k_1)$ for all $p^s - 1 \leq t_1 \leq t$ and $t_1 < k_1 \leq k$.

Proof We use an induction on $t_1 + k_1$. First let $t_1 = p^s - 1$ and $k_1 = p^s$. From LS$[p](t, t + 1, p^{s+1} + t)$ and Theorem 2.1 we obtain LS$[p](t_1, k_1, p^{s+1} + t_1)$. Therefore, we are done by Theorems 4.2 and 7.7.

Now suppose that $2p^s - 1 < t_1 + k_1 \le t + k$, $t_1 \le t$ and $t_1 < k_1$. Suppose that ℓ_1 is the smallest positive integer such that $(k_1/p^{\ell_1}) > t_1$. Assume that we have shown that

$$p^{\ell_1} + t_1 \in A[p](t_1, k'), \text{ for all } t_1 + 1 \le k' \le \min(k_1, (p^{\ell_1} + t_1)/2). \tag{8.1}$$

Let $v \in B[p](t_1, k_1)$. By Theorem 4.1, there exists $r \ge \ell_1$ such that $t_1 \le (v/p^r) < (k_1/p^r)$. We have

$$[v/p^r]p^r + t_1 \in A[p](t_1, [k_1/p^r]\, p^r + j),$$

for all $(k_1/p^r) - (v/p^r) + t_1 \le j \le (k_1/p^r)$. Because, if $j < (k_1/p^r)$, we are done by the induction hypothesis. If $j = (k_1/p^r)$, then it holds by (8.1) and Theorem 7.7. Hence, by Theorem 6.1, $v = [v/p^r]p^r + (v/p^r) \in A[p](t_1, k_1)$.

Now we prove (8.1). If $k_1 > (p^{\ell_1} + t_1)/2$, then it follows from the induction hypothesis. So let

$$k_1 \le \frac{p^{\ell_1} + t_1}{2}. \tag{8.2}$$

By the induction hypothesis, it is sufficient to establish the existence of an $\mathrm{LS}[p](t_1, k_1, p^{\ell_1} + t_1)$. From (8.2), we have $[k_1/p^{\ell_1}] = 0$. Therefore, $\ell_1 \ge s+1$. If $\ell_1 = s+1$, then by (i), we can obtain $\mathrm{LS}[p](t_1, k_1, p^{s+1} + t_1)$ from $\mathrm{LS}[p](t, \max(t+1, k_1), p^{s+1} + t)$ using Theorem 2.1. So suppose that $\ell_1 > s+1$. Let $[k_1/p^{\ell_1 - 1}] = i$ and $(k_1/p^{\ell_1-1}) = j$. Clearly $j \le t_1 \le t$. By (8.2), we also obtain that $i \le (p-1)/2$. Now $\mathrm{LS}[p](t, ip^{\ell_1 - 1} + j, p^{\ell_1} + t)$, which exists by (ii), can be employed to find an $\mathrm{LS}[p](t_1, ip^{\ell_1-1} + j, p^{\ell_1} + t_1)$ via Theorem 2.1. □

9 More results on large sets of sizes two and three

In the last two section we presented some recursive constructions and theorems for large sets of prime sizes. It is possible to find more comprehensive results for large sets of sizes two and three. We will give the existence results obtained by the following theorems in the next section.

Theorem 9.1 **[2]** *Let t, k, f and n be positive integers such that $f < n$, $t \le 2^{f-2}$ and $2^{n-1} \le k < 2^n$. Suppose that $A[2](t,i) = B[2](t,i)$ for $t < i < 2^f$. Then*

(i) $B[2](t,k) \setminus A[2](t,k) \subset \{2^n + j \mid t \le j < t2^{n-f}\}$,
(ii) *If $2^{n-1} + t2^{n-f} \le k < 2^n$, then $A[2](t,k) = B[2](t,k)$.*

Theorem 9.2 **[35]** *Let t, k, f and n be positive integers such that $f < n$, $t \le 3^{f-2}$ and $3^{n-1} \le k < 3^n$. Suppose that $A[3](t,i) = B[3](t,i)$ for $t < i < 3^f$. Then*

(i) $B[3](t,k) \setminus A[3](t,k) \subset \{3^n + j \mid t \le j < t3^{n-f}\}$,
(ii) *If $2.3^{n-1} + t3^{n-f} \le k < 3^n$, then $A[3](t,k) = B[3](t,k)$.*

Theorems 8.1 and 8.2 indicate that one can construct all possible large sets of sizes two and three from the root cases LS[2]$(t, 2^n + j_1, 2^{n+1} + t)$ and LS[3]$(t, 3^n + j_2, 3^{n+1} + t)$, respectively, where j_1, j_2, and n are nonnegative integers such that $j_1 \leq t/2$ and $j_2 \leq t$. It is quite interesting that we can introduce different classes of root cases which are not related to t and say the story for all t. These classes are identified in the following theorems.

Theorem 9.3 *If there exists an LS*$[2](2^n - 2, 2^n - 1, 2^{n+1} - 2)$ *for every positive integer* n*, then* $A[2](t, k) = B[2](t, k)$ *for any* t *and* k*.*

Theorem 9.4 *If there exists an LS*$[3](3^n - 2, 3^n - 1, 2.3^n - 2)$ *for every positive integer* n*, then* $A[3](t, k) = B[3](t, k)$ *for any* t *and* k*.*

Finally, we note that by Theorem 3.4, large sets LS[2]$(2^n - 2, 2^n - 1, 2^{n+1} - 2)$ and LS[3]$(3^n - 2, 3^n - 1, 2.3^n - 2)$ can be considered as the extensions of LS[2]$(2^n - 3, 2^n - 2, 2^{n+1} - 3)$ and LS[3]$(3^n - 3, 3^n - 2, 2.3^n - 3)$, respectively. Therefore, it is possible to consider these latter classes as root cases which have to be constructed.

10 Existence results

In 1987, Hartman [17] conjectured that the necessary conditions (2.3) are sufficient for the existence of large sets of size 2. Later Khosrovshahi extended this conjecture to large sets of sizes 3 and 4 [4]. These conjectures have not yet been settled and their proofs seem to be far from reach. Note that Theorems 9.1 and 9.2 indicate that for given t if these conjectures are true for some small values of k, then they will be true for infinitely many values of k. By now, the best known result concerning these conjectures is due to Ajoodani-Namini who showed that Hartman's conjecture is true for $t = 2$ [1]. By Theorem 8.1, to establish this result, one should construct two families of large sets LS[2]$(2, 2^n + 1, 2^{n+1} + 2)$ and LS[2]$(2, 2^n, 2^{n+1} + 2)$. The first family exists according to Corollary 3.1. Ajoodani-Namini has also constructed the second family by the use of $(2, 2)$-partitionable sets. His construction is long and complicated (see [1] or [2]). We note that Ajoodani-Namini has also shown that Hartman's conjecture is true asymptotically for $k = t + 1$ [2]. He uses the approach of partitionable sets and Teirlink's methods in his proof. For large sets of size 3, we know that $A[3](2, k) = B[3](2, k)$ for $k \leq 80$ and also for infinitely many values of k [20, 35]. We now summarize the results which have been obtained by the approach of partitionable sets in the following theorem.

Theorem 10.1 *The following results are obtained through partitionable sets.*

(1) $A[2](2, k) = B[2](2, k)$ *for all* $k \geq 2$ [1].

(2) *If* $3 \leq t \leq 5$ *and* $k \leq 15$ *or,* $t = 6$ *and* $k = 7, 8, 9$*, then* $A[2](t, k) = B[2](t, k)$ [1, 5, 17, 26].

(3) *If* $2^{n-1} + 3.2^{n-4} \leq k < 2^n$ *for a positive integer* $n > 4$*, then* $A[2](3, k) = B[2](3, k)$ [2].

(4) *If* $k \leq 80$*, then* $A[3](2, k) = B[3](2, k)$ [20].

(5) *If* $t \leq 4$ *and* $k \leq 8$*, then* $A[3](t, k) = B[3](t, k)$ [34].

(6) *If* $2.3^{n-1} + 2.3^{n-4} \leq k < 3^n$ *for a positive integer* $n > 4$*, then* $A[3](2, k) = B[3](2, k)$ [35].

(7) *If* $k \leq 5$*, then* $A[5](2, k) = B[5](2, k) \setminus \{7\}$ [27].

(8) *If* $k \leq 5$*, then* $A[5](3, k) = B[5](3, k) \setminus \{8\}$ [27].

(9) *If* $k \leq 6$*, then* $A[7](2, k) = B[7](2, k)$ [27].

(10) *If* $k \leq 10$*, then* $A[11](2, k) = B[11](2, k)$ [27].

(11) *If* $k \leq 5$*, then* $A[29](2, k) = B[29](2, k)$ [27].

11 Open problems

As the previous sections suggest there are many unsolved problems on large sets of t-designs. We list some open problems here for further researches.

Problem 1 Construct an LS[3](5, 6, 14). There are five 5-(14, 6, 3) designs known [30], but the existence of LS[3](5, 6, 14) is in doubt. In the case of nonexistence, it will be a counterexample for Khosrovshahi's conjecture on large sets of size 3.

Problem 2 Is it possible to find an LS[2](6, 7, 14) through partitionable sets? All known examples of this large set have been found by prescribing some groups as automorphism group of designs.

Problem 3 Construct LS[3]$(2, 3^n + j, 3^{n+1} + 2)$ for $j = 0, 1, 2$ and for any $n > 3$. If these exist, then we will have $A[3](2, k) = B[3](2, k)$ for all $k \geq 2$.

Problem 4 Prove or disprove the existence of LS[2]$(2^n - 2, 2^n - 1, 2^{n+1} - 2)$ for $n > 4$. If these large sets exist, then Hartman's conjecture will be true.

Problem 5 Prove or disprove the existence of LS[3]$(3^n - 2, 3^n - 1, 2.3^n - 2)$ for $n > 1$. If these large sets exist, then Khosrovshahi's conjecture on large sets of size 3 will be true.

Problem 6 Determine root cases for large sets of any sizes. In particular, determine root cases for large sets of prime power sizes.

Problem 7 Are there general theorems similar to Theorems 7.1 and 7.2 for large sets of prime power sizes.

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